Integrable spin-1 model with magnetic impurity

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We propose an integrable spin-1 chain with a magnetic impurity whose spin is $1/2$. The integrability of the model is based on an operator solution of the associated reflection equation. Due to the existence of the impurity, the SU(3) symmetry of the bulk is broken. By using the nested algebraic Bethe ansatz, we obtain the exact solution of the system. The eigenvalues, eigenstates, and Bethe ansatz equations are given explicitly. The method provided in this paper can be generalized to other high-rank quantum integrable systems with magnetic impurities where the spins of the bulk and of impurities are different.

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I. INTRODUCTION

Low-dimensional quantum many-body problems play an important role in the study of condensed matter, theoretical physics, and statistical mechanics [1]. Due to the strong interactions, it is very hard to study this kind of system because the mean-field approximation and perturbation theory cannot be applied. In order to overcome this difficulty, some numerical methods such as the density matrix renormalization group and tensor network [2,3] have been developed. Then a benchmark is needed to check the validity of numerical simulations and the new physical mechanism. The first candidate is the exact solutions because they can provide believable results. Fortunately, there do exist some quantum many-body systems which can be solved exactly such as one-dimensional interacting particles with contact potential [4], the Hubbard model [5], the supersymmetric t-J model [6], and the Heisenberg spin chain [7]. The coordinate Bethe ansatz [8,9], algebraic Bethe ansatz [10–13], and T-Q relation [14] are the typical methods to calculate the exact solutions of interacting systems. Later, in order to study high-rank quantum integrable systems, the nested algebraic Bethe ansatz was proposed [15–17].

Generic integrable boundary conditions include periodic, antiperiodic, and open boundary conditions. Recently, exactly strongly correlated systems with boundary reflections have attracted attention, because they have extensive applications in the theories of quantum magnetism, topological physics, stochastic processes in nonequilibrium statistics, and open AdS/CFT duality. Many interesting phenomena such as Kondo problems, spiral phases, novel magnetic ordered states, and zero modes induced by boundary fields or magnetic impurities have been found [18–32].

For systems with open boundary conditions, besides the Yang-Baxter equation satisfied by two-body scattering matrices, integrability of the systems requires that the boundary reflection matrices should satisfy the reflection equation or its dual one [33–35].

Ordinarily, the solution of the reflection equation is a constant matrix [36–39]. Interestingly, the reflection equation can have operator solutions whose matrix elements are the operators [29,32,40]. These operators can be used to characterize the intrinsic degrees of freedom of magnetic impurities such as spin. Based on the exact solutions, the boundary bound states [29,40] and novel screened phases [32] are found, where the spins of impurities and of bulk particles are the same.

In this paper, we obtain a new operator solution of the reflection equation. Based on it, we construct a new exactly solvable model, where the spin of particles in the bulk is set at 1, while the spin of the boundary impurity is set at $1/2$. By using the nested algebraic Bethe ansatz, we obtain the exact solution of the system. The method of constructing a new integrable Hamiltonian given in this paper can be generalized to other strongly correlated electronic systems with magnetic impurities.

We should note that the spin-1 bilinear biquadratic Heisenberg chain is very important. Some famous physical pictures such as the Haldane conjecture, where the biquadratic coupling is absent, and symmetry-protected topological phases, where free spin-$1/2$ is living on the boundaries in the gapped regime, are found [41–43]. Another interesting finding is that at the AKLT point [44,45], the ground state of the system is

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the valence bond solid state. Bulk-boundary correspondence [44,45], a topological nature [46], and massless edge modes carrying a topological quantum number [47] are also obtained in this model.

This paper is organized as follows. In Sec. II, we introduce the integrable model Hamiltonian and show the details of construction. In Sec. III, we investigate the exact solution of the model. The commutative relations, energy spectrum, eigenstates, and Bethe ansatz equations are given explicitly. Section IV is devoted to concluding remarks.

II. THE MODEL AND ITS INTEGRABILITY

The integrable impurity model studied in this paper is characterized by the Hamiltonian

$$H=\sum_{j=1}^{N-1}[\vec{S}_j\cdot\vec{S}_{j+1}+(\vec{S}_j\cdot\vec{S}_{j+1})^2]+\frac{\sqrt{2}}{4}\vec{S}_1\cdot\vec{\sigma}_\tau -\frac{\sqrt{2}}{4}\left[S_1^z(\vec{S}_1\cdot\vec{\sigma}_\tau)+(\vec{S}_1\cdot\vec{\sigma}_\tau)S_1^z\right] -\frac{1}{4}S_1^z-\frac{1}{4}\left(S_1^z\right)^2+\frac{1}{2}\sigma_\tau^z+\frac{1-\sqrt{2}}{4}S_1^z\sigma_\tau^z +\frac{-3+2\sqrt{2}}{4}\left(S_1^z\right)^2\sigma_\tau^z, \quad (1)$$

where S_j^α ($\alpha=x,y,z$) is the spin-1 operator of the bulk and σ_τ^α is the spin-1/2 operator of the impurity on one side. Obviously, the interactions among particles in the bulk have SU(3) symmetry. This symmetry is broken due to the couplings between the boundary spin and the impurity spin. Meanwhile, system (1) is a high-spin system, thus the high-order coupling terms should be included to ensure integrability.

