Letter

Continuous time reversal and equality in the thermodynamic uncertainty relation

Andreas Dechant and Shin-ichi Sasa
Department of Physics 1, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

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We introduce a continuous time-reversal operation which connects the time-forward and time-reversed trajectories in the steady state of an irreversible Markovian dynamics via a continuous family of stochastic dynamics. This continuous time reversal allows us to derive a tighter version of the thermodynamic uncertainty relation (TUR) involving observables evaluated relative to their local mean value. Moreover, the family of dynamics realizing the continuous time reversal contains an equilibrium dynamics halfway between the time-forward and time-reversed dynamics. We show that this equilibrium dynamics, together with an appropriate choice of the observable, turns the inequality in the TUR into an equality. We demonstrate our findings for the example of a particle diffusing in a tilted periodic potential.

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The behavior of a system under time reversal is one of its fundamental physical properties. While most microscopic laws of physics are invariant under time reversal, this is generally not true for macroscopic systems. In addition to the energy-driven transitions between microscopic states, we also have to account for entropy, i.e., the number of microscopic states that are compatible with a certain macroscopic state. Thus, even if two macroscopic states are energetically equivalent, the likelihood of observing them may be vastly different, and the transitions from less likely to more likely macroscopic states lead to a breaking of time-reversal symmetry and an increase in entropy.

Irreversibility is made explicit in the framework of stochastic thermodynamics; there, the entropy production $\Delta S^{\text{irr}}_\tau$ during a time interval $[0, \tau]$ is defined via the probabilities $\mathbb{P}_\tau(\Gamma)$ and $\mathbb{P}^\dagger_\tau(\Gamma)$ of observing a given trajectory Γ of the system forward and time-reversed process, respectively [1,2],

$$\Delta S^{\text{irr}}_\tau = D_{\text{KL}}(\mathbb{P}_\tau \| \mathbb{P}^\dagger_\tau) = \int d\Gamma\, \mathbb{P}_\tau(\Gamma) \ln\left(\frac{\mathbb{P}_\tau(\Gamma)}{\mathbb{P}^\dagger_\tau(\Gamma)}\right). \tag{1}$$

D_{KL} denotes the Kullback-Leibler (KL) divergence. The entropy production is positive, except when the system is symmetric under time reversal, $\mathbb{P}_\tau(\Gamma) = \mathbb{P}^\dagger_\tau(\Gamma)$. The definition Eq. (1) agrees with the thermodynamic definition of entropy for systems in contact with a heat bath, and also implies a stochastic entropy production along a single trajectory [1,2],

$$\Sigma_\tau(\Gamma) = \ln\left(\frac{\mathbb{P}_\tau(\Gamma)}{\mathbb{P}^\dagger_\tau(\Gamma)}\right), \tag{2}$$

such that its average is $\langle \Sigma_\tau \rangle = \Delta S^{\text{irr}}_\tau$.

Intuitively, the entropy production $\Delta S^{\text{irr}}_\tau$ should also control to what degree physical observables can exhibit irreversibility. This connection is made explicit in the thermodynamic uncertainty relation (TUR) [3–7]. The TUR, which applies to steady states of irreversible Markovian dynamics, is an inequality between the average and fluctuations of an observable time-integrated current J_τ [see Eq. (13)] and the entropy production $\Delta S^{\text{irr}}_\tau$,

$$\frac{(\langle J_\tau \rangle)^2}{\text{Var}(J_\tau)} \leqslant \frac{1}{2} \Delta S_{\text{irr},\tau}. \tag{3}$$

Here, $\langle J_\tau \rangle$ denotes the average accumulated current up to time τ [see Eq. (14)] and $\text{Var}(J_\tau) = \langle J_\tau^2 \rangle - \langle J_\tau \rangle^2$ is the variance. The TUR is a tradeoff relation between precision and dissipation [5,8,9]: For a fixed average amount of physical quantity (particles, work, heat, etc.) being transported, the product of fluctuations and dissipation cannot be less than the bound Eq. (3); thus small fluctuations imply large dissipation.

The TUR relates the statistics of a current, which is odd under time reversal, $\langle J_\tau \rangle^\dagger = -\langle J_\tau \rangle$, to the entropy production, which quantifies the asymmetry of the trajectories under time reversal. This suggests that this symmetry may be responsible for Eq. (3). A variant of the TUR, in which the right-hand side is proportional to the exponential of the entropy production, was derived from this symmetry in Ref. [10]. However, this bound is generally less tight than Eq. (3) [11]. In this Letter, we show that, indeed, the TUR is the consequence of the symmetry under a different type of time-reversal operation.

