

Classification of magnetic vortices by angular momentum conservation

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Superfluid vortices are quantum excitations carrying a quantized amount of orbital angular momentum in a phase where global symmetry is spontaneously broken. We address a question whether magnetic vortices in superconductors with dynamical gauge fields carry nonzero orbital angular momentum or not. We provide the answer in view of the angular momentum conservation in several distinct classes of examples from interdisciplinary fields of physics across condensed matter, dense nuclear systems, and cosmology. The angular momentum carried by gauge field configuration around the magnetic vortex plays a crucial role in satisfying the principle of the angular momentum conservation. Based on various ways on how the angular momentum conservation is realized, we provide a general scheme of classifying magnetic vortices in different phases of matter.

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I. INTRODUCTION

When a superfluid with a macroscopic condensate of identical bosons is under rotation, superfluid vortices emerge. Each microscopic constituent in a superfluid vortex carries the same amount of quantized orbital angular momentum, i.e., an integer multiple of fundamental quantum \hbar . This is a remarkable way to store macroscopic amount of angular momentum in a highly coherent quantum state. Not only in table-top physical systems of superfluids such as ^4He , the superfluid vortex can also be an important constituent in rotating nuclear matter found inside a neutron star, where the extremely high matter density causes a nonzero order parameter that signifies spontaneous breaking of global baryon number $U(1)_B$ symmetry. In a more interesting scenario of dense quark matter in quantum chromodynamics (QCD), this order parameter is also responsible for the superconducting phase of color gauge interactions, most likely the color-flavor-locked (CFL) superconducting phase [1]. There exist interesting nontrivial vortices in the CFL phase, called non-Abelian CFL vortices [2], that involve dynamics of both the global baryon and the local color gauge symmetries; see Ref. [3] for a comprehensive review.

As seen in the example of a non-Abelian CFL vortex, the symmetries involved in vortex contents are often entirely

or partially gauge symmetries in many interesting physical systems. The prototypical example is of course the magnetic vortex in type-II superconductors. In these cases the vortex profile is fundamentally different from that of the purely superfluid vortex; a magnetic flux is threaded into the vortex core. Among many differences between a superfluid vortex and a gauged magnetic vortex, one may ask a question of the angular momentum they carry. Surprisingly to us, we find that this simple question has not been properly addressed in the literature. As we try to answer the question in various examples across different fields of physics, we discover surprisingly diverse situations. It is the purpose of this article to present a compelling list of examples where the answers are quite different from each other, and also to provide an overarching physics explanation of why the answers can be so diverse. We will demonstrate that the angular momentum conservation offers a key guiding principle to understand the physics origin of the different answers. Our detailed analysis in the main text shows that the angular momentum, carried by not only the matter sector of the system but also by the dynamical gauge fields surrounding the vortex, should be considered in order to fulfill the principle of angular momentum conservation. Building upon this principle, we attempt a general classification scheme of magnetic vortices in different phases of matter that can hopefully be applied to other physical systems.

A natural starting point of our discussion is the vortices in type-II superconductors. Quite generally, it is easy to see that the vortex should carry a nonvanishing orbital angular momentum. Due to one of the Maxwell equations $\nabla \times \mathbf{B} = \mathbf{j}$ (where we choose a natural unit in which the magnetic permeability μ_0 is the unity), a smooth and finite ranged profile of magnetic flux means the existence of the azimuthal component of the current density \mathbf{j} . Under a fairly general assumption that the charge and momentum carriers are non-relativistic quasielectrons in the conduction band forming the

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Fermi surface with a superconducting gap, the current and the momentum are linearly related as an operator relation, holding for all states: $\mathbf{j} = -\frac{e}{m}\mathbf{P}$, where \mathbf{P} is the momentum density operator, m is the effective mass of conduction electrons, and $-e$ is the charge of electrons. Since $\mathbf{j} \neq \mathbf{0}$, we have $\mathbf{P} \neq \mathbf{0}$ and the finite sized vortex should carry a finite angular momentum by $\mathbf{L} = \int_V \mathbf{x} \times \mathbf{P} \neq \mathbf{0}$.

The linear relation between the current and the momentum for nonrelativistic electrons is a consequence of Galilean invariance, and is not necessarily universal. Even though the dispersion relation deviates from the nonrelativistic Galilean invariant one, the current and the momentum are still negatively correlated, and there is no reason to exclude nonvanishing angular momentum. This discussion also implies that the quantization of the angular momentum in units of \hbar may not be universal. To complicate the situation more nontrivially, some vortices may also carry a localized electric charge [4,5], and the resulting electromagnetic (EM) field around such a vortex gives rise to a nonvanishing Poynting vector around the vortex core axis. The total angular momentum should then include a contribution from the EM field around the vortex.

Although the above features are robust, one can consider the following thought experiment that is somewhat similar to the Feynman's angular momentum paradox. One places a solenoid below a superconductor sample, and turns on an external magnetic field to create magnetic vortices piercing the superconductor. The process can be implemented in an azimuthal symmetric way, and should not change the total angular momentum which is zero initially. Since the created vortices carry a finite amount of angular momentum, where can the compensating angular momentum be found?

The answer to this question is easy to guess: the background of solid crystal and the electrons in valence bands should carry the compensating angular momentum. Their inertia is infinitely large and their rotation may not be detectable, but the torque acting on them during the vortex creation process should impart to them precisely the negative amount of the angular momentum of the created vortices. In Sec. II A we will be able to confirm this expectation quantitatively, in a concrete model which is simple and yet general enough to carry out the analysis of charged magnetic vortices. In this case the angular momentum carried by EM field also needs to be counted in the total angular momentum, and the angular momentum conservation holds true quite nontrivially only after including this EM contribution. We note that the EM field is localized and attached to the vortex, so one should think of it as a part of the vortex profile.

As a continuation of the above thought experiment, we ask an obvious next question: what would happen in a system where there is no background matter to absorb the angular momentum? A concrete example of such a system is provided by a relativistic field theory which is self-consistent by itself without any other degrees of freedom: it could be identified as the electroweak sector of the standard model with Higgs field condensate, or more simply, a theoretical model by Nielsen and Olesen [6]. For this class of examples, our previous argument of angular momentum conservation becomes powerful enough to dictate that any magnetic vortices, either charged or not, should carry zero angular momentum. We call them

“spinless vortices.” We will show that this statement is indeed true for the Nielsen-Olesen model. In showing this for the charged vortex case, it is again critical to include the EM or gauge field contribution to the total angular momentum. For a similar conclusion for the dyonic solitons, see Refs. [7–9]. We make a remark that Appendix C of the well-known reference, Ref. [10], contains an erroneous statement on this for the charged vortex case, which we will rectify in Sec. III A. This showcases the nontriviality of our argument based on the angular momentum conservation.

What would happen if the vortex is a combination of magnetic vortex of a gauge symmetry and superfluid vortex of a global symmetry? To answer this question, we study in Sec. III B the previously mentioned example of the “non-Abelian vortex” [2,11] in the CFL superconducting phase of dense quark matter, which may be relevant for the physics of neutron star cores. The non-Abelian CFL vortices also play a role in the idea of quark-hadron continuity [12] in the high density region of a QCD phase diagram [13–16] (see, however, Refs. [17–19] for recent debates on the idea of quark-hadron continuity). For such an object of composite nature, one can imagine a creation process by an external magnetic field for the gauge symmetry, together with a physical rotation to create the superfluid vortex part of the global symmetry. Our angular momentum conservation argument then predicts that the total angular momentum should be given only by the superfluid part of global symmetry without any contribution from the gauged symmetry part. We will explicitly confirm this expectation in a highly nontrivial manner.

There is a logical exception to the above argument for spinless vortices in a system with no background. During the above considered creation process by an external solenoid, a finite amount of angular momentum may diffuse away to spatial infinity, resulting in an opposite amount of angular momentum localized around the vortex. The total angular momentum is conserved and zero, but the part at infinity is not visible, and should not be thought of as a contribution to the angular momentum of the localized vortex. The vortex then carries a leftover angular momentum that is finite. What distinguishes this case from our first case with background matter is that the opposite angular momentum to the one carried by the vortex strictly resides at spatial infinity, or more precisely at the boundary of the system far away from the vortex. This makes a contrast to the previous case with background matter, where the bulk of the background absorbs a finite angular momentum.

In Sec. II B we discuss an instructive example of this class of vortex that is provided by the magnetic vortex on the surface of a time-reversal invariant (i.e., T invariant) strong topological insulator (TI) in a setup recently studied in Ref. [20]. Although the authors of Ref. [20] considered an interface between TI and a superconductor, we will focus on the TI part to account clearly for the physics origin of the net fractional (in units of \hbar) angular momentum of the vortex. We will show that the total angular momentum solely arises from the gauge field configuration surrounding the vortex on the TI surface, without any TI matter contributions. We will argue for this peculiar feature that the topological nature of the TI is responsible for moving apart a finite angular momentum to the (infinitely separated) boundary, which characterizes this class

of example. Ubiquitous topological vortices with fractional angular momentum in topological phases of matter as found in Refs. [21–23] should belong to this class of magnetic vortex.

The final class of vortex in our classification is from the last logical possibility: a vortex may *not* be created by our thought experiment with an external solenoid in a way that conserves angular momentum, and additional operations to violate angular momentum conservation must be performed to create a vortex. This class of vortex is rather exotic and rarely found in the literature: one example we address in this paper is an object called the “charged semilocal vortex” of Abraham [24]. Since this class of vortex simply falls outside of our principle of angular momentum conservation, they may or may not carry an angular momentum: in our example the charged semilocal vortex carries a finite angular momentum. It is an inhomogeneous profile along the vortex axis that makes it impossible to create this kind of vortices by simply piercing an external magnetic flux: an additional “twisting” or “spinning” along the axis is needed to create such a vortex profile.

In summary, we have the following distinct classes of vortices in regard to angular momentum conservation and their creation processes:

(1) Class Ia (spinful vortices): They carry a finite angular momentum due to the existence of background matter that can absorb angular momentum. Examples are the vortices in type-II superconductors.