The integrability of Hamiltonian (1) is related to the R matrix

$$R_{i,j}(u)=\frac{u}{u+1}+\frac{1}{u+1}P_{i,j}, \quad (2)$$

where u is the spectral parameter, and $P_{i,j}$ is the permutation operator with the matrix elements $[P_{i,j}]_{ac}^{bd}=\delta_{ad}\delta_{bc}$. Throughout this paper, we use standard notation. Let V denote a three-dimensional linear space and V_τ denote the two-dimensional linear space which belongs to the impurity. For any matrix $A\in\mathrm{End}(V)$, A_j is an embedding operator in the tensor space $V\otimes V\otimes\ldots$, which acts as A in the jth space and as an identity in the other factor spaces. For a matrix $R\in\mathrm{End}(V\otimes V)$, $R_{i,j}$ is an embedding operator defined in the same tensor space, which acts as an identity in the factor spaces except for the ith and jth ones.

The R matrix, (2), has the following properties.

$$\text{Initial condition:}\quad R_{i,j}(0)=P_{i,j}. \quad (3)$$

$$\text{Unitarity:}\quad R_{i,j}(u)R_{j,i}(-u)=\mathrm{id}. \quad (4)$$

$$\text{Crossing unitarity:}\quad R_{i,j}^{t_i}(u)R_{j,i}^{t_i}(-u-3)=\varphi(u)\times\mathrm{id},\qquad \varphi(u)=\frac{u(u+3)}{(u+1)(u+2)}. \quad (5)$$

$$\text{Fusion condition:}\quad R_{i,j}(u)=(u+1)\mathcal{R}_{i,j}(u),\qquad \mathcal{R}_{i,j}(-1)=-1+P_{i,j}=-2P_{i,j}^{(-)}. \quad (6)$$

Here t_i denotes the transposition in the ith space, $P_{i,j}$ is the permutation operator, and $R_{j,i}=P_{i,j}R_{i,j}P_{i,j}$. The operator $P_{i,j}^{(-)}$ is actually a three-dimensional projector operator given by $P_{i,j}^{(-)}=|\psi_1\rangle\langle\psi_1|+|\psi_2\rangle\langle\psi_2|+|\psi_3\rangle\langle\psi_3|$, and the corresponding basis vectors are $|\psi_1\rangle=\frac{1}{\sqrt{2}}(|-1,0\rangle-|0,-1\rangle)$, $|\psi_2\rangle=\frac{1}{\sqrt{2}}(|-1,1\rangle-|1,-1\rangle)$, and $|\psi_3\rangle=\frac{1}{\sqrt{2}}(|0,1\rangle-|1,0\rangle)$, where $|1\rangle$, $|0\rangle$, and $|-1\rangle$ are the eigenstates of the spin-1 operator S^z with eigenvalues 1, 0, and -1, respectively. The R matrix, (2), satisfies the Yang-Baxter equation

$$R_{i,j}(u-v)R_{i,k}(u)R_{j,k}(v)=R_{j,k}(v)R_{i,k}(u)R_{i,j}(u-v). \quad (7)$$

For integrable systems with open boundaries, besides the R matrix, we should also consider the reflection matrix $K^-(u)$ and its dual one $K^+(u)$. The reflection matrix $K^-(u)$ satisfies the reflection equation

$$R_{1,2}(u-v)K_1^-(u)R_{2,1}(u+v)K_2^-(v) = K_2^-(v)R_{1,2}(u+v)K_1^-(u)R_{2,1}(u-v), \quad (8)$$

while the dual one $K^+(u)$ satisfies the dual reflection equation

$$R_{1,2}(v-u)K_1^+(u)R_{2,1}(-u-v-3)K_2^+(v) = K_2^+(v)R_{1,2}(-u-v-3)K_1^+(u)R_{2,1}(v-u). \quad (9)$$