In general, time reversal is a discrete operation, replacing the time-forward with the time-reversed process. Our main result is that for the systems satisfying the TUR, there also exists a continuous time-reversal operation. This operation describes a family of processes, parametrized by $\theta \in [-1, 1]$, which connects the time-forward process at $\theta = 1$ to the time-reversed process at $\theta = -1$. For any value of θ, we have $\Sigma^\theta_\tau = \theta \Sigma_\tau$ and $\langle J_\tau \rangle^\theta = \theta \langle J_\tau \rangle$, so that the stochastic entropy production Eq. (2) and the symmetry of currents both extend

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in a natural way to the continuous case. Further, every member of the family has the same steady state $p^\theta_{\rm st} = p_{\rm st}$. Intuitively, the continuous time-reversal operation can be thought of as adiabatically changing the direction of the irreversible flows in the system: First, we reduce the magnitude of the flows while keeping the steady state fixed; at $\theta = 0$, the flows vanish and the system is in equilibrium. Then, we increase the magnitude of the flows in the opposite direction until at $\theta = -1$, all the flows have the same magnitude but opposite direction.

The continuous nature of this time-reversal operation allows us to derive tighter inequalities, as compared to a discrete operation. Instead of an exponential bound [10], we obtain the linear inequality Eq. (3). As our second main result, we further obtain two variants of the TUR,

$$\frac{(\langle J_\tau\rangle)^2}{\mathrm{Var}^0(J_\tau)} \leqslant \frac{1}{2}\Delta S^{\rm irr}_\tau, \tag{4}$$

$$\frac{(\langle J_\tau\rangle)^2}{\mathrm{Var}(\delta J_\tau)} \leqslant \frac{1}{2}\Delta S^{\rm irr}_\tau. \tag{5}$$

Compared to Eq. (3), the difference is in the denominator on the left-hand side. In Eq. (4), the fluctuations of the current are replaced by the fluctuations in the equilibrium process at $\theta = 0$. In Eq. (5), on the other hand, we consider the fluctuations $\delta J_\tau = J_\tau - \bar{J}_\tau$ relative to the local mean current $\bar{J}_\tau$, which is the current for a particle moving with the local mean velocity [see Eq. (24)]. We refer to Eq. (5) as the relative TUR (RTUR). In most cases of interest, we have $\mathrm{Var}^0(J_\tau) < \mathrm{Var}(J_\tau)$ and $\mathrm{Var}(\delta J_\tau) < \mathrm{Var}(J_\tau)$ and both Eqs. (4) and (5) are tighter inequalities than the TUR (3). In particular, both Eqs. (4) and (5) reduce to an equality when we choose the stochastic entropy production as the observable, $J_\tau = \Sigma_\tau$. This is in contrast to the TUR, which reduces to the Fano-factor inequality derived in Ref. [12].

Continuous time reversal. For simplicity, we focus on the case of an overdamped Langevin dynamics in the following. We consider a system of N degrees of freedom $\boldsymbol{x}(t) = [x_1(t), \dots, x_N(t)]$ whose motion is described by the overdamped Langevin equation during the time interval $t \in [0, \tau]$ [13],

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{a}(\boldsymbol{x}(t)) + \boldsymbol{G}\cdot\boldsymbol{\xi}(t). \tag{6}$$

Here, $\boldsymbol{a}(\boldsymbol{x})$ is the drift vector, and we assume that the matrix $\boldsymbol{G}$ has full rank such that the diffusion matrix $\boldsymbol{B} = \boldsymbol{G}\boldsymbol{G}^{\rm T}/2$, where the superscript T denotes transposition, is positive definite. $\boldsymbol{\xi}(t)$ is a vector of uncorrelated Gaussian white noises. The extension to a coordinate-dependent matrix $\boldsymbol{G}(\boldsymbol{x})$ is provided in the Supplemental Material (SM) [14]. The paradigmatic example is a system of N particles with systematic forces $\boldsymbol{f}(\boldsymbol{x})$, which diffuse in an environment described by a mobility μ and a temperature T. In this case, we have $\boldsymbol{a}(\boldsymbol{x}) = \mu\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{B} = \mu k_{\rm B}T\boldsymbol{1}$. We assume that $\boldsymbol{a}(\boldsymbol{x})$ and $\boldsymbol{B}$ give rise to a time-independent state in the long-time limit, i.e., that the solution of the associated Fokker-Planck equation for the probability density $p(\boldsymbol{x}, t)$ [13],