(2) Class Ib (topological vortices): They carry a finite angular momentum, but no background matter exists in the bulk. The angular momentum resides only on the boundary. The angular momentum carried by the surrounding gauge fields must be counted for the total angular momentum. Examples are the vortices on the surface of topological phases of matter.

(3) Class II (spinless vortices): They do not carry a net angular momentum due to the angular momentum conservation. The angular momentum carried by the surrounding gauge fields should be included. Examples are the vortices in relativistic field theories and cosmology.

(4) Class III (exotic vortices): They have an inhomogeneous profile along the vortex axis, so that they cannot be created by a simple procedure of piercing magnetic flux. They may or may not carry angular momentum. An example is the charged semilocal vortex.

In the following sections we present detailed analysis on the concrete examples that belong to each of the above classes.

II. CLASS I: CASE STUDY OF MAGNETIC VORTICES WITH NONZERO ANGULAR MOMENTUM

In this section we discuss the case of magnetic vortices that carry a nonzero angular momentum. Because the angular momentum should be conserved as long as rotational symmetry is preserved, the angular momentum of magnetic vortices, if it is nonzero, should be balanced by other contributions. According to the types of such balancing contributions, we further classify them into two distinct subclasses; namely, class Ia and class Ib.

A. Class Ia: Incomplete cancellation due to background matter

The most familiar magnetic vortices in a type-II superconductor belong to class Ia. The magnetic vortices can carry a

nonzero angular momentum but its value is not quantized in units of \hbar , unlike the angular momentum carried by superfluid vortices. Explicit calculations as shown below make clear where the difference appears.

For an explicit demonstration we shall consider a relativistic scalar field theory in the Higgs phase of $U(1)$ symmetry, so that gauged magnetic vortices emerge. We then take the nonrelativistic reduction and find the equations of motion that are familiar in condensed matter physics describing magnetic vortices in type-II superconductors. The Lagrangian density we study in the natural unit system ($\hbar = c = 1$) reads as

$$\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - U(\Phi^\dagger \Phi) + \frac{1}{2} \mathbf{E}^2 - \frac{1}{2} \mathbf{B}^2 - qA_0, \tag{2.1}$$

where $D_\mu = \partial_\mu + ien_e A_\mu$ is the covariant derivative, with n_e being the electric charge carried by Φ in units of $e > 0$. As usual, $\mathbf{E} = -\nabla A_0 - \partial_0 \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ are electric and magnetic fields, respectively. We should choose $n_e = -2$ for the Cooper pair in electron superconductivity. The last term qA_0 , with a background charge density $q(x)$, is introduced to keep the total electric charge neutrality, which we will simply refer to as the “background” in the following. For example, in a solid with conduction electrons, positively charged ions in the crystal and other electrons in valence bands neutralize the whole system. We also note that a finite chemical potential μ is introduced by replacing $ien_e A_0 \rightarrow ien_e A_0 - i\mu$. The potential $U(\Phi^\dagger \Phi)$ is chosen to have a nonzero condensate of Φ in the Higgs phase, the simplest choice of which would be a polynomial form:

$$U(\Phi^\dagger \Phi) = -\lambda_2 \Phi^\dagger \Phi + \frac{\lambda_4}{2} (\Phi^\dagger \Phi)^2. \tag{2.2}$$

The equations of motion from the Lagrangian are given by

$$-(D_\mu D^\mu) \Phi + \lambda_2 \Phi - \lambda_4 (\Phi^\dagger \Phi) \Phi = 0, \tag{2.3}$$

$$-\partial_0 \mathbf{E} + \nabla \times \mathbf{B} + ien_e [(\mathbf{D}\Phi)^\dagger \Phi - \Phi^\dagger (\mathbf{D}\Phi)] = 0, \tag{2.4}$$

$$\nabla \cdot \mathbf{E} + ien_e [(D_0 \Phi)^\dagger \Phi - \Phi^\dagger D_0 \Phi] - q = 0, \tag{2.5}$$

with $(\mathbf{D})^i \equiv D^i$, where we note that $\partial^i = \partial/\partial x_i = -\partial/\partial x^i$ in our metric convention $(+, -, -, -)$. The magnetic vortices we consider are the static solutions of the above equations of motion, so we drop the time derivative terms in the below. Then Eq. (2.3) takes a form of

$$[(\mu - en_e A^0)^2 + \mathbf{D}^2 + \lambda_2] \Phi - \lambda_4 (\Phi^\dagger \Phi) \Phi = 0. \tag{2.6}$$

Instead of solving this equation directly, we would like to make the problem close to a more conventional situation in condensed matter physics, by considering the nonrelativistic reduction. Because the nonrelativistic energy is measured from the rest mass energy m , we should split the mass term and rescale the field as

$$\mu \rightarrow m + \tilde{\mu}, \quad \lambda_2 \rightarrow -m^2 + 2m\tilde{\lambda}_2, \quad \Phi \rightarrow \frac{\psi}{\sqrt{2m}}, \tag{2.7}$$

where $\tilde{\mu}$ denotes the nonrelativistic chemical potential. Equation (2.6) multiplied by $1/\sqrt{2m}$ then becomes

$$(\tilde{\lambda}_2 + \tilde{\mu} - en_e A^0) \psi + \frac{\mathbf{D}^2}{2m} \psi - \frac{\lambda_4}{4m^2} |\psi|^2 \psi = 0, \tag{2.8}$$

where we have dropped a subleading term proportional to $(\tilde{\mu} - en_e A^0)^2 / (2m)$.

We should solve Eq. (2.8) together with Eqs. (2.4) and (2.5) for EM fields. The Gauss law (2.5) reads

$$\nabla^2 A^0 + en_e |\psi|^2 + q = -\frac{en_e}{m} (\tilde{\mu} - en_e A^0) |\psi|^2 \simeq 0, \quad (2.9)$$

where we again drop the last term which is subleading according to the approximation made in Eq. (2.8), while we still keep the kinetic term $D^2 / (2m)$ in Eq. (2.8). For notational brevity, let us rename our variables as follows:

$$\tilde{\lambda}_2 + \tilde{\mu} \rightarrow \mu, \quad \frac{\lambda_4}{4m^2} \rightarrow g, \quad A^0 = \frac{\mu}{en_e} a, \quad \psi \rightarrow \sqrt{\frac{\mu}{g}} \Psi. \quad (2.10)$$

Here we note that this μ is different from the original one in Eq. (2.6). Together with the Maxwell equation for \mathbf{A} in Eq. (2.4), our equations finally become

$$(1 - a)\Psi + \frac{1}{m_H^2} (\nabla - ien_e \mathbf{A})^2 \Psi - |\Psi|^2 \Psi = 0, \quad (2.11)$$

$$\nabla \times (\nabla \times \mathbf{A}) + m_V^2 \left[\mathbf{A} |\Psi|^2 - \frac{i}{2en_e} (\Psi \nabla \Psi^\dagger - \Psi^\dagger \nabla \Psi) \right] = 0, \quad (2.12)$$

$$\nabla^2 a + 2m^2 \frac{m_V^2}{m_H^2} (|\Psi|^2 + \tilde{q}) = 0, \quad (2.13)$$

where $\tilde{q} \equiv (g/en_e \mu)q$, and we also introduce the two typical mass scales as

$$m_H^2 \equiv 2m\mu, \quad m_V^2 \equiv \frac{(en_e)^2 \mu}{mg}. \quad (2.14)$$

Physically, $1/m_H$ represents the coherent length of the field Ψ , while $1/m_V$ represents the penetration length of the magnetic field. If the penetration length is smaller than the coherent length, $m_V > m_H$, the Meissner screening effect is dominant and the phase separation is more preferred than forming magnetic vortices, which corresponds to type-I superconductivity. We are interested in type-II superconductivity in the opposite regime with $m_H > m_V$.

The ansatz for the vortex solution with the winding number ν is

$$\Psi = f(r) e^{i\nu\varphi}, \quad a = a(r), \quad A^i = -\frac{\nu}{en_e} \varepsilon^{ij} \frac{x^j}{r^2} [1 - h(r)], \quad (2.15)$$

where $r \equiv \sqrt{x^2 + y^2}$ and $\tan \varphi \equiv y/x$. Introducing a dimensionless radial coordinate $\rho \equiv m_V r$, we can rewrite the differential equations (with $' \equiv \frac{d}{d\rho}$) as

$$-(\rho f')' + \frac{\nu^2 h^2}{\rho} f + \lambda \rho f (f^2 - 1 + a) = 0, \quad (2.16)$$

$$\rho \left(\frac{h'}{\rho} \right)' - f^2 h = 0, \quad (2.17)$$

$$\frac{1}{\rho} (\rho a')' + \frac{2}{\lambda} \frac{m^2}{m_V^2} (f^2 + \tilde{q}) = 0, \quad (2.18)$$

where $\lambda \equiv m_H^2 / m_V^2 > 1$. For the total charge neutrality condition, we impose the condition

$$\int_x \tilde{q} = - \int_x f^2. \quad (2.19)$$

Here \int_x refers to the two-dimensional integration on the plane perpendicular to the vortex axis. This neutrality condition is demanded by the fact that the static potential would behave as $a(\rho \gg 1) = \frac{Q}{2\pi} \log \rho$ if the total net charge Q is nonzero. The combination of $(\mu - en_e A^0)$ appears in the equations of motion and it plays a role of an effective chemical potential. To have a well-defined effective chemical potential at spatial infinity, we should impose $Q = 0$.

We can numerically solve these differential equations with appropriate boundary conditions. Let us first consider the conventional ‘‘locally neutral’’ vortex solution without coupling to electric field, so that $a(\rho) = 0$ simply. This can be achieved by choosing a space dependent background charge density $\tilde{q}(x)$ that locally neutralizes the net charge; that is, $f^2 + \tilde{q} = 0$, leading to $a(\rho) = 0$ from Eq. (2.18). Most type-II vortices behave this way, but there are examples where this does not happen in general; see Refs. [4,5]. The regularity of Ψ at $\rho = 0$ requires $f(0) = 0$, and at infinity it should approach the vacuum value of $f(\infty) = 1$. In the absence of a , then the boundary conditions should be

$$f(0) = 0, \quad f(\infty) = 1, \quad h(0) = 1, \quad h(\infty) = 0. \quad (2.20)$$

We can easily obtain the numerical solutions using a technique called the ‘‘shooting method’’ to satisfy these boundary conditions. The left panel of Fig. 1 shows an example of the profile of the magnetic vortex for $\lambda = 1.5$. We see that $h(\rho)$ extends more widely than $f(\rho)$, reflecting $m_H > m_V$.