The general solutions of the reflection equations, (8) and (9), have been obtained, where all the matrix elements are constant [48–50]. In this paper, we seek the operator solutions. After some struggle, we obtain

$$K^-(u)=\begin{pmatrix}1-u^2 & 0 & 0\\ 0 & 1+u^2+u(1+\sigma_\tau^z) & 2u\sigma_\tau^-\\ 0 & 2u\sigma_\tau^+ & 1+u^2+u(1-\sigma_\tau^z)\end{pmatrix}\equiv\begin{pmatrix}k_{11}^-(u) & 0 & 0\\ 0 & k_{22}^-(u) & k_{23}^-(u)\\ 0 & k_{32}^-(u) & k_{33}^-(u)\end{pmatrix}, \quad (10)$$

where $\sigma_\tau^\pm=\frac{1}{2}(\sigma_\tau^x\pm i\sigma_\tau^y)$. We note that the reflection matrix $K^-(u)$ acts on the three-dimensional auxiliary space whose elements are the operators acting on the Hilbert space of the impurity. It is clear that $K^-(u)$ is a block matrix and the impurity survives in the two-dimensional subspace, which is denoted V_τ. The operator solution $K^+(u)$ of the dual reflection equation, (9), can be obtained similarly. In fact, the operator form of the dual reflection matrix can also be achieved by the mapping of $K^+(u)=K^-(-u-\frac{3}{2})$.

For simplicity, we consider the case where there is only one impurity in the system, thus the dual reflection matrix $K^+(u)$ is chosen as the identity matrix

$$K^+(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

The monodromy matrix $T_0(u)$ and the reflecting one $\hat{T}_0(u)$ are defined as

$$T_0(u) = R_{0,N}(u)R_{0,N-1}(u)\dots R_{0,1}(u),$$
$$\hat{T}_0(u) = R_{1,0}(u)R_{2,0}(u)\dots R_{N,0}(u), \tag{12}$$

where V_0 is the auxiliary space, $V_1 \otimes V_2 \otimes \dots \otimes V_N$ is the physical or quantum space, and N is the total number of sites. The monodromy matrices satisfy the Yang-Baxter relations

$$R_{1,2}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{1,2}(u-v),$$
$$R_{1,2}(u-v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{1,2}(u-v). \tag{13}$$

The double-row monodromy matrix is

$$\mathcal{T}_0(u) = T_0(u)K_0^-(u)\hat{T}_0(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}_1(u) & \mathcal{B}_2(u) \\ \mathcal{C}_1(u) & \mathcal{D}_{11}(u) & \mathcal{D}_{12}(u) \\ \mathcal{C}_2(u) & \mathcal{D}_{21}(u) & \mathcal{D}_{22}(u) \end{pmatrix}. \tag{14}$$

The double-row monodromy matrix satisfies the reflection equation

$$R_{1,2}(u-v)\mathcal{T}_1(u)R_{1,2}(u+v)\mathcal{T}_2(v) = \mathcal{T}_2(v)R_{1,2}(u+v)\mathcal{T}_1(u)R_{1,2}(u-v). \tag{15}$$

The transfer matrix is

$$t(u) = tr_0[K_0^+(u)\mathcal{T}(u)] = \mathcal{A}(u) + \mathcal{D}_{11}(u) + \mathcal{D}_{22}(u). \tag{16}$$

From the Yang-Baxter equation, (7), and reflection equation, (8), as well as the dual one, (9), one can prove that transfer matrices with different spectral parameters commutate with each other, $[t(u), t(v)] = 0$. Thus $t(u)$ serves as the generating functional of all the conserved quantities, which ensures the integrability of the system. The model Hamiltonian, (1), is constructed by

$$H = \frac{1}{6}\frac{d}{du}t(u)\bigg|_{u=0} + 2N - \frac{11}{6}. \tag{17}$$

III. EXACT SOLUTION

A. Commutative relations

Now we solve model (1) by the nested algebraic Bethe ansatz, which includes two critical steps. One is the commutative relations among the matrix elements of the double-row monodromy matrix and the other is the vacuum state.

From the reflection equation, (15), we obtain

$$\mathcal{B}_i(u)\mathcal{B}_j(v) = \frac{r(u-v)_{lk}^{ij}}{u-v+1}\mathcal{B}_k(v)\mathcal{B}_l(u), \tag{18}$$

$$\mathcal{A}(u)\mathcal{B}_j(v) = \frac{(u-v-1)(u+v)}{(u+v+1)(u-v)}\mathcal{B}_j(v)\mathcal{A}(u) - \frac{1}{u+v+1}\mathcal{B}_i(u)\tilde{\mathcal{D}}_{ij}(v) + \frac{2v}{(u-v)(2v+1)}\mathcal{B}_j(u)\mathcal{A}(v), \tag{19}$$

$$\tilde{\mathcal{D}}_{ij}(u)\mathcal{B}_k(v) = \frac{r(u+v+1)_{ef}^{id}r(u-v)_{kj}^{fg}}{(u+v+1)(u-v)}\mathcal{B}_d(v)\tilde{\mathcal{D}}_{eg}(u) - \frac{r(2u+1)_{ej}^{id}}{(2u+1)(u-v)}\mathcal{B}_d(u)\tilde{\mathcal{D}}_{ek}(v) + \frac{2v}{2u+1}\frac{r(2u+1)_{kj}^{id}}{(2v+1)(u+v+1)}\mathcal{B}_d(u)\mathcal{A}(v), \tag{20}$$

where the repeated indices should be summed and

$$\tilde{\mathcal{D}}_{ij}(u) = \mathcal{D}_{ij}(u) - \delta_{ij}\frac{1}{2u+1}\mathcal{A}(u), \tag{21}$$

$$r(u) = \begin{pmatrix} u+1 & 0 & 0 & 0 \\ 0 & u & 1 & 0 \\ 0 & 1 & u & 0 \\ 0 & 0 & 0 & u+1 \end{pmatrix}. \tag{22}$$