$$\partial_t p(\boldsymbol{x}, t) = -\boldsymbol{\nabla}\cdot[\boldsymbol{\nu}(\boldsymbol{x}, t)p(\boldsymbol{x}, t)] \tag{7}$$

with

$$\boldsymbol{\nu}(\boldsymbol{x}, t) = \boldsymbol{a}(\boldsymbol{x}) - \boldsymbol{B}\nabla\ln p(\boldsymbol{x}, t),$$

tends, as $t \to \infty$, towards a steady-state solution $p_{\rm st}(\boldsymbol{x})$ with local mean velocity $\boldsymbol{\nu}_{\rm st}(\boldsymbol{x})$. Physically, the local mean velocity $\boldsymbol{\nu}_{\rm st}(\boldsymbol{x})$ characterizes the irreversible local flows in the system [15,16]. Since, generally, the system described by Eq. (6) is out of equilibrium, these flows do not vanish even in the steady state. We use the local mean velocity to write the drift vector as

$$\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{\nu}_{\rm st}(\boldsymbol{x}) + \boldsymbol{B}\nabla\ln p_{\rm st}(\boldsymbol{x}). \tag{8}$$

Equation (8) may be viewed as a decomposition of the drift vector into an irreversible part $\boldsymbol{\nu}_{\rm st}(\boldsymbol{x})$ and a reversible part [15–19]. We introduce a modified drift vector,

$$\boldsymbol{a}^\theta(\boldsymbol{x}) = \theta\boldsymbol{\nu}_{\rm st}(\boldsymbol{x}) + \boldsymbol{B}\nabla\ln p_{\rm st}(\boldsymbol{x}), \tag{9}$$

with a parameter $\theta \in [-1, 1]$, and consider the corresponding Langevin dynamics

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{a}^\theta(\boldsymbol{x}(t)) + \boldsymbol{G}\cdot\boldsymbol{\xi}(t). \tag{10}$$

Compared to Eq. (8), we have rescaled the irreversible part of the drift vector, while leaving the reversible part unchanged. It is straightforward to verify that the steady-state solution for Eq. (10) is given by $p^\theta_{\rm st}(\boldsymbol{x}) = p_{\rm st}(\boldsymbol{x})$ and $\boldsymbol{\nu}^\theta_{\rm st}(\boldsymbol{x}) = \theta\boldsymbol{\nu}_{\rm st}(\boldsymbol{x})$, i.e., we obtain the same steady-state density as Eq. (6) and a local mean velocity scaled by a factor θ. The family of dynamics Eq. (10) was previously studied in Ref. [20], where it was shown to lead to generalized fluctuation theorems. Here and in the following, we use a superscript θ to refer to quantities evaluated in the dynamics with drift vector Eq. (10); quantities without a superscript refer to Eq. (6). For each value of θ, Eq. (10) generates a path probability density $\mathbb{P}^\theta_\tau[\hat{\boldsymbol{x}}]$, which measures the probability of observing a specific trajectory $\hat{\boldsymbol{x}} = [\boldsymbol{x}(t)]_{t\in[0,\tau]}$. For each trajectory, we can also consider its time-reversed version $\hat{\boldsymbol{x}}^\dagger = [\boldsymbol{x}(\tau - t)]_{t\in[0,\tau]}$. For the dynamics Eq. (10) in the steady state, this time-reversed trajectory defines the path probability of the reverse process, $\mathbb{P}^{\theta,\dagger}_\tau[\hat{\boldsymbol{x}}] = \mathbb{P}^\theta_\tau[\hat{\boldsymbol{x}}^\dagger]$. A technical but straightforward calculation [see Eq. (S50) of the SM [14]] shows that the time-reversed path probability satisfies

$$D_{\rm KL}\big(\mathbb{P}^{-\theta}_\tau[\hat{\boldsymbol{x}}]\|\mathbb{P}^{\theta,\dagger}_\tau[\hat{\boldsymbol{x}}]\big) = 0. \tag{11}$$