As a nontrivial example where the local charge density and the electric field are nonvanishing, let us consider a constant background charge density \tilde{q} that is determined by the total charge neutrality condition (2.19) as

$$\tilde{q} = -\frac{1}{S} \int_x f^2, \quad (2.21)$$

with $S \equiv \int_x$ as the transverse area. In the limit of an infinitely large system \tilde{q} would approach the negative unity. In the present case we should revise the boundary conditions accordingly. That is, f needs not be unity at large ρ , but $f^2 - 1 + a$ should be vanishing as ρ gets large. Also, we physically require vanishing electric field at $\rho = 0$ and $\rho \rightarrow \infty$. Therefore, we impose the following boundary conditions:

$$f(0) = 0, \quad f(\infty) = \sqrt{1 - a(\infty)}, \quad h(0) = 1, \quad h(\infty) = 0, \quad a'(0) = 0, \quad a'(\infty) = 0. \quad (2.22)$$

Actually, these boundary conditions are not sufficient to determine the numerical solution uniquely, but a shift of $a(\rho) \rightarrow a(\rho) + c$ with a constant c still exists. This shift would change the value of μ , and the magnitude of condensate would also be modified, which would result in a different value of \tilde{q} in Eq. (2.21). In other words, we can adjust \tilde{q} to make a constant shift on $a(\rho)$. To fix this freedom, a natural condition to impose would be to set $a \rightarrow 0$ at large ρ , so that the effective chemical potential at infinity, by definition, remains to be μ . We choose $\lambda = 1.5$ and $m^2 / m_V^2 = 1$ to find that $a(\infty) \rightarrow 0$

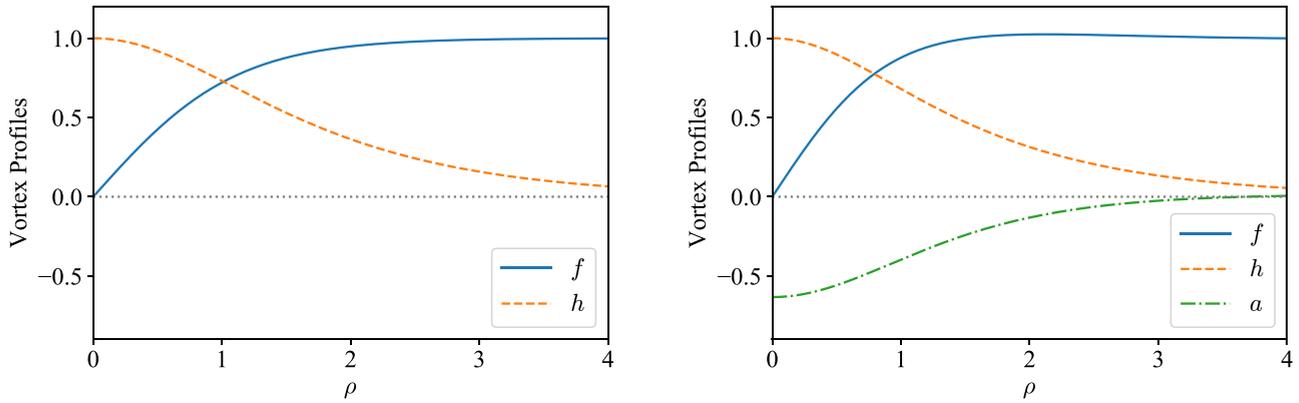


FIG. 1. (Left panel) Profile of the conventional elementary ($v = 1$) magnetic vortex; f and h without coupling to a for $\lambda = 1.5$. (Right panel) Profile of the elementary vortex with the electric field; f , h , and a for $\lambda = 1.5$ and $m^2/m_v^2 = 1$.

is realized with $\tilde{q} \simeq -0.985$. In the right panel of Fig. 1 we present the numerical solution with these parameters. This explicitly demonstrates that nontrivial solutions with nonzero local charge density and electric field certainly exist. We see that the profile of the condensate slightly shrinks as compared to the locally neutral case shown in the left panel.

Let us now compute the angular momentum carried by the matter and the EM fields. The matter part of the angular momentum per unit vortex length is

$$L_z^{\text{matter}} = \int_x \psi^\dagger \left(\frac{\hbar}{i} D_\varphi \right) \psi, \quad D_\varphi \equiv \partial_\varphi - \frac{ien_e}{\hbar} A_\varphi, \quad A_\varphi \equiv \epsilon^{ij} x^i A^j = \frac{v\hbar}{en_e} [1 - h(r)], \quad (2.23)$$

where we reinstate \hbar as a common unit for the angular momentum and also change the variables back to r and φ . We note that the boundary condition (2.22) guarantees $D_\varphi[f(r)e^{iv\varphi}] \rightarrow 0$ as $r \rightarrow \infty$, and the above integral is convergent. With the explicit forms of the vortex profile and the associated vector potential, the matter part of the angular momentum per unit vortex length becomes

$$L_z^{\text{matter}} = v(2\pi\hbar) \frac{\mu}{g} \int_0^R dr r h(r) f^2(r). \quad (2.24)$$

Let us next consider the EM contribution, i.e.,

$$L_z^{\text{gauge}} = \int_x [\mathbf{x} \times (\mathbf{E} \times \mathbf{B})]_z. \quad (2.25)$$

Plugging the explicit forms of \mathbf{E} and \mathbf{B} into the above, we find L_z^{gauge} as

$$L_z^{\text{gauge}} = -(2\pi\hbar) \frac{v\mu}{(en_e)^2} \int_0^R dr \left[r \frac{da(r)}{dr} \right] \frac{dh(r)}{dr}. \quad (2.26)$$

In Fig. 2 the integrand corresponding to the local angular momentum density is plotted, where the variables are made dimensionless again. The angular momentum distribution is peaked around $\rho \sim 1$ and decays at large ρ . We can perform an integration by part and use the equation of motion to

transform the above expression into

$$L_z^{\text{gauge}} = -(2\pi\hbar) \frac{v\mu}{(en_e)^2} \left\{ r \frac{da(r)}{dr} h(r) \Big|_0^R + 2 \frac{m^2}{\lambda} \int_0^R dr r h(r) [f^2(r) + \tilde{q}(r)] \right\}. \quad (2.27)$$

Because of the boundary conditions (2.22), the surface contribution vanishes. Using $\lambda = m_H^2/m_V^2$ we can simplify the above expression into

$$L_z^{\text{gauge}} = -v(2\pi\hbar) \frac{\mu}{g} \int_0^R dr r h(r) [f^2(r) + \tilde{q}(r)]. \quad (2.28)$$

Comparing with the matter contribution L_z^{matter} in Eq. (2.24), the first term is remarkably equal to $-L_z^{\text{matter}}$, and the total kinetic angular momentum is thus

$$L_z^{\text{total}} = -v(2\pi\hbar) \frac{\mu}{g} \int_0^R dr r h(r) \tilde{q}(r). \quad (2.29)$$

We see that L_z^{total} is proportional to \tilde{q} and this nonzero value of the total angular momentum is attributed to the presence of the background. If we had no background, $\tilde{q} = 0$, then L_z^{matter} and L_z^{gauge} would have perfect cancellation, but we then allow

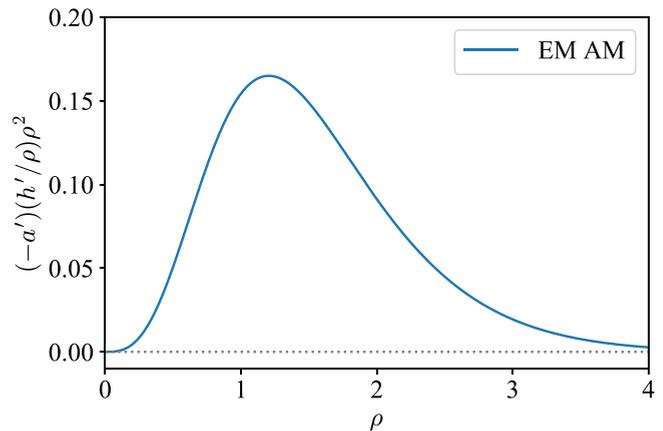


FIG. 2. The integrand of Eq. (2.26) in terms of dimensionless variables for $\lambda = 1.5$ and $m^2/m_v^2 = 1$, which represents the local distribution of the EM angular momentum.

for a “charged” magnetic vortex. This might be possible due to finiteness of the system bounded by R . A natural realization of this possibility will be discussed as class II in the next section.

B. Class Ib: Incomplete cancellation due to topological boundary

Our next example for incomplete cancellation has been motivated by the physical setup discussed in Ref. [20] where a fractional angular momentum in the units of \hbar is found to be carried by the magnetic vortex at the interface between a superconductor and a T -invariant strong topological insulator (TI).¹ We will consider a simplified situation that still demonstrates the essential physics involved; we will show that a nonzero and fractional angular momentum is localized around a magnetic flux on the boundary surface between a bulk TI and the vacuum outside. Let us think of this situation from a different perspective. In the same way we discussed class Ia in the previous section, we can imagine a procedure to turn on the magnetic field gradually from zero, and yet the angular momentum conservation guarantees zero total angular momentum of the whole system. The only way our result of fractional angular momentum can be consistent with the angular momentum conservation is that the other compensating angular momentum should be located in the other part of the TI-vacuum boundary where the magnetic flux leaves out from the bulk TI. If this boundary region is far separated from the place where the original incoming flux enters the TI, we can reasonably neglect this far-away region, and focus only on the angular momentum localized on the incoming flux alone. This angular momentum indeed takes a fractional value, as we confirm in the following discussions. We can say that the fractional angular momentum is transported from the boundary at infinity to the incoming magnetic flux; this characterizes the magnetic vortices of class Ib in our classification. Such magnetic vortices with fractional angular momenta are not peculiar, but rather ubiquitous in topological phases of matter; see, for example, Refs. [21–23]. A deeper insight to the angular momentum conservation from our discussion should be useful for better understanding of these systems.