We note that $r(u)$ is the R matrix of the spin-1/2 Heisenberg model. According to definition (21), the transfer matrix, (16), reads

$$t(u) = \frac{2u+3}{2u+1}\mathcal{A}(u) + \tilde{\mathcal{D}}_{11}(u) + \tilde{\mathcal{D}}_{22}(u). \tag{23}$$

B. Vacuum state

The vacuum state of the system takes the form

$$|\Phi\rangle = \prod_{j=1}^{N}|0\rangle_j|0\rangle_\tau \equiv |\Omega\rangle|0\rangle_\tau, \tag{24}$$

where $|0\rangle_j = (1,0,0)^{t_j}$ and $|0\rangle_\tau = (1,0)^{t_\tau}$ are the vectors in the Hilbert spaces of the jth spin and of the impurity, respectively. Obviously, $|0\rangle_j$ is the eigenstate of the spin operator S_j^z with eigenvalue 1 and $|0\rangle_\tau$ is the eigenstate of impurity spin σ_τ^z with eigenvalue 1. Here we should note that the double-row monodromy matrix $\mathcal{T}_0(u)$ and the reflection matrix $K_0^-(u)$ are defined in the three-dimensional auxiliary space $|0\rangle_0$. Thus the effect induced by the boundary magnetic impurity cannot be embodied in this step.

Acting the double-row monodromy matrix, (14), to the above product state, we obtain

$$\mathcal{A}(u)|\Phi\rangle = k_{11}^-(u)|\Phi\rangle,$$
$$\mathcal{D}_{11}(u)|\Phi\rangle = \Big[\frac{k_{11}^-(u)}{2u+1} + \Big(k_{22}^-(u) - \frac{k_{11}^-(u)}{2u+1}\Big)b_0(u)\Big]|\Phi\rangle,$$
$$\mathcal{D}_{22}(u)|\Phi\rangle = \Big[\frac{k_{11}^-(u)}{2u+1} + \Big(k_{33}^-(u) - \frac{k_{11}^-(u)}{2u+1}\Big)b_0(u)\Big]|\Phi\rangle,$$
$$\mathcal{D}_{12}(u)|\Phi\rangle = k_{23}^-(u)b_0(u)|\Phi\rangle,$$
$$\mathcal{D}_{21}(u)|\Phi\rangle = k_{32}^-(u)b_0(u)|\Phi\rangle,$$
$$\mathcal{B}_n(u)|\Phi\rangle \neq 0, \quad \mathcal{C}_n|\Phi\rangle = 0, \quad n = 1, 2, \tag{25}$$

where

$$b_0(u) = \left(\frac{u}{u+1}\right)^{2N}.$$

From Eq. (25), we see that the operators $\mathcal{A}(u)$ and $\{\mathcal{D}_{ii}(u)\}$ acting on the vacuum state give the eigenvalues. The operators $\{\mathcal{B}_i(u)\}$ acting on the vacuum state generate other states and thus can be regarded as the creation operators. The operators $\{\mathcal{C}_i(u)\}$ acting on the vacuum state give 0 and thus can be regarded as the annihilation operators. From Eqs. (21) and (25), we also know that

$$\tilde{\mathcal{D}}_{11}(u)|\Phi\rangle = \left(k^-_{22}(u) - \frac{k^-_{11}(u)}{2u+1}\right)b_0(u)|\Phi\rangle,$$

$$\tilde{\mathcal{D}}_{22}(u)|\Phi\rangle = \left(k^-_{33}(u) - \frac{k^-_{11}(u)}{2u+1}\right)b_0(u)|\Phi\rangle. \tag{26}$$

C. Eigenstates

Assume that the eigenstate of the system is

$$|u_1,\ldots,u_M|F\rangle = \mathcal{B}_{a_1}(u_1)\mathcal{B}_{a_2}(u_2)\ldots\mathcal{B}_{a_M}(u_M)|\Phi\rangle F^{a_1\ldots a_M}, \tag{27}$$

where $\{u_l|l=1,\ldots,M\}$ are the Bethe roots, M is the number of Bethe roots, and $F^{a_1\ldots a_M}$ are the related nested algebraic Bethe states including the impurity effect, which is determined in the next step. We expect that the transfer matrix $t(u)$ acting on the assumed eigenstate, (27), can give the eigenvalue. Because we only know the behaviors of the transfer matrix $t(u)$ acting on the vacuum state, (24), we should exchange the order between $\mathcal{B}_{a_l}(u_l)$ and $\mathcal{A}(u)$, $\{\tilde{\mathcal{D}}_{nn}(u)\}$ until the transfer matrix $t(u)$ can act on the vacuum state, (24).