If the KL divergence between two probability densities vanishes, then the two probability densities are equivalent: Any average evaluated with respect to either of them yields the same result. From this, we can conclude that the dynamics Eq. (10) at $-\theta$ is equivalent to the time-reversed dynamics at θ. In particular, Eq. (10) for $\theta = -1$ yields the time-reversed dynamics of Eq. (6) (see Ref. [19]). Thus, for a general nonequilibrium dynamics, Eq. (10) provides a continuous interpolation between the original, time-forward dynamics for $\theta = 1$ and the time-reversed dynamics for $\theta = -1$. For $\theta = 0$, the irreversible part of the drift vanishes and Eq. (10) describes an equilibrium system. However, this does not necessarily correspond to the intuitive, "physical" equilibrium. The reason is that, when driving a system out of equilibrium by applying a nonconservative force, the state density $p_{\rm st}(\boldsymbol{x})$ is generally different from the equilibrium state $p_{\rm eq}(\boldsymbol{x})$ in the absence of the driving. By contrast, for Eq. (10) with $\theta = 0$, the steady state $p_{\rm st}(\boldsymbol{x})$ is the equilibrium state.

Thermodynamic uncertainty relation. While Eq. (11) provides a relation between the time-reversed dynamics at $-\theta$

and the time-forward dynamics at θ, we also obtain a relation between the time-forward dynamics at two different values of θ [see Eq. (S32) of the SM [14]],

$$D_{\mathrm{KL}}\big(\mathbb{P}_\tau^\theta[\hat{\boldsymbol{x}}]\|\mathbb{P}_\tau^{\theta'}[\hat{\boldsymbol{x}}]\big) = \tfrac{1}{4}(\theta'-\theta)^2\Delta S_{\mathrm{irr},\tau}. \tag{12}$$

For $\theta = 1$ and $\theta' = -1$, this is precisely Eq. (1). Surprisingly, the entropy production not only characterizes the difference between the forward and reverse dynamics, but also between any two members of the family of dynamics Eq. (10). Next, we establish the connection between Eq. (10) and time-integrated currents. The latter are defined as

$$J_\tau = \int_0^\tau dt\, \boldsymbol{w}(\boldsymbol{x}(t)) \circ \dot{\boldsymbol{x}}(t), \tag{13}$$

where $\boldsymbol{w}(\boldsymbol{x})$ is a weighting function and $\circ$ is the Stratonovich product. If $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{e}$ is a constant vector of unit length, then J_τ is the displacement along the direction $\boldsymbol{e}$. Another physically relevant choice is $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})$, in which case J_τ is the heat dissipated into the surrounding environment. The steady-state average of Eq. (13) is given by

$$\langle J_\tau\rangle = \tau \int d\boldsymbol{x}\, \boldsymbol{w}(\boldsymbol{x}(t)) \cdot \boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x}) p_{\mathrm{st}}(\boldsymbol{x}). \tag{14}$$

Since this is proportional to the local mean velocity, the average current in the dynamics with Eq. (10) exhibits the same scaling,

$$\langle J_\tau\rangle^\theta = \theta\langle J_\tau\rangle, \tag{15}$$

and we have $\langle J_\tau\rangle^0 = 0$ and $\langle J_\tau\rangle^{-1} = -\langle J_\tau\rangle$; the average current vanishes in equilibrium and time reversal changes its sign. Now, we return to Eq. (12) and focus on the case $\theta' = \theta + d\theta$ with $d\theta \ll 1$. Using the fluctuation-response inequality for linear response derived in Ref. [21], we have

$$\frac{\big(\langle J_\tau\rangle^{\theta+d\theta} - \langle J_\tau\rangle^\theta\big)^2}{2\mathrm{Var}^\theta(J_\tau)} \leqslant D_{\mathrm{KL}}\big(\mathbb{P}_\tau^\theta[\hat{\boldsymbol{x}}]\|\mathbb{P}_\tau^{\theta+d\theta}[\hat{\boldsymbol{x}}]\big). \tag{16}$$

Using Eqs. (12) and (15), this yields

$$\frac{(\langle J_\tau\rangle)^2}{\mathrm{Var}^\theta(J_\tau)} \leqslant \frac{1}{2}\Delta S_\tau^{\mathrm{irr}}. \tag{17}$$

Since this is valid for any value of $\theta \in [-1, 1]$, we may also maximize the left-hand side over θ, which yields

$$\frac{(\langle J_\tau\rangle)^2}{\inf_\theta[\mathrm{Var}^\theta(J_\tau)]} \leqslant \frac{1}{2}S_\tau^{\mathrm{irr}}. \tag{18}$$

This bound is tighter than Eq. (3); further, any value of θ yields a valid bound. In particular, we may choose $\theta = 1$ and obtain Eq. (3) or $\theta = 0$ and obtain Eq. (4). We remark that Eq. (17) is conceptually different from previous formulations of the TUR, since it relates observables evaluated in different dynamics.