Let us consider a situation where we have a TI bulk in the $z > 0$ region and the vacuum in $z < 0$, with an interface at the $z = 0$ surface, as illustrated in Fig. 3. It is well known that the boundary of TI supports massless surface states that can be described by a single Dirac fermion field. For our purpose, let us assume that there are T -violating magnetic impurities on the surface that opens a mass gap for the surface fermions. Integrating out the massive surface fermion gives rise to a new term in the effective action in the low energy limit for the EM fields, which is the Chern-Simons action with a half integer level $\nu = \frac{1}{2}$ [25] (which should not be confused with the winding number in the previous subsection). To capture the essential physics of our discussion, we will consider an idealized situation that this is the only response of

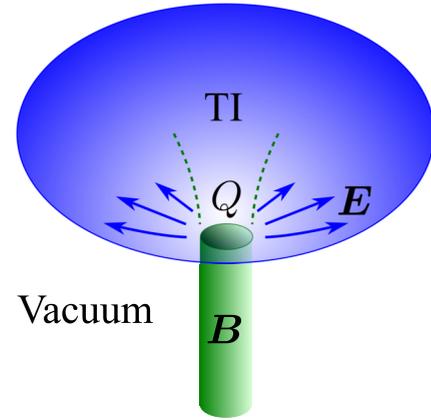


FIG. 3. Interface between the TI and the vacuum with a localized flux of magnetic field B . The electric charge Q is stored at the interface which produces the electric field E .

the TI surface (with T -violating impurities) to an externally applied EM field. At least in long wavelength and time limit, the Chern-Simons term becomes dominant over other higher derivative terms in the action.

From the Chern-Simons action, the charge current in response to an applied EM field is obtained as

$$j^\mu = -\frac{\nu}{2} \frac{e^2}{\hbar} \epsilon^{\mu\nu\alpha} F_{\nu\alpha} = -\frac{e^2}{8\pi\hbar} \epsilon^{\mu\nu\alpha} F_{\nu\alpha}, \quad (2.30)$$

which in components reads as

$$Q = \frac{e^2}{4\pi\hbar} B_z, \quad j_x = \frac{e^2}{4\pi\hbar} E_y, \quad j_y = -\frac{e^2}{4\pi\hbar} E_x. \quad (2.31)$$

where Q and $j_{x,y}$ are the charge density and the quantum Hall effect (QHE) current, respectively. Here $j_{x,y}$, B_z , and $E_{x,y}$ in Eq. (2.31) represent three-dimensional (3D) vector components without distinction between upper and lower indices.

We consider a magnetic flux that is vertically piercing the interface and is cylindrically symmetric: $B = B_z(r)\hat{z}$, where r is the radius in the x - y plane. We further assume that $B_z(r)$ is localized for $r \leq R$, so we can regard it as a flux tube like a magnetic vortex. In fact, we may realize such a magnetic profile by an external superconducting vortex as postulated in Ref. [20]. As seen from Eq. (2.31) the magnetic flux induces a surface charge density Q and this charge gives rise to a nonzero electric field according to the Gauss law. It is easy to understand that a nonzero angular momentum emerges from the resulting EM fields which are indicated by arrows in Fig. 3.

Before going into the computation of the angular momentum carried by the EM field, we first show that the angular momentum contribution from the TI matter part at $z > 0$ is generally vanishing. The easiest way to confirm this is to compute the angular momentum that may be transferred to the TI surface states as we increase the magnetic flux from zero. This is because the TI bulk is gapped, and only the surface states may carry angular momentum in response to the applied EM fields in the system. During the process of turning on the magnetic flux, we have a tangential electric

¹We note that our result derived in the following is different by a factor 1/2 from Ref. [20]. We have identified where this difference stems from, but it is not essential for our present argument, so we will not go into that detail.

field $E_\varphi = (xE_y - yE_x)/r$ from Faraday's law,

$$2\pi r E_\varphi(r) = -2\pi \int_0^r dr' r' \frac{\partial B_z(r', t)}{\partial t}. \quad (2.32)$$

Then, according to Eq. (2.31) in the cylindrical coordinates, we have the QHE current as $j_r = (xj_x + yj_y)/r = \frac{e^2}{4\pi\hbar} E_\varphi$. From this we can compute the torque from the EM force acting on the surface states along the φ direction. The EM force reads

$$F_\varphi = QE_\varphi - j_r B_z, \quad (2.33)$$

where the second term represents the Lorentz force of $\mathbf{j} \times \mathbf{B}$. Using $Q = \frac{e^2}{4\pi\hbar} B_z$ and $j_r = \frac{e^2}{4\pi\hbar} E_\varphi$, we see that the force vanishes identically, that is, the surface states do not experience any tangential force, or torque, by the Chern-Simons term. In fact, it is easy to verify that this result generally holds for any geometry. We conclude that no angular momentum is carried by the TI matter and its surface states. The angular momentum of the whole system resides solely in the EM sector.

Now let us return back to the angular momentum in the EM sector. For static fields satisfying $\nabla \times \mathbf{E} = \mathbf{0}$ and the vector potential $\mathbf{A} = A_\varphi \hat{\varphi}/r$ satisfying $\nabla \cdot \mathbf{A} = 0$, we can rewrite the angular momentum of EM fields as

$$L_z = \int_{\mathbf{x}} [\mathbf{x} \times (\mathbf{E} \times \mathbf{B})]_z = \int_{\mathbf{x}} (\nabla \cdot \mathbf{E}) A_\varphi - \int A_\varphi (\mathbf{E} \cdot d\mathbf{S}), \quad (2.34)$$

where the last term is the surface integral on the exterior boundary. Using the Stokes theorem and the cylindrical symmetry, we find the vector potential with the boundary condition $A_\varphi(0) = 0$ to be

$$2\pi A_\varphi(r) = 2\pi \int_0^r dr' r' B_z(r'). \quad (2.35)$$

Then, with the Gauss law, $\nabla \cdot \mathbf{E} = Q\delta(z)$, the first term in the above expression of L_z becomes

$$\begin{aligned} \int_{\mathbf{x}} (\nabla \cdot \mathbf{E}) A_\varphi &= \frac{e^2}{2\hbar} \int_0^\infty dr r B_z(r) \int_0^r dr' r' B_z(r') \\ &= \frac{e^2}{4\hbar} \left[\int_0^\infty dr r B_z(r) \right]^2 = \frac{e^2}{16\pi^2 \hbar} \Phi_0^2, \end{aligned} \quad (2.36)$$

where Φ_0 is the total magnetic flux. For the second term we consider a cylindrical boundary at $r = R$, and the Stokes theorem leads to

$$2\pi A_\varphi(R) = \Phi_0, \quad (2.37)$$

which takes a constant value along the boundary. Then the vector potential can be taken out from the integrand, which gives

$$\begin{aligned} - \int A_\varphi (\mathbf{E} \cdot d\mathbf{S}) &= -\frac{\Phi_0}{2\pi} \int \mathbf{E} \cdot d\mathbf{S} = -\frac{\Phi_0}{2\pi} Q_{\text{tot}} \\ &= -\frac{e^2}{8\pi^2 \hbar} \Phi_0^2, \end{aligned} \quad (2.38)$$

where $Q_{\text{tot}} = \frac{e^2}{4\pi\hbar} \Phi_0$ from Eq. (2.31) is used. Summing the above two terms, we get the total angular momentum as

$$L_z = -\frac{e^2}{16\pi^2 \hbar} \Phi_0^2, \quad (2.39)$$

with the right sign that can easily be confirmed. We note that the original integral of the angular momentum is convergent by itself as $B_z(r)$ is of finite range, and the above way to split it into two terms is just a mathematical manipulation for convenience.

We shall suppose that the magnetic flux is quantized as if it were provided by an adjacent superconducting vortex of the winding number ν considered in Ref. [20]. We note that the magnetic vortex in superconductivity does not carry a finite net angular momentum except for the background contribution, so that the total angular momentum of our interest is still given by the above formula. The flux quantization gives $\frac{2e}{\hbar} \Phi_0 = 2\pi\nu$, where the factor 2 of $\frac{2e}{\hbar}$ originates from the Cooper pair. This finally leads to

$$L_z = -\frac{\nu^2}{16} \hbar. \quad (2.40)$$

Therefore, the EM field surrounding the magnetic vortex between a TI and a superconductor carries a nonzero angular momentum given in Eq. (2.40).

The conservation of the total angular momentum during the process of turning on the magnetic flux requires the existence of an opposite and compensating angular momentum somewhere else. To identify where this compensating component is, let us consider a global geometry of the bulk TI and its closed boundary. For simplicity we assume that the bulk TI (which is a blue shaded region in Fig. 3) is a large ball of radius R and the boundary surface is a sphere of radius R . A localized magnetic tube with a flux Φ_0 enters the TI at $\theta = \pi$, where θ is the polar angle in 3D spherical coordinates. The same amount of flux goes out of the TI at other places of the surface in cylindrically symmetric (i.e., φ independent) way. Let the radial component of the magnetic field at $r_{3\text{D}} = R$ be $B_{r_{3\text{D}}}(\theta)$ as a function of θ , where $r_{3\text{D}}^2 = r^2 + z^2$ is the 3D radius. The flux conservation results in

$$\int_0^\pi d\theta \sin\theta B_{r_{3\text{D}}}(\theta) = 0. \quad (2.41)$$

We consider turning on the magnetic field adiabatically from zero, and the time-dependent magnetic field gives E_φ as well as the QHE current j_θ , but the net force on the surface states is vanishing as we saw before. Therefore, the total angular momentum resides in the EM fields only.