Repeatedly using the commutative relations (18)–(20) and considering properties (25) and (26), after some tedious calculations, we arrive at

$$t(u)|u_1,\ldots,u_M|F\rangle = \left\{\frac{(2u+3)k^-_{11}(u)}{2u+1}\prod_{j=1}^{M}\frac{(u-u_j-1)(u+u_j)}{(u-u_j)(u+u_j+1)} + b_0(u)\prod_{j=1}^{M}\frac{1}{(u+u_j+1)(u-u_j)}t^{(1)}(u,\{u_l\})\right\}|u_1,\ldots,u_M|F\rangle$$
$$+\sum_{j=1}^{M}\Lambda_j(u)\mathcal{B}_{a_1}(u_1)\ldots\mathcal{B}_{a_{j-1}}(u_{j-1})\mathcal{B}_{a_j}(u)\mathcal{B}_{a_{j+1}}(u_{j+1})\ldots\mathcal{B}_{a_M}(u_M)|\Phi\rangle F^{a_1\ldots a_M}. \tag{28}$$

Here $t^{(1)}(u,\{u_l\})$ is the nested transfer matrix with the form

$$t^{(1)}(u,\{u_l\}) = tr_0[\bar{K}^+_0(u)r_{0,1}(u+u_1+1)\ldots r_{0,M}(u+u_M+1)\bar{K}^-_0(u)r_{M,0}(u-u_M)\ldots r_{1,0}(u-u_1)], \tag{29}$$

where $\bar{K}^{\pm}(u)$ are the nested reflection matrices

$$\bar{K}^+(u) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \bar{K}^-(u) = \begin{pmatrix}k^-_{22}(u) - \frac{k^-_{11}(u)}{2u+1} & k^-_{23}(u)\\ k^-_{32}(u) & k^-_{33}(u) - \frac{k^-_{11}(u)}{2u+1}\end{pmatrix}, \tag{30}$$

and $r_{0,l}(u)$ is the nested R matrix given by (22). We diagonalize the nested transfer matrix $t^{(1)}(u,\{u_l\})$ in the next subsection. Denote the eigenvalue of $t^{(1)}(u,\{u_l\})$ as $\Lambda^{(1)}(u,\{u_l\})$, which is given by Eq. (46) in the following. $\Lambda_j(u)$ is the unwanted term

$$\Lambda_j(u) = \frac{2u+3}{(u+u_j+1)(u-u_j)}\frac{1}{2u_j+3}\left[\frac{2(2u_j+3)u_j(1-u_j^2)}{2u_j+1}\prod_{l\neq j}^{M}\frac{(u_j-u_l-1)(u_j+u_l)}{(u_j-u_l)(u_j+u_l+1)}\right.$$
$$\left.-b_0(u_j)\prod_{l\neq j}^{M}\frac{1}{(u_j+u_l+1)(u_j-u_l)}\Lambda^{(1)}(u_j,\{u_l\})\right]. \tag{31}$$

From Eq. (28), we see that if the assumed state, (27), is indeed the eigenstate of the transfer matrix $t(u)$, the unwanted terms in Eq. (28) should vanish, which gives the constraints of the Bethe roots $\{u_l\}$. Putting $\Lambda_j(u)=0$, we obtain the first set of Bethe ansatz equations

$$\frac{(2u_j+3)(1-u_j^2)}{2u_j+1}\prod_{l=1}^{M}(u_j-u_l-1)(u_j+u_l) = -b_0(u_j)\Lambda^{(1)}(u_j,\{u_l\}),$$
$$j=1,2,\ldots,M, \tag{32}$$

where $\Lambda^{(1)}(u_j,\{u_l\})$ is the value of $\Lambda^{(1)}(u,\{u_l\})$ at the point u_j. The corresponding eigenvalues $\Lambda(u)$ of the transfer matrix $t(u)$ are

$$\Lambda(u) = \frac{(2u+3)(1-u^2)}{2u+1}\prod_{j=1}^{M}\frac{(u-u_j-1)(u+u_j)}{(u-u_j)(u+u_j+1)} + b_0(u)\prod_{j=1}^{M}\frac{1}{(u+u_j+1)(u-u_j)}\Lambda^{(1)}(u,\{u_l\}), \tag{33}$$

where the Bethe roots $\{u_l\}$ should satisfy the Bethe ansatz equations, (32).