The variance is the second cumulant of the current. However, if the distribution of the current is not Gaussian, the current also possesses nonvanishing higher-order cumulants. These can be calculated from the cumulant generating function

$$K_{J_\tau}^\theta(h) = \ln \int d\hat{\boldsymbol{x}}\, e^{hJ_\tau[\hat{\boldsymbol{x}}]}\mathbb{P}_\tau^\theta[\hat{\boldsymbol{x}}], \tag{19}$$

in terms of which the nth cumulant $\kappa_{J_\tau}^{(n),\theta}$ is defined as $\partial_h^n K_{J_\tau}^\theta(h)|_{h=0}$. Since the currents are odd under time reversal, this satisfies

$$K_{J_\tau}^{-\theta}(h) = K_{J_\tau}^\theta(-h), \tag{20}$$

that is, even cumulants are invariant under the change $\theta \to -\theta$, while odd cumulants change sign. This implies

$$\kappa_{J_\tau}^{(n),0} = 0 \quad \text{for } n \text{ odd}, \tag{21}$$

and all odd cumulants vanish in the equilibrium state at $\theta = 0$. As demonstrated in Eq. (S84) of the SM [14], we may also use the higher-order cumulants to obtain a generalization of Eq. (18),

$$\Delta S_\tau^{\mathrm{irr}} \geqslant \sup_{h,\theta}\left[\frac{h^2(\langle J_\tau\rangle)^2}{K_{J_\tau}^\theta(h) - h\theta\langle J_\tau\rangle}\right]. \tag{22}$$

This reduces to Eq. (18) in the limit $h \to 0$, but yields a tighter bound if the higher-order cumulants of the current are known. In particular, for $\theta = 1$, we obtain a higher-order TUR,

$$\Delta S_\tau^{\mathrm{irr}} \geqslant \sup_h\left[\frac{h^2(\langle J_\tau\rangle)^2}{K_{J_\tau}(h) - h\langle J_\tau\rangle}\right]. \tag{23}$$

Current fluctuations. We define the local mean value $\bar{J}_\tau$ of the current Eq. (13) by replacing the velocity with its local mean value,

$$\bar{J}_\tau = \int_0^\tau dt\, \boldsymbol{w}(\boldsymbol{x}(t)) \cdot \boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x}(t)), \tag{24}$$

and the current relative to the local mean value $\delta J_\tau = J_\tau - \bar{J}_\tau$. From the definition, it is clear that $\langle\bar{J}_\tau\rangle = \langle J_\tau\rangle$ and $\langle\delta J_\tau\rangle = 0$, i.e., only $\bar{J}_\tau$ contributes to the average current. Evaluating the average of δJ_τ in the dynamics Eq. (10), we obtain

$$\langle\delta J_\tau\rangle^\theta = (\theta-1)\langle J_\tau\rangle \Rightarrow \langle\delta J_\tau\rangle^{\theta+d\theta} - \langle\delta J_\tau\rangle^\theta = d\theta\langle J_\tau\rangle. \tag{25}$$

Using this in Eq. (16), we obtain the inequality

$$\frac{(\langle J_\tau\rangle)^2}{\mathrm{Var}^\theta(\delta J_\tau)} \leqslant \frac{1}{2}S_\tau^{\mathrm{irr}}. \tag{26}$$

For $\theta = 1$, we find the RTUR (5) involving the current relative to its local mean value. Generally, $\mathrm{Var}(\delta J_\tau)$ may be larger or smaller than $\mathrm{Var}(J_\tau)$, and thus, either the TUR or the RTUR may be tighter. However, the relation $\mathrm{Var}(J_\tau) \geqslant \mathrm{Var}(\delta J_\tau)$ often holds in practice, where Eq. (5) thus provides a tighter bound than Eq. (3). We provide an example for this behavior below.