To compute the EM part of the angular momentum, we follow the same steps as before. Previously we considered only the contribution from the incoming magnetic flux, but if we perform the same computation including the whole TI boundary surface, the total L_z turns out to be zero as we show in the following that is consistent with our angular momentum conservation argument. From the spherical symmetry of the TI geometry and the cylindrical symmetry of the vector potential, we have

$$2\pi A_\varphi(\theta) = 2\pi R^2 \int_0^\theta d\theta' \sin\theta' B_{r_{3\text{D}}}(\theta'). \quad (2.42)$$

The Gauss law gives

$$\nabla \cdot \mathbf{E} = \frac{e^2}{4\pi\hbar} B_{r_{3D}}(\theta) \delta(r_{3D} - R). \quad (2.43)$$

Then we find the first term in Eq. (2.34) to be

$$\begin{aligned} \int_x (\nabla \cdot \mathbf{E}) A_\varphi &= 2\pi R^4 \frac{e^2}{4\pi\hbar} \int_0^\pi d\theta \sin\theta B_{r_{3D}}(\theta) \\ &\quad \times \int_0^\theta d\theta' \sin\theta' B_{r_{3D}}(\theta') \\ &= 2\pi R^4 \frac{e^2}{4\pi\hbar} \frac{1}{2} \left[\int_0^\pi d\theta \sin\theta B_{r_{3D}}(\theta) \right]^2 = 0 \end{aligned} \quad (2.44)$$

using Eq. (2.41). For the second term in Eq. (2.34) we can still employ Eq. (2.38) with different Q_{tot} . Previously we took account of Q_{tot} around the incoming magnetic flux only, but if we sum up all the contributions from the whole TI surface, it should amount to $Q_{\text{tot}} = 0$ due to Eq. (2.41). In this way we see that the second term is zero as well. We emphasize that the original expression of the angular momentum is localized in the region where $\mathbf{B} \neq \mathbf{0}$ and $\mathbf{E} \neq \mathbf{0}$, that is, it is localized around the TI boundary where the magnetic flux either enters or leaves the TI. Therefore, the fractional angular momentum localized around the magnetic tube at $\theta = \pi$ is compensated by the angular momentum carried by the outgoing flux in other places of the TI boundary which can be taken infinitely away.

III. CLASS II: CASE STUDY OF MAGNETIC VORTICES WITH ZERO ANGULAR MOMENTUM

In this section we consider magnetic vortices in relativistic field theory as examples of self-consistent systems without any background matter or boundary that could absorb angular momentum. Such vortices could appear in the standard model and extensions of the standard model. They have been considered in the context of high energy physics and cosmology. A faithful application of our angular momentum conservation argument to these vortices then dictates that they should be spinless. We will confirm this claim also in a nontrivial example where the angular momentum carried by surrounding localized gauge fields is essential for the cancellation of the total angular momentum. We emphasize that these localized gauge fields around the vortex core should be considered as a part of the magnetic vortex configuration under consideration.

A. Example 1: Relativistic Nielsen-Olesen vortices

Let us illustrate our main points in the simplest example of Nielsen-Olesen vortices in relativistic scalar theory that we already treated in the previous section. The formulation of the theory presented below is a standard one, but we would like to pay a special attention to the cases with nonvanishing charge density. Consequently, nonzero electric fields accompany the vortices; we then have a precise description of the charged vortices, taking proper account of the Gauss law constraint. This endeavor, which we did not find in the literature in full generality as we present here, turns out to be crucial to show

the exact cancellation of total angular momenta carried by the matter and the gauge field parts.

We write down the Lagrangian as

$$\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - U(\Phi^\dagger \Phi) + \frac{1}{2} \mathbf{E}^2 - \frac{1}{2} \mathbf{B}^2, \quad (3.1)$$

which is Eq. (2.1) without background, i.e., $q = 0$. Here we take $n_e = 1$ and $D_\mu \Phi = (\partial_\mu + ieA_\mu)\Phi$ and, as defined in Sec. II A, we adopt a convention of $(\mathbf{D})^i = D^i$. We use the unit system with $c = \hbar = 1$ in this section. We also take a conventional form of the potential same as in the previous section; $U(\Phi^\dagger \Phi) = -\lambda_2 \Phi^\dagger \Phi + \frac{\lambda_4}{2} (\Phi^\dagger \Phi)^2$. A finite chemical potential μ is introduced by replacing $eA_0 \rightarrow eA_0 - \mu$ (see Appendix A for introducing a chemical potential via Hamiltonian formulation). The equations of motion and the Gauss law from this action is the same as Eqs. (2.3)–(2.5) with $q = 0$, $n_e = 1$.

We do not take the nonrelativistic reduction, so that the Gauss law reads

$$\nabla^2 A_0 = -2e(\mu - eA_0)\Phi^\dagger \Phi. \quad (3.2)$$

Equations (2.6) and (3.2) together with the Maxwell equation for \mathbf{A} , i.e.,

$$\nabla \times (\nabla \times \mathbf{A}) + ie[(\mathbf{D}\Phi)^\dagger \Phi - \Phi^\dagger (\mathbf{D}\Phi)] = \mathbf{0}, \quad (3.3)$$

constitute our final set of closed equations for Φ , \mathbf{A} , and A_0 to be solved for classical configurations in relativistic theory. If $\mu = 0$, then it is consistent with $A_0 = 0$, and there is no electric field. This situation at $\mu = 0$ corresponds to charge neutral vortices in our problem. For $\mu \neq 0$ there exists nonvanishing charge and the electric field in the solution, which we will refer to as ‘‘charged Nielsen-Olesen vortices.’’

We could estimate the angular momentum in the matter part using an expression like Eq. (2.23), but here we shall show an alternative physical approach. To compute the matter contribution to the angular momentum, we need the linear momentum density, i.e., \mathbf{P} , obtained from T^{0i} component of the energy-momentum tensor. From the Noether method we see that T^{0i} for the static solution is given by

$$\begin{aligned} \mathbf{P}^i &= T^{0i} = [(\mathbf{D}^i \Phi)^\dagger (D^0 \Phi) + (D^0 \Phi)^\dagger (\mathbf{D}^i \Phi)] \\ &= -i[(\mathbf{D}^i \Phi)^\dagger (\mu - eA_0)\Phi - \Phi^\dagger (\mu - eA_0)(\mathbf{D}^i \Phi)]. \end{aligned} \quad (3.4)$$

The kinetic angular momentum carried by the matter part is then

$$L_z^{\text{matter}} = \int_x [\mathbf{x} \times \mathbf{P}]_z. \quad (3.5)$$

On the other hand, the gauge field contribution to the angular momentum is easily found from the Poynting vector, namely from Eq. (2.25).

The charge neutral case at $\mu = 0$ makes the essential difference from the nonrelativistic case in the previous section. The above equations of motion for relativistic vortices are mathematically identical, up to trivial scaling of parameters, to those of ‘‘locally’’ neutral nonrelativistic vortices without coupling to electric field: both of them are known as Nielsen-Olesen vortices. For the charge neutral case, it is algebraically trivial to see $\mathbf{P} = \mathbf{E} = \mathbf{0}$ from $\mu = A_0 = 0$, and both the matter and gauge field contributions to the total angular momentum are zero, but this conclusion is physically nontrivial;

Neutral Nielsen-Olesen Vortex

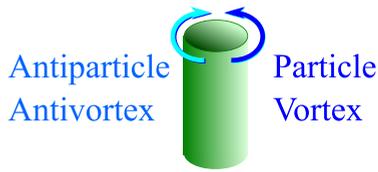


FIG. 4. Schematic illustration of the charge neutral Nielsen-Olesen vortex composed from a particle vortex and an antiparticle antivortex.

vortices are circulating configurations and yet they have no angular momentum. Intuitively, the absence of matter contribution to the angular momentum in the relativistic theory can be understood as a cancellation between particle vortex and antiparticle antivortex as follows: recall that $\Phi \sim a + b^\dagger$ and the conjugate momentum $\Pi \equiv D_0\Phi \sim i(a - b^\dagger)$ where a and b are annihilation operators for particle and antiparticle, respectively. In the superfluid phase of a large occupation number, we can regard a and b as c numbers as usual. A vortex profile with winding number ν can be viewed as a superposition of a particle vortex of $a \sim e^{i\nu\varphi}$ and an antiparticle antivortex of $b \sim e^{-i\nu\varphi}$. In the charge neutral case of $\Pi = 0$, their amplitudes are precisely equal, i.e., $a = b^\dagger$, and the antiparticle antivortex contribution to the angular momentum is precisely opposite to that of the particle vortex as schematically illustrated in Fig. 4. The absence of antiparticles in the nonrelativistic vortex in the previous section is the major difference from the relativistic theory discussed in this section.

For the charged case at $\mu \neq 0$, the two systems of equations are different by terms that we previously neglected in the nonrelativistic reduction. In addition to this difference for the charged case, a background charge density that we introduced as q in the previous section is also absent here. This means that the net charge of a charged Nielsen-Olesen vortices is not zero, and the electric field grows logarithmically at large distance in two-dimensional space perpendicular to the vortex string in three dimensions. This implies that the line density of energy of a charged vortex is divergent in infinite space, and a sensible solution would exist only in a finite transverse volume. As we will focus on azimuthally symmetric vortex configurations to apply our angular momentum conservation argument, we consider a spatial cutoff in transverse space at a certain distance from the origin, that is, $r \leq R$ in the radial direction. We will show that the total angular momentum of a charged vortex within the volume $r \leq R$ is always zero for any cutoff R , when we sum the contributions of both matter part and the gauge fields.

Our conclusion rectifies a misleading statement in the Appendix C of the well-known literature, Ref. [10], that a charged Nielsen-Olesen vortex carries a nonzero angular momentum. Our result of vanishing angular momentum even for charged vortices is independent of the issue of diverging line energy density in infinite space. Later, we will more precisely point out where the misleading conclusion in Ref. [10] stems from.