D. Nested eigenvalue problem

The rest of the task is to determine the value of $\Lambda^{(1)}(u,\{u_l\})$. Define the nested single-row monodromy matrices $T_0^{(1)}(u)$ and $\hat{T}_0^{(1)}(u)$ as

$$T_0^{(1)}(u) = r_{0,1}(u+u_1+1)\dots r_{0,M}(u+u_M+1),$$
$$\hat{T}_0^{(1)}(u) = r_{M,0}(u-u_M)\dots r_{1,0}(u-u_1).$$

The nested double-row monodromy matrix is

$$\mathcal{T}_0^{(1)}(u) = T_0^{(1)}(u)\bar{K}_0^-(u)\hat{T}_0^{(1)}(u) = \begin{pmatrix} \mathcal{A}^{(1)}(u) & \mathcal{B}^{(1)}(u) \\ \mathcal{C}^{(1)}(u) & \mathcal{D}^{(1)}(u) \end{pmatrix}, \tag{34}$$

which satisfies the reflection equation

$$r_{1,2}(u-v)\mathcal{T}_1^{(1)}(u)r_{2,1}(u+v+1)\mathcal{T}_2^{(1)}(v)$$
$$= \mathcal{T}_2^{(1)}(v)r_{1,2}(u+v+1)\mathcal{T}_1^{(1)}(u)r_{2,1}(u-v). \tag{35}$$

Then the nested transfer matrix, (29), reads

$$t^{(1)}(u,\{u_l\}) = tr_0[\bar{K}_0^+(u)\mathcal{T}_0^{(1)}(u)] = \mathcal{A}^{(1)}(u) + \mathcal{D}^{(1)}(u). \tag{36}$$

From the reflection equation, (35), we obtain the commutative relations among the elements of the nested double-row monodromy matrix

$$\mathcal{B}^{(1)}(u)\mathcal{B}^{(1)}(v) = \mathcal{B}^{(1)}(v)\mathcal{B}^{(1)}(u),$$
$$\mathcal{A}^{(1)}(u)\mathcal{B}^{(1)}(v) = \frac{(u+v+1)(u-v-1)}{(u-v)(u+v+2)}\mathcal{B}^{(1)}(v)\mathcal{A}^{(1)}(u) + \frac{u+v+1}{(u-v)(u+v+2)}\mathcal{B}^{(1)}(u)\mathcal{A}^{(1)}(v) - \frac{1}{u+v+2}\mathcal{B}^{(1)}(u)\mathcal{D}^{(1)}(v),$$
$$\mathcal{D}^{(1)}(u)\mathcal{B}^{(1)}(v) = \frac{(u-v+1)(u+v+2)}{(u-v)(u+v+1)}\mathcal{B}^{(1)}(v)\mathcal{D}^{(1)}(u) + \frac{u-v+1}{(u-v)(u+v+1)}\mathcal{A}^{(1)}(v)\mathcal{B}^{(1)}(u)$$
$$-\frac{u+v+2}{(u-v)(u+v+1)}\mathcal{B}^{(1)}(u)\mathcal{D}^{(1)}(v) - \frac{1}{(u-v)(u+v+1)}\mathcal{A}^{(1)}(u)\mathcal{B}^{(1)}(v). \tag{37}$$

Define

$$\tilde{\mathcal{D}}^{(1)}(u) = \mathcal{D}^{(1)}(u) - \frac{1}{2u+2}\mathcal{A}^{(1)}(u); \tag{38}$$

then we have

$$\mathcal{A}^{(1)}(u)\mathcal{B}^{(1)}(v) = \frac{(u+v+1)(u-v-1)}{(u-v)(u+v+2)}\mathcal{B}^{(1)}(v)\mathcal{A}^{(1)}(u) + \frac{2v+1}{2(u-v)(v+1)}\mathcal{B}^{(1)}(u)\mathcal{A}^{(1)}(v) - \frac{1}{u+v+2}\mathcal{B}^{(1)}(u)\tilde{\mathcal{D}}^{(1)}(v),$$
$$\tilde{\mathcal{D}}^{(1)}(u)\mathcal{B}^{(1)}(v) = \frac{(u-v+1)(u+v+3)}{(u+v+2)(u-v)}\mathcal{B}^{(1)}(v)\tilde{\mathcal{D}}^{(1)}(u) - \frac{2u+3}{2(u-v)(u+1)}\mathcal{B}^{(1)}(u)\tilde{\mathcal{D}}^{(1)}(v)$$
$$+\frac{(2u+3)(2v+1)}{4(u+1)(v+1)(u+v+2)}\mathcal{B}^{(1)}(u)\mathcal{A}^{(1)}(v). \tag{39}$$