Entropy fluctuations. An important case of a time-integrated current Eq. (13) is $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{B}^{-1}\boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x})$, for which $J_\tau = \Sigma_\tau$ is equal to the stochastic entropy production Eq. (2),

$$\Sigma_\tau[\hat{\boldsymbol{x}}] = \ln\frac{\mathbb{P}_\tau[\hat{\boldsymbol{x}}]}{\mathbb{P}_\tau^\dagger[\hat{\boldsymbol{x}}]}. \tag{27}$$

The equivalence between Eq. (13) with $\boldsymbol{w}(\boldsymbol{x})$ as above and Eq. (27) is established in Sec. SI B of the SM [14]. Written in this way Σ_τ explicitly depends on the path statistics of the entire ensemble. For the dynamics Eq. (10) we may similarly write

$$\Sigma_\tau^\theta[\hat{\boldsymbol{x}}] = \ln\frac{\mathbb{P}_\tau^\theta[\hat{\boldsymbol{x}}]}{\mathbb{P}_\tau^{\theta,\dagger}[\hat{\boldsymbol{x}}]}. \tag{28}$$

Using the definition of Σ_τ^θ in terms of $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{B}^{-1}\boldsymbol{v}_{\mathrm{st}}^\theta(\boldsymbol{x})$ together with the scaling of the local mean velocity $\boldsymbol{v}_{\mathrm{st}}^\theta(\boldsymbol{x}) = \theta \boldsymbol{v}_{\mathrm{st}}(\boldsymbol{x})$, we immediately find

$$\Sigma_\tau^\theta[\hat{\boldsymbol{x}}] = \theta \Sigma_\tau[\hat{\boldsymbol{x}}]. \tag{29}$$

This means that the parameter θ determines the relative likelihood of observing a trajectory as a forward or reverse trajectory in the dynamics with Eq. (10), with both possibilities being equally likely at $\theta = 0$. Evaluating the variance of Σ_τ [see Ref. [12] and Eq. (S72) of the SM [14]], we find

$$\mathrm{Var}(\Sigma_\tau) = \mathrm{Var}(\bar{\Sigma}_\tau) + \mathrm{Var}(\delta\Sigma_\tau). \tag{30}$$

Formally, Eq. (30) is equivalent to the introduction of an entropic time in Ref. [12]: The quantity $\bar{\Sigma}_\tau$ can be interpreted as a dimensionless, stochastic time coordinate. Then, $\delta\Sigma_\tau$ is equal to the entropy production measured in units of the stochastic time. As was shown in Ref. [12], this implies that the distribution of $\delta\Sigma_\tau$ is Gaussian. Further, we have the identities

$$\mathrm{Var}(\delta\Sigma_\tau) = 2\Delta S_\tau^{\mathrm{irr}} = \mathrm{Var}^0(\Sigma_\tau). \tag{31}$$

Comparing the first identity to the RTUR (5), we see that the latter turns into an equality. The second identity turns Eq. (4) into an equality. Using this, we may write the variational expression

$$\sup_{J,\theta}\left[\frac{(\langle J_\tau\rangle)^2}{\mathrm{Var}^\theta(J_\tau)}\right] = \sup_{J}\left[\frac{(\langle J_\tau\rangle)^2}{\mathrm{Var}(\delta J_\tau)}\right] = \frac{1}{2}\Delta S_\tau^{\mathrm{irr}}, \tag{32}$$

which characterizes the equality condition for the TUR (3) and the RTUR (5). Close to equilibrium, we have $\mathrm{Var}(\Sigma_\tau) \simeq \mathrm{Var}(\delta\Sigma_\tau)$ and the TUR turns into an equality by choosing the stochastic entropy production as an observable [22,23]. Indeed, the relation $\mathrm{Var}(\Sigma_\tau) \simeq 2S_\tau^{\mathrm{irr}}$ follows from the fluctuation-dissipation theorem [24,25]. Far from equilibrium, this breaks down, and there is generally no observable that turns the TUR into an equality; to realize the equality, we have to replace the current fluctuations with their equilibrium value at $\theta = 0$. This suggests that the presence of excess fluctuations out of equilibrium prohibits equality in the TUR. On the other hand, equality in the RTUR (5) may always be realized by choosing the stochastic entropy production as an observable. Just as the velocity relative to the local mean velocity recovers the equilibrium fluctuation-dissipation theorem [16], the current relative to the local mean value recovers the equilibrium equality condition for the TUR.