The computation in the charged vortex case is more delicate than the neutral case, and the detailed mechanism for

cancellation is similar to that in the previous section. First of all, since $\Pi \neq 0$, the particle vortex and the antiparticle antivortex have different amplitudes, and the net matter angular momentum no longer cancels to be zero. From the radial electric field $\mathbf{E} \neq \mathbf{0}$, the gauge fields also contribute to the total angular momentum. We take the following ansatz:

$$\Phi = f(r)e^{i\nu\varphi}, \quad A_0 = a(r), \quad A_\varphi = \frac{\nu}{e}[1 - h(r)], \quad (3.6)$$

with the boundary condition for vanishing magnetic flux, i.e., $h(R) = 0$ at sufficiently large boundary $r = R$. Then the equations of motion and the Gauss law become (with $' \equiv \frac{d}{dr}$)

$$\frac{1}{r}(rf')' - \frac{\nu^2}{r^2}h^2f + (\lambda_2 - \lambda_4f^2)f + (ea - \mu)^2f = 0, \quad (3.7)$$

$$\left(\frac{h'}{r}\right)' - \frac{2e^2}{r}f^2h = 0, \quad (3.8)$$

$$\frac{1}{r}(ra')' - 2e(ea - \mu)f^2 = 0. \quad (3.9)$$

The matter part of the angular momentum is computed from Eq. (3.5) as

$$L_z^{\text{matter}} = 4\pi\nu \int_0^R dr rh(r)f^2(r)[-ea(r) + \mu] \quad (3.10)$$

and the gauge field contribution from Eq. (2.25) as

$$\begin{aligned} L_z^{\text{gauge}} &= -\frac{2\pi\nu}{e} \int_0^R dr ra'(r)h'(r) \\ &= \frac{2\pi\nu}{e} \int_0^R dr [ra'(r)]'h(r), \end{aligned} \quad (3.11)$$

where in the last equality we performed the integration by part and used the boundary condition at $r = R$ as in the previous section. Using the Gauss law (3.9) to replace $[ra'(r)]'$, we arrive at

$$L_z^{\text{gauge}} = -4\pi\nu \int_0^R dr rh(r)f^2(r)[-ea(r) + \mu] = -L_z^{\text{matter}}, \quad (3.12)$$

which precisely cancels the matter contribution. As a result the total angular momentum is vanishing. In Appendix C of Ref. [10], the surface term of Eq. (C3) that was neglected is nonzero: this can be seen from the description of the solution below Eq. (C6) with $Q \neq 0$. It can be shown that Eq. (C3) precisely cancels Eq. (C4), so that the total angular momentum is zero. This cancellation has its origin in the angular momentum conservation, and it holds for any R regardless of an issue of infinite line energy density of the charged solution.

We finish this subsection by pointing out that our finding of zero angular momentum for magnetic vortices in the $U(1)$ Abelian Higgs model is consistent with the particle-vortex duality in $2 + 1$ dimensions [22,23], where the magnetic vortices in the Abelian Higgs model are mapped to the elementary excitations of a dual complex scalar field which are clearly spinless. Checking the spins of other excitations in the web of dualities [22] would be interesting.

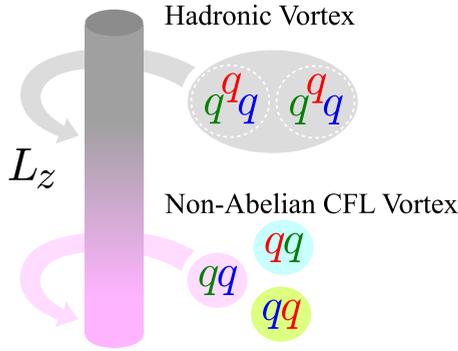


FIG. 5. Schematic illustration of the continuity between the dibaryon vortex in the hadronic phase and the non-Abelian CFL vortex in CFL quark matter in QCD.

B. Example 2: Non-Abelian CFL vortices

We can test our assertion in a more nontrivial example of non-Abelian vortices [2, 11] in the CFL color-superconducting phase of QCD quark matter at high baryon density and low temperature. The diquark condensates in the CFL phase break both QCD gauge symmetry and the global $U(1)_B$ baryon number symmetry. The non-Abelian vortices arise from coupled dynamics of color fields and $U(1)_B$ superfluidity, and carry fractional winding numbers for both gauge and global symmetries, such that the total winding number for each color component of the diquark condensate field is an integer. One might think that the non-Abelian CFL vortex is peculiar to QCD, but similar structures can also be found in multicomponent superconductivity, see Ref. [26] for example. The minimal non-Abelian vortex carries only $1/2$ of the $U(1)_B$ winding number (that is equivalent to $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$ winding number for the diquark field), so that the non-Abelian CFL vortices can be considered as fractionalized $U(1)_B$ vortices. In the hadronic phase, on the other hand, the minimal dibaryon Cooper-pair superfluid vortex also carries the same winding number $\frac{1}{2}$, so that across the two phases the dibaryon vortex should transmute to the non-Abelian CFL vortex [15], which is schematically illustrated in Fig. 5 (see also Ref. [16] for an alternative scenario). Since the angular momentum must be conserved during this transmutation process, we expect the angular momenta of the two vortices to be equal. The minimal dibaryon vortex of $1/2$ of the $U(1)_B$ winding number is a usual superfluid vortex and carries the angular momentum $L_z = N_B/2$ where N_B is the total baryon number. In contrast, the non-Abelian CFL vortex is also accompanied by color gauge fields, and in general, its total angular momentum receives contribution from these localized color fields. It is a nontrivial check to see that the total angular momentum of the non-Abelian CFL vortex from both matter part and the gauge fields is indeed $L_z = N_B/2$, i.e., the same as in the hadronic phase, as we will show below. Essentially, this means that the color-magnetic part of the non-Abelian CFL vortex does not contribute to the angular momentum, and only the $U(1)_B$ superfluid part makes a finite contribution. This situation provides another example of confirming our assertion that the gauged magnetic vortex does not carry angular momentum.

The diquark condensate in the CFL phase is described by a 3×3 matrix field $\Phi = \Phi^{i\alpha}$, where i and α are color and flavor

indices, respectively. More precisely, there are two such fields for left-handed and right-handed diquarks, and we assume that they share the same configuration in a vortex solution. We can always perform suitable color rotations, such that the profile of the non-Abelian CFL vortex appears only in the global $U(1)_B$ and the eighth component of the color field A_μ^8 with the generator $t^8 = \frac{1}{\sqrt{12}} \text{diag}(-2, 1, 1)$.² Therefore, we will show expressions only in these parts in the following. The QCD covariant derivative with A_μ^8 only is

$$D_\mu \Phi = (\partial_\mu - igA_\mu^8 t^8 - 2i\delta_\mu^0 \mu_B/3) \Phi, \quad (3.13)$$

where g is the QCD coupling constant, and μ_B is the baryon chemical potential. The coefficient is understood from the baryon charge $2/3$ carried by the diquark field Φ . We will work with the Lagrangian in terms of the diquark field given by

$$\mathcal{L} = \text{tr}[(D_\mu \Phi)^\dagger (D^\mu \Phi)] - V[\text{tr}(\Phi^\dagger \Phi)] + \frac{1}{2} \mathbf{E}^8 \cdot \mathbf{E}^8 - \frac{1}{2} \mathbf{B}^8 \cdot \mathbf{B}^8, \quad (3.14)$$

where $\mathbf{E}^8 = -\nabla A_0^8 - \partial_0 \mathbf{A}^8$ and $\mathbf{B}^8 = \nabla \times \mathbf{A}^8$. The concrete shape of the potential V is not important for our purpose. The ensuing analysis is very similar to that in the previous subsection, and we will highlight only the important differences and the major results. The color charge that appears in the Gauss law constraint is given by

$$\nabla^2 A_0^8 = 2g \text{tr}[\Phi^\dagger t^8 (gA_0^8 t^8 + 2\mu_B/3) \Phi], \quad (3.15)$$

which is easily obtained from the equation of motion for A_0^8 . In the above the term $\propto \mu_B$ should be understood as the unity matrix in color space. The other equations of motion are

$$D^2 \Phi - \Phi V'[\text{tr}(\Phi^\dagger \Phi)] + (gA_0^8 t^8 + 2\mu_B/3)^2 \Phi = 0 \quad (3.16)$$

and

$$\nabla \times (\nabla \times \mathbf{A}^8) = ig \text{tr}[(D\Phi)^\dagger t^8 \Phi - \Phi^\dagger t^8 (D\Phi)]. \quad (3.17)$$

Equations (3.15), (3.16), and (3.17) form a closed set to solve for the vortex profile of Φ , \mathbf{A}^8 , and A_0^8 . The non-Abelian CFL vortex solution has the following form [2]:

$$A_0^8 = a(r), \quad A_\varphi^8 = \frac{\nu}{g} \frac{\sqrt{12}}{3} [1 - h(r)], \quad (3.18)$$

with $\mathbf{A}^8 = A_\varphi^8 \hat{\varphi}/r$ and

$$\Phi = \begin{pmatrix} f(r) e^{i\nu\varphi} & 0 & 0 \\ 0 & b(r) & 0 \\ 0 & 0 & b(r) \end{pmatrix}. \quad (3.19)$$

The boundary condition is $h(\infty) = 0$ which ensures

$$-D\Phi = (\nabla + igA^8 t^8) \Phi \rightarrow i \frac{\nu}{3} \frac{\hat{\varphi}}{r} \Phi \quad \text{as } r \rightarrow \infty. \quad (3.20)$$

This signifies that the vortex carries a superfluid winding number $\nu/3$ with respect to the diquark global $U(1)$ [which is equivalent to $\nu/2$ with respect to $U(1)_B$ symmetry]. To

²This matrix representation is an unconventional choice; in later discussions we will focus on the u -quark sector and this choice is good for that purpose.

see how the color-magnetic vortex is embedded in the above solution, we can factorize it as follows:

$$\Phi = e^{i\frac{\nu}{3}\varphi} \begin{pmatrix} e^{i\frac{2\nu}{3}\varphi} & 0 & 0 \\ 0 & e^{-i\frac{\nu}{3}\varphi} & 0 \\ 0 & 0 & e^{-i\frac{\nu}{3}\varphi} \end{pmatrix} \begin{pmatrix} f(r) & 0 & 0 \\ 0 & b(r) & 0 \\ 0 & 0 & b(r) \end{pmatrix}, \quad (3.21)$$

where the overall phase corresponds to the global $U(1)$ and the middle matrix $e^{-i\nu\frac{\sqrt{12}}{3}t^8\varphi}$ belongs to $SU(3)$, and this is why this configuration as implemented in Eq. (3.19) is called a “non-Abelian” vortex.