We choose as the nested vacuum state the product state

$$|\Phi^{(1)}\rangle = \prod_{l=1}^{M}|0^{(1)}\rangle_l|0\rangle_\tau, \tag{40}$$

where $|0^{(1)}\rangle_l = (1,0)^{t_l}$ and $|0\rangle_\tau = (1,0)^{t_\tau}$ are the spin states of the lth site and of the impurity, respectively. Direct calculations imply

$$\mathcal{A}^{(1)}(u)|\Phi^{(1)}\rangle = \frac{2u(u+1)(u+2)}{2u+1}a_0^{(1)}(u)|\Phi^{(1)}\rangle,$$
$$\mathcal{D}^{(1)}(u)|\Phi^{(1)}\rangle = \left[u^2b_0^{(1)}(u) + \frac{u(u+2)}{2u+1}a_0^{(1)}\right]|\Phi^{(1)}\rangle,$$
$$\mathcal{C}^{(1)}(u)|\Phi^{(1)}\rangle = 0, \quad \mathcal{B}^{(1)}(u)|\Phi^{(1)}\rangle \neq 0, \tag{41}$$

where

$$a_0^{(1)}(u) = \prod_{l=1}^{M}(u+u_l+2)(u-u_l+1), \quad b_0^{(1)}(u) = a_0^{(1)}(u-1).$$

Again, the operator $\mathcal{C}^{(1)}(u)$ can be regraded as the annihilation operator and the operator $\mathcal{B}^{(1)}(u)$ acting on the nested vacuum state, (40), will generate other states.

Assume that the eigenstate of the nested transfer matrix $t^{(1)}(u,\{u_l\})$ is

$$|F\rangle = \mathcal{B}^{(1)}(\mu_1)\ldots\mathcal{B}^{(1)}(\mu_L)|\Phi^{(1)}\rangle, \tag{42}$$

where $\{\mu_k|k=1,\ldots,L\}$ are the nested Bethe roots. The nested transfer matrix, (36), acting on the assumed state, (42), gives

$$\begin{aligned} t^{(1)}(u,\{u_l\})|F\rangle &= \left\{\frac{u(u+2)(2u+3)}{2u+1}\prod_{k=1}^{L}\frac{(u-\mu_k-1)(u+\mu_k+1)}{(u-\mu_k)(u+\mu_k+2)}a_0^{(1)}(u) + u^2\prod_{k=1}^{L}\frac{(u+\mu_k+3)(u-\mu_k+1)}{(u-\mu_k)(u+\mu_k+2)}b_0^{(1)}(u)\right\}|F\rangle \\ &\quad + \sum_{k=1}^{L}\Lambda_k^{(1)}(u)\mathcal{B}^{(1)}(\mu_1)\ldots\mathcal{B}^{(1)}(\mu_{k-1})\mathcal{B}^{(1)}(u)\mathcal{B}^{(1)}(\mu_{k+1})\ldots\mathcal{B}^{(1)}(\mu_L)|\Phi^{(1)}\rangle, \end{aligned} \tag{43}$$

where $\Lambda_k^{(1)}(u)$ denotes the unwanted term with the form

$$\begin{aligned} \Lambda_k^{(1)}(u) &= \frac{(2u+3)\mu_k}{(u-\mu_k)(u+\mu_k+2)}\Bigg[(\mu_k+2)\prod_{m\neq k}^{L}\frac{(\mu_k+\mu_m+1)(\mu_k-\mu_m-1)}{(\mu_k-\mu_m)(\mu_k+\mu_m+2)} \\ &\quad \times a_0^{(1)}(\mu_k) - \mu_k\prod_{m\neq k}^{L}\frac{(\mu_k-\mu_m+1)(\mu_k+\mu_m+3)}{(\mu_k-\mu_m)(\mu_k+\mu_m+2)}b_0^{(1)}(\mu_k)\Bigg]. \end{aligned} \tag{44}$$

From Eq. (43), we know that if all the unwanted terms are 0, the assumed state, (42), is indeed the eigenstate of the nested transfer matrix $t^{(1)}(u,\{\bar{u}_l\})$. Putting $\Lambda_k^{(1)}(u)=0$, we obtain the second set of Bethe ansatz equations

$$\frac{(\bar{\mu}_k+i)(\bar{\mu}_k+\frac{1}{2}i)}{(\bar{\mu}_k-i)(\bar{\mu}_k-\frac{1}{2}i)}\prod_{m=1}^{L}\frac{(\bar{\mu}_k-\bar{\mu}_m-i)(\bar{\mu}_k+\bar{\mu}_m-i)}{(\bar{\mu}_k-\bar{\mu}_m+i)(\bar{\mu}_k+\bar{\mu}_m+i)} = -\prod_{j=1}^{M}\frac{(\bar{\mu}_k+\bar{u}_j-\frac{1}{2}i)(\bar{\mu}_k-\bar{u}_j-\frac{1}{2}i)}{(\bar{\mu}_k+\bar{u}_j+\frac{1}{2}i)(\bar{\mu}_k-\bar{u}_j+\frac{1}{2}i)}, \quad k=1,2,\ldots,L, \tag{45}$$

where we have used the notation $u_j = i\bar{u}_j - \frac{1}{2}$, $\mu_k = i\bar{\mu}_k - 1$ and i is the imaginary unit. The corresponding eigenvalue reads