Demonstration: Tilted periodic potential. We illustrate our results using a paradigmatic example of a nonequilibrium steady state. We consider a Brownian particle in one dimension with mobility μ and at temperature T, which moves in a periodic potential $U(x+L) = U(x)$. This situation is described by the Langevin equation

$$\dot{x}(t) = \mu[-U'(x(t)) + F] + \sqrt{2\mu k_{\mathrm{B}}T}\,\xi(t). \tag{33}$$

The system is driven out of equilibrium by the constant bias force F. The (periodic) steady-state probability density and local mean velocity for this system may be computed explicitly (see Ref. [26] and Sec. S IV of the SM [14]). Since the steady-state probability density differs from the Boltzmann-Gibbs density $p_{\mathrm{eq}}(x) \propto e^{-U(x)/(k_{\mathrm{B}}T)}$, the equilibrium state for $\theta = 0$ does not coincide with the physical equilibrium at $F = 0$. In the following, we focus on the displacement z_τ of the particle with $w(x) = 1$ in Eq. (13). In the long-time limit, displacement behaves diffusively $\mathrm{Var}(z_\tau) \simeq 2D_z\tau$; an explicit expression for the diffusion coefficient D_z was derived in Ref. [26]. One remarkable feature appears for low temperatures and a bias force close to the critical value F_{crit}, at which the minima of the tilted potential disappear. Under these conditions, the diffusion coefficient can be orders of magnitude larger than the free-diffusion coefficient in the absence of the periodic potential $D_{z,\mathrm{free}} = \mu k_{\mathrm{B}}T$ [26]. As a function of the bias, the diffusion coefficient is small for small bias, reaches a maximum near critical tilt, and then decreases towards the free value (see Fig. 1). However, this enhancement of diffusion is absent in the displacement relative to the local mean value: The corresponding diffusion coefficient $D_{\delta z}$ increases monotonously towards the free value and is always smaller than D_z (see Fig. 1). As a consequence, we have

$$\eta_z = \frac{2(\langle z_\tau\rangle)^2}{\mathrm{Var}(z_\tau)S_\tau^{\mathrm{irr}}} \leqslant \eta_{\delta z} = \frac{2(\langle z_\tau\rangle)^2}{\mathrm{Var}(\delta z_\tau)S_\tau^{\mathrm{irr}}} \leqslant 1, \tag{34}$$

i.e., the RTUR (5) is tighter than the TUR (3). For small bias (near equilibrium), η_z approaches unity. For large bias,

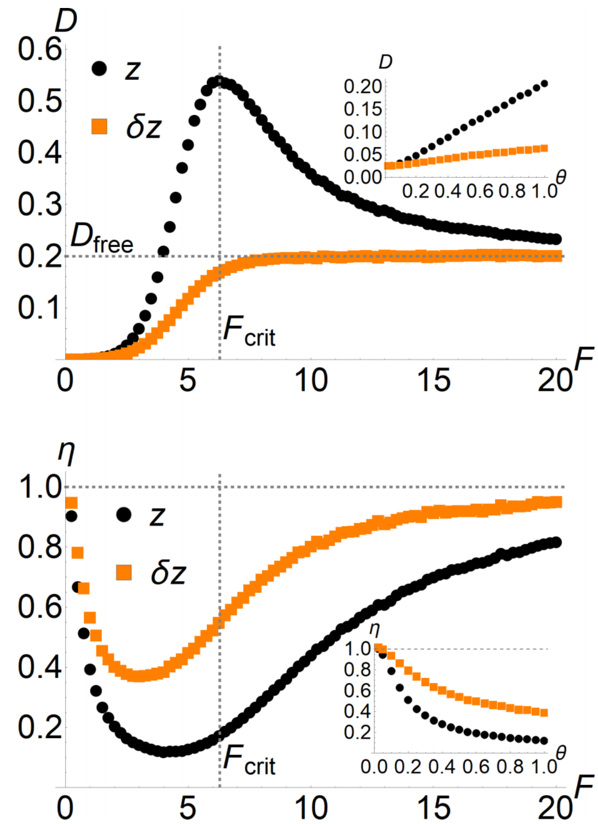

FIG. 1. The diffusion coefficient (top) and the transport efficiency (bottom) as a function of the bias force (main panel) and the parameter θ for $F = 4$ (inset). Black dots show the respective quantity for the displacement, while the orange squares correspond to the fluctuations of the displacement around its local mean value. The data were obtained using Langevin simulations in a sine potential $U(x) = U_0\sin(2\pi x/L)$ with $U_0 = 1$, $L = 1$. The temperature and mobility were set to $T = 0.2$ and $\mu = 1$.