Using equations of motion (3.15), (3.16), and (3.17) we find that the matter and gauge parts of the angular momentum are

$$L_z^{\text{matter}} = \frac{2\pi\nu}{3} \int_0^R dr r \left[(1+2h)f^2 \left(-\frac{4g}{\sqrt{12}}a + \frac{4}{3}\mu_B \right) + 4(1-h)b^2 \left(\frac{g}{\sqrt{12}}a + \frac{2}{3}\mu_B \right) \right], \quad (3.22)$$

$$L_z^{\text{gauge}} = (2\pi\nu) \frac{\sqrt{12}}{3} \int_0^R dr r \left[\frac{g}{3}a(2f^2 + b^2) + \frac{4}{3\sqrt{3}}\mu_B(-f^2 + b^2) \right] h. \quad (3.23)$$

For details of the derivation, see Appendix B. Summing up L_z^{matter} in Eq. (3.22) and L_z^{gauge} in Eq. (3.23), we get the total angular momentum per unit vortex length to be

$$L_z^{\text{tot}} = 2\pi\nu \int_0^R dr r \left[f^2 \left(-\frac{2g}{3\sqrt{3}}a + \frac{4}{9}\mu_B \right) + b^2 \left(\frac{2g}{3\sqrt{3}}a + \frac{8}{9}\mu_B \right) \right]. \quad (3.24)$$

One might think that the above result is an involved expression, but there is an elegant physical interpretation. To this end, we shall compute the baryon charge density as

$$Q_B = -\frac{2}{3}i \text{tr}[(D_0\Phi)^\dagger\Phi - \Phi^\dagger(D_0\Phi)] = \frac{4}{3}\text{tr}[\Phi^\dagger(gA_0^8 t^8 + 2\mu_B/3)\Phi], \quad (3.25)$$

from which the total baryon charge per unit vortex length reads

$$N_B = 2\pi \int_0^R dr r \left[f^2 \left(-\frac{4g}{3\sqrt{3}}a + \frac{8}{9}\mu_B \right) + b^2 \left(\frac{4g}{3\sqrt{3}}a + \frac{16}{9}\mu_B \right) \right]. \quad (3.26)$$

Comparing L_z^{tot} and N_B , we see that the following relation holds:

$$L_z^{\text{tot}} = \frac{\nu}{2}N_B. \quad (3.27)$$

This confirms that the total angular momentum of the non-Abelian vortex in the CFL phase contains only the contribution from the global $U(1)_B$ vortex; the total angular momentum (in the unit of \hbar) is ν times the number of the Cooper pairs. We note that this result completely agrees with

Eq. (14) in Ref. [15], but in Ref. [15] only the $U(1)_B$ contribution to the angular momentum was postulated without rigorous justification.

IV. CLASS III: CASE STUDY WITHOUT ANGULAR MOMENTUM CONSERVATION

The last logical possibility in our classification is that magnetic vortices cannot be created by simply turning on external magnetic flux in an azimuthally symmetric way. What distinguishes this case from all previous cases is that the principle of angular momentum conservation does not naively apply in the vortex creation process. The vortices classified in this class are characterized by inhomogeneous profiles along the vortex axis, which means that not only the external magnetic flux but also something else are needed to create the vortices: roughly speaking, a kind of twisting along the axis would be required. Such vortices do exist as we discuss below, although they seem to be rare in the literature.

An example that belongs to this class is provided by an object called “charged semilocal vortex” as constructed by Abraham in Ref. [24]. This Abraham vortex is an extension of the semilocal vortex [27] that has been discussed in the context of electroweak strings in cosmology (see Ref. [28] for a review). They also appear quite commonly as topological BPS (Bogomol’nyi-Prasad-Sommerfield) objects in supersymmetric gauge theories. The simplest model of the Abraham vortex consists of two charged scalar fields Φ_a ($a = 1, 2$), with the equal charge, and a $U(1)$ gauge field A_μ . The Hamiltonian in the critical limit reads

$$H = \sum_{a=1,2} (|D_0\Phi_a|^2 + |\mathbf{D}\Phi_a|^2) + \frac{g^2}{2} \left(\sum_{a=1,2} |\Phi_a|^2 - v^2 \right)^2 + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \quad (4.1)$$

where $D_\mu\Phi_a = (\partial_\mu - igA_\mu)\Phi_a$, and the Gauss law is

$$\nabla \cdot \mathbf{E} = ig \sum_{a=1,2} [(D_0\Phi_a)^\dagger\Phi_a - \Phi_a^\dagger(D_0\Phi_a)]. \quad (4.2)$$

The vortex solution relies on the following Bogomol’nyi bound:

$$H = \sum_{a=1,2} (|D_0\Phi_a \pm D_3\Phi_a|^2 + |D_1\Phi_a \pm iD_2\Phi_a|^2) + \frac{1}{2}|E_x \mp B_y|^2 + \frac{1}{2}|E_y \pm B_x|^2 + \frac{1}{2} \left[B_z \mp g \left(\sum_{a=1,2} |\Phi_a|^2 - v^2 \right) \right]^2 \mp \alpha Q_2 \mp v^2 g B_z \mp \nabla \cdot (\mathbf{E}A_z). \quad (4.3)$$

Here $\mathbf{A} = (A_x, A_y, A_z)$, $\mathbf{E} = (E_x, E_y, E_z)$, $\mathbf{B} = (B_x, B_y, B_z)$ and we imposed additional conditions that $\partial_3\Phi_2 = i\alpha\Phi_2$ with a constant α and all other ∂_3 is vanishing. We defined Q_2 as

$$Q_2 = i[(D_0\Phi_2)^\dagger\Phi_2 - \Phi_2^\dagger(D_0\Phi_2)]. \quad (4.4)$$

The equations we obtain from this, for the upper sign, are

$$D_0\Phi_a + D_3\Phi_a = 0, \quad D_1\Phi_a + iD_2\Phi_a = 0, \quad E_i = \epsilon_{ij}B_j, \\ B_z - g\left(\sum_{a=1,2} |\Phi_a|^2 - v^2\right) = 0. \quad (4.5)$$

It can be checked that these solve the original equations of motion. In Ref. [24] it was also shown that these equations admit nice solutions with zero net gauged $U(1)$ charge but nonzero Q_2 , which are somewhat misleadingly called “charged” semilocal vortices. These solutions have finite line energy density, due to the fact that the total $U(1)$ charge is zero. The solutions are possible only for $\alpha \neq 0$, that corresponds to a “twisting” along the vortex axis. Due to this, the vortex string carries a net linear momentum along the axis direction. Although the total $U(1)$ charge is zero, the charge density profile in space is nonzero, and there exists nontrivial profile of local electric and magnetic fields. This leads to a nonvanishing contribution of the electromagnetic fields to the total angular momentum. As pointed out in Ref. [24], the solutions carry nonzero total angular momentum, but we would not go into technicality here, and the readers can directly consult Ref. [24]. The twisting, parametrized by α , can be considered as spinning the vortex to give a finite angular momentum. This is an extra operation that would be needed to create such a vortex profile by hand, and the angular momentum conservation cannot be applied to the situation. In other words, in this peculiar system belonging to this class, the angular momentum conservation is not satisfied by L_z^{matter} or L_z^{gaugc} or their sum.

V. CONCLUSION

In this work we apply the principle of angular momentum conservation to understand the origin of angular momentum carried by magnetic vortices in various physical systems in condensed matter, high energy, nuclear physics, and cosmology. We find that this simple principle is powerful enough to allow us an overarching scheme of classifying the known examples, according to how the principle of angular momentum conservation is satisfied. We find the four distinct classes of examples in our classification scheme; spinful (class Ia), topological (class Ib), spinless (class II), and exotic (class III) vortices, as already summarized in the Introduction. We present detailed analyses for these examples, and emphasize that the angular momentum carried by localized gauge fields around the vortex core plays a crucial role in satisfying the angular momentum conservation. We believe that our study gives a clear answer to the seemingly confusing, but surprisingly rich, question of angular momentum carried by magnetic vortices that are ubiquitous in many branches of physics.

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APPENDIX A: HAMILTONIAN FORMULATION

In this Appendix we introduce the chemical potential via the Hamiltonian formulation of $U(1)$ gauge theory with the Lagrangian (3.1). The canonical conjugate field is given by

$$\Pi^\dagger = \frac{\delta\mathcal{L}}{\delta\partial_0\Phi} = (D_0\Phi)^\dagger. \quad (A1)$$

It should be noted that in our convention the above expression defines Π^\dagger , not Π . The charge density from the Noether method is

$$Q = -i[(D_0\Phi)^\dagger\Phi - \Phi^\dagger(D_0\Phi)] = -i(\Pi^\dagger\Phi - \Phi^\dagger\Pi), \quad (A2)$$

and the Gauss law takes the form of

$$\nabla \cdot \mathbf{E} = eQ = -ie(\Pi^\dagger\Phi - \Phi^\dagger\Pi). \quad (A3)$$

The Hamiltonian density from the Legendre transformation (including the EM sector) is obtained as

$$H = \Pi^\dagger(\partial_0\Phi) + \Pi(\partial_0\Phi)^\dagger - \mathbf{E}(\partial_0\mathbf{A}) - \mathcal{L} \\ = \Pi^\dagger\Pi + (\mathbf{D}\Phi)^\dagger(\mathbf{D}\Phi) + U(\Phi^\dagger\Phi) + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \\ - ieA_0(\Pi^\dagger\Phi - \Phi^\dagger\Pi) - A_0\nabla \cdot \mathbf{E} \\ = \Pi^\dagger\Pi + (\mathbf{D}\Phi)^\dagger(\mathbf{D}\Phi) + U(\Phi^\dagger\Phi) + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \quad (A4)$$

where we dropped the total derivative term $\nabla \cdot (A_0\mathbf{E})$ in the second line, and from the second to the last line, we used the Gauss law to have cancellation between the last two terms. This should be the case since A_0 is not a dynamical degrees of freedom in the Hamiltonian formulation of gauge theory.

For our convenience we introduce a chemical potential μ via the free energy to be minimized; $F = H - \mu Q$. This is equivalent to introducing μ in the covariant derivative, once F in this section is identified as the Hamiltonian density in the previous section. The free energy is explicitly given by

$$F = H - \mu Q = \Pi^\dagger\Pi + (\mathbf{D}\Phi)^\dagger(\mathbf{D}\Phi) + U(\Phi^\dagger\Phi) \\ + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + i\mu(\Pi^\dagger\Phi - \Phi^\dagger\Pi). \quad (A5)$$

This, together with the Gauss law, constitutes a precise formulation of gauged Ginzburg-Landau description for the cases with nonzero charge distributions. From the Gauss law we see that \mathbf{E} is not independent but generated through Π and Φ , albeit in a nonlocal way. The variables Π , Φ , and \mathbf{A} are considered as independent degrees of freedom, with respect to which the free energy F should be extremized to obtain the equations of motion.

We are interested in the stationary configurations where magnetic field is static; $\partial_0\mathbf{B} = 0$. In this case, as is familiar in classical electromagnetism, we can introduce an auxiliary function or static potential A_0 such that $\mathbf{E} = -\nabla A_0$ and, with a proper boundary condition at spatial infinity, the Gauss law can be solved nonlocally as

$$A_0 = ie\frac{1}{\nabla^2}(\Pi^\dagger\Phi - \Phi^\dagger\Pi), \quad A_0(\infty) = 0. \quad (A6)$$

This boundary condition is necessary, since a nonzero $A_0(\infty)$ would shift our definition of chemical potential μ , that is, the true chemical potential is $\mu - eA_0(\infty)$, as we have already seen in the previous section. Using this, one of the terms in F , that is, the electric field energy is expressed as

$$\frac{1}{2}E^2 = \frac{e^2}{2}(\Pi^\dagger\Phi - \Phi^\dagger\Pi)\frac{1}{\nabla^2}(\Pi^\dagger\Phi - \Phi^\dagger\Pi), \quad (\text{A7})$$

which is nothing but the Coulomb energy induced by the charge distributions. The resulting expression for the free energy F involves only the independent variables Π , Φ , and \mathbf{A} , from which we can proceed to obtain the equations of motion.

From the variation with respect to Π^\dagger , we get

$$\Pi + i\mu\Phi + e^2\Phi\frac{1}{\nabla^2}(\Pi^\dagger\Phi - \Phi^\dagger\Pi) = 0. \quad (\text{A8})$$

Using the expression for A_0 , this can be written as

$$\Pi + i\mu\Phi - ieA_0\Phi = 0 \quad \text{or} \quad \Pi = -i(\mu - eA_0)\Phi. \quad (\text{A9})$$

Recalling the relation $\Pi = D_0\Phi = (\partial_0 + ieA_0)\Phi$, this gives the well-known Josephson relation

$$\partial_0\Phi = -i\mu\Phi. \quad (\text{A10})$$

Since F is quadratic in Π , one may choose to insert back the solution for Π from Eq. (A8) into F to get a more conventional form of the free energy in terms of Φ and \mathbf{A} only. It is explicitly given by

$$F = (\mathbf{D}\Phi)^\dagger(\mathbf{D}\Phi) + U(\Phi^\dagger\Phi) - \mu(\mu - eA_0)\Phi^\dagger\Phi + \frac{1}{2}\mathbf{B}^2. \quad (\text{A11})$$

One should keep in mind that A_0 in the above expression is a functional of Φ that should be obtained by solving the Gauss law (A6) together with Eq. (A8), i.e., Eq. (3.2), which is in general nonvanishing for $\mu \neq 0$ corresponding to nonzero charge distributions in a solution. A more practical way to approach this problem is indeed what we have described in the preceding paragraphs, i.e., keeping Π as an independent degree of freedom.

The variation of the free energy with respect to Φ^\dagger gives

$$-\mathbf{D}^2\Phi + \Phi U'(\Phi^\dagger\Phi) - i\mu\Pi - e^2\Pi\frac{1}{\nabla^2}(\Pi^\dagger\Phi - \Phi^\dagger\Pi) = 0, \quad (\text{A12})$$

which, upon using the expression for A_0 , is equivalent to

$$-\mathbf{D}^2\Phi + U'(\Phi^\dagger\Phi)\Phi - i(\mu - eA_0)\Pi = 0, \quad (\text{A13})$$

and using the solution for Π , we finally get to

$$-\mathbf{D}^2\Phi + U'(\Phi^\dagger\Phi)\Phi - (\mu - eA_0)^2\Phi = 0. \quad (\text{A14})$$

This, as it should, agrees with the equation of motion for Φ obtained from the Lagrangian, after using the Josephson relation $\partial_0\Phi = -i\mu\Phi$.

APPENDIX B: DERIVATION OF THE ANGULAR MOMENTUM IN EQS. (3.22) AND (3.23)

We show the detailed derivation of the angular momentum in Eqs. (3.22) and (3.23). The Gauss law (3.15), with the

ansatz Eqs. (3.18) and (3.19), becomes

$$\frac{1}{r}(ra')' - \frac{g^2}{3}a(2f^2 + b^2) - \frac{4g}{3\sqrt{3}}\mu_B(-f^2 + b^2) = 0. \quad (\text{B1})$$

To compute the matter part of the angular momentum, we need the momentum density,

$$\mathbf{P}^i = T^{0i} = \text{tr}[(D_0\Phi)^\dagger(\mathbf{D}^i\Phi) + (\mathbf{D}^i\Phi)^\dagger(D_0\Phi)], \quad (\text{B2})$$

for which we obtain

$$\begin{aligned} \mathbf{P} &= P_\varphi\hat{\varphi} = -i\text{tr}[(\mathbf{D}\Phi)^\dagger(gA_0^8t^8 + 2\mu_B/3)\Phi \\ &\quad - \Phi^\dagger(gA_0^8t^8 + 2\mu_B/3)(\mathbf{D}\Phi)] \\ &= \left[\frac{\nu}{3r}(1+2h)f^2\left(-\frac{4g}{\sqrt{12}}a + \frac{4}{3}\mu_B\right) \right. \\ &\quad \left. + \frac{4\nu}{3r}(1-h)b^2\left(\frac{g}{\sqrt{12}}a + \frac{2}{3}\mu_B\right) \right]\hat{\varphi}. \end{aligned} \quad (\text{B3})$$

Thus we obtain the matter part of the angular momentum (3.22),

$$\begin{aligned} L_z^{\text{matter}} &= 2\pi \int_0^R dr r(rP_\varphi) \\ &= \frac{2\pi\nu}{3} \int_0^R dr r \left[(1+2h)f^2\left(-\frac{4g}{\sqrt{12}}a + \frac{4}{3}\mu_B\right) \right. \\ &\quad \left. + 4(1-h)b^2\left(\frac{g}{\sqrt{12}}a + \frac{2}{3}\mu_B\right) \right]. \end{aligned} \quad (\text{B4})$$

The color gauge field contribution is computed as

$$\begin{aligned} L_z^{\text{gauge}} &= \int_{\mathbf{x}} [\mathbf{x} \times (\mathbf{E}^8 \times \mathbf{B}^8)]_z \\ &= -(2\pi)\frac{\sqrt{12}\nu}{3g} \int_0^R dr rd'(r)h'(r). \end{aligned} \quad (\text{B5})$$

Integrating by part and using the Gauss law (B1), we arrive at Eq. (3.23),

$$\begin{aligned} L_z^{\text{gauge}} &= (2\pi\nu)\frac{\sqrt{12}}{3} \int_0^R dr r \left[\frac{g}{3}a(2f^2 + b^2) \right. \\ &\quad \left. + \frac{4}{3\sqrt{3}}\mu_B(-f^2 + b^2) \right]h. \end{aligned} \quad (\text{B6})$$

APPENDIX C: CANONICAL ANGULAR MOMENTUM

The matter part of angular momentum we use in Eq. (2.23) with covariant derivative D_φ corresponds to the *kinetic angular momentum*, that is the angular momentum carried by matter alone. We could have defined the *canonical angular momentum* using ∂_φ . It is straightforward to find

$$\begin{aligned} L_z^{\text{can,matter}} &= \int_{\mathbf{x}} \psi^\dagger \left(\frac{\hbar}{i} \partial_\varphi \right) \psi = \nu(2\pi\hbar)\frac{\mu}{g} \int_0^R dr r f^2(r) \\ &= \nu\hbar N, \end{aligned} \quad (\text{C1})$$

where $N \equiv \frac{\mu}{g} \int_{\mathbf{x}} f^2$ is the total number of particles per unit vortex length, and R is the size of the system in radial direction. This expression is identical to the well-known one for the

quantized angular momentum of a superfluid vortex. There is also a contribution from the canonical angular momentum of gauge field:

$$L_z^{\text{can.gauge}} = \int_x [\mathbf{E} \cdot \partial_\varphi \mathbf{A} + (\mathbf{E} \times \mathbf{A})_z], \quad (\text{C2})$$

which vanishes in the vortex configuration (2.15). The sum of $L_z^{\text{can.matter}}$ and $L_z^{\text{can.gauge}}$ gives the total canonical angular momentum L_z^{can} , which is conserved. On the other hand, in the main text we instead have considered the other conserved

angular momentum defined by the sum of the gauge invariant contributions: L_z^{matter} and L_z^{gauge} , i.e., $L_z^{\text{total}} = L_z^{\text{matter}} + L_z^{\text{gauge}}$. The difference between L_z^{can} and L_z^{total} is only a boundary term, and which of L_z^{can} or L_z^{total} is the relevant angular momentum depends on the physical setup. In our present setup we can gradually turn on the magnetic field, so that the magnetic vortex emerges. In this case, it makes sense to consider L_z^{total} , not L_z^{can} . For more discussions on the canonical angular momentum, we refer the readers to a concrete analysis in Ref. [29] and also a general consideration in Ref. [30].

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