the potential becomes negligible and the system behaves as a biased diffusion, where η_z likewise approaches unity. For intermediate bias, on the other hand, η_z is significantly smaller than unity. In this regime, the bound involving δz_τ is considerably tighter, indicating that the decrease in η_z is partly due to the enhancement of the diffusion coefficient. Note that, in general, the definition of the local mean current Eq. (24) involves the local mean velocity and may thus be difficult to compute in cases where the latter is not explicitly known. However, for one-dimensional systems, we have the relation

$$\nu_{\mathrm{st}}(x) = \frac{\langle \dot{z} \rangle}{L p_{\mathrm{st}}(x)}, \tag{35}$$

and thus $\nu_{\mathrm{st}}(x)$ and δz can be evaluated by measuring the steady-state probability density. Finally, we remark that, while both $F = 0$ and $\theta = 0$ (for finite F) correspond to an equilibrium dynamics, the nonmonotonic behavior in D_z and η_z only appears as a function of F. By contrast, D_z (η_z) increases (decreases) monotonically when changing θ from 0 to 1 (see the insets of Fig. 1).

Discussion. The dynamics Eq. (10) provide a natural way to interpolate between the time-forward and the time-reversed dynamics, replacing a discrete operation with a continuous one. A continuous operation can be represented by a series of infinitesimal steps, which can then be analyzed individually, reconstructing the entire operation from the individual steps. In the present context, this allows us to apply the linear-response fluctuation-response inequality Eq. (16), providing a tighter inequality than can be obtained by directly comparing the time-forward and time-reversed process (see also Sec. S III of the SM [14]).

In many applications, nonequilibrium states are obtained by driving an equilibrium system, for example, by applying nonconservative force. In this case, the nonequilibrium system has a natural equilibrium counterpart. However, for a given nonequilibrium state, this equilibrium is not unique; the same nonequilibrium state may be obtained by driving two different systems in different ways. Thus, knowledge about the equilibrium system may not necessarily tell us anything about the nonequilibrium state. By contrast, the continuous time-reversal operation continuously connects a nonequilibrium steady state to a unique equilibrium system with the same steady state. As demonstrated in the insets of Fig. 1, the physical properties of the system change in a much more controlled fashion between this unique equilibrium and the nonequilibrium state, when compared with the physical equilibrium state. If this type of behavior can be shown to be generic, this may provide an alternate approach of characterizing nonequilibrium states in terms of equilibrium states and their well-understood properties.

A practical application of the TUR Eq. (3) is to estimate the entropy production and thus dissipation by measuring a current in the system [27–30]. Since the dissipation is often not directly accessible in experiments, relating it to measurable quantities is crucial. Then, an obvious question is how good the lower estimate Eq. (3) on the entropy production can be. The generally tighter bounds Eqs. (4) and (5) restrict the quality of this estimate in terms of the fluctuations of the current. If we have $\mathrm{Var}(J_\tau) \geqslant \mathrm{Var}^0(J_\tau), \mathrm{Var}(\delta J_\tau)$, then this immediately implies that the estimate from the TUR will be too small by at least this amount.

While in this work we focused on overdamped Langevin dynamics, the notion of continuous time reversal and the results of this Letter also apply to Markov jump dynamics, as we will discuss in an upcoming publication [31]. Since the TUR follows explicitly as a consequence of the continuity, we speculate that finding a continuous time-reversal symmetry may serve as a way to extend the TUR to other classes of dynamics. Whether such an operation exists depends on the dynamics; for example, it is known that the TUR can be violated in the presence of magnetic fields which transform in a discrete manner under time reversal [32]. Similarly, it would be interesting to explore whether recent extensions of the TUR to nonsteady initial states [33], time-periodic [34–36], or arbitrary time-dependent driving [37] can be connected to the existence of a generalized continuous time-reversal operation.

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