$$\begin{aligned} \Lambda^{(1)}(u,\{\bar{u}_l\}) &= \frac{u(u+2)(2u+3)}{2u+1}\prod_{j=1}^{M}\left(u+\bar{u}_j i+\frac{3}{2}\right)\left(u-\bar{u}_j i+\frac{1}{2}\right)\prod_{k=1}^{L}\frac{(u+\bar{\mu}_k i)(u-\bar{\mu}_k i-2)}{(u-\bar{\mu}_k i-1)(u+\bar{\mu}_k i+1)} \\ &\quad + u^2\prod_{j=1}^{M}\left(u+\bar{u}_j i+\frac{1}{2}\right)\left(u-\bar{u}_j i-\frac{1}{2}\right)\prod_{k=1}^{L}\frac{(u-\bar{\mu}_k i)(u+\bar{\mu}_k i+2)}{(u-\bar{\mu}_k i-1)(u+\bar{\mu}_k i+1)}. \end{aligned} \tag{46}$$

Substituting Eq. (46) into Eq. (32), we obtain the explicit form of the first set of Bethe ansatz equations

$$\frac{\bar{u}_j-\frac{3}{2}i}{\bar{u}_j+\frac{3}{2}i}\left(\frac{\bar{u}_j+\frac{1}{2}i}{\bar{u}_j-\frac{1}{2}i}\right)^{2N+1}\prod_{l=1}^{M}\frac{(\bar{u}_j-\bar{u}_l-i)(\bar{u}_j+\bar{u}_l-i)}{(\bar{u}_j-\bar{u}_l+i)(\bar{u}_j+\bar{u}_l+i)} = \prod_{k=1}^{L}\frac{(\bar{u}_j-\bar{\mu}_k-\frac{1}{2}i)(\bar{u}_j+\bar{\mu}_k-\frac{1}{2}i)}{(\bar{u}_j-\bar{\mu}_k+\frac{1}{2}i)(\bar{u}_j+\bar{\mu}_k+\frac{1}{2}i)}, \quad j=1,2,\ldots,M. \tag{47}$$

Substituting Eq. (46) into Eq. (33), we obtain the eigenvalues $\Lambda(u)$ of the transfer matrix $t(u)$

$$\begin{aligned} \Lambda(u) &= \frac{(2u+3)(1-u^2)}{2u+1}\prod_{j=1}^{M}\frac{(u-\bar{u}_j i-\frac{3}{2})(u+\bar{u}_j i-\frac{1}{2})}{(u+\bar{u}_j i+\frac{1}{2})(u-\bar{u}_j i-\frac{1}{2})} + u^2 b_0(u)\prod_{k=1}^{L}\frac{(u+\bar{\mu}_k i+2)(u-\bar{\mu}_k i)}{(u-\bar{\mu}_k i-1)(u+\bar{\mu}_k i+1)} \\ &\quad + \frac{u(u+2)(2u+3)}{2u+1}b_0(u)\prod_{j=1}^{M}\frac{(u+\bar{u}_j i+\frac{3}{2})(u-\bar{u}_j i+\frac{1}{2})}{(u+\bar{u}_j i+\frac{1}{2})(u-\bar{u}_j i-\frac{1}{2})}\prod_{k=1}^{L}\frac{(u-\bar{\mu}_k i-2)(u+\bar{\mu}_k i)}{(u-\bar{\mu}_k i-1)(u+\bar{\mu}_k i+1)}, \end{aligned} \tag{48}$$

where the Bethe roots $\{\bar{u}_j\}$ and $\{\bar{\mu}_k\}$ should satisfy the Bethe ansatz equations, (45) and (47). The eigenenergy of Hamiltonian (1) is

$$E = \frac{1}{6}\frac{d\Lambda(u)}{du}\bigg|_{u=0} + 2N - \frac{11}{6} = -\sum_{j=1}^{M}\frac{1}{\bar{u}_j^2+\frac{1}{4}} + 2N - \frac{5}{2}. \tag{49}$$

IV. CONCLUSION

In this paper, we construct the operator solution of the reflection equation. Based on it, we propose an integrable quantum spin chain with a magnetic impurity. By using the nested algebraic Bethe ansatz, we obtain the exact solution of the system.

The scheme given in this paper can be generalized. For example, if we start from the trigonometric or elliptic R matrix, we can construct an integrable anisotropic model with an impurity. Then the anisotropic effects of the magnetic impurity can be studied. Meanwhile, the bulk of model (1) has SU(3) symmetry. If we start from the R matrix of the SU(2)-invariant high-spin Heisenberg model, we can put the integrable impurity into the SU(2)-invariant high-spin Heisenberg model. We also note that if both sides of the open systems have impurities, the corresponding exact solution can also be studied similarly.

Another interesting issue is that the reflection equation may have other forms of operator solutions. One particular case is that the reflection matrix may have nondiagonal elements, which will change the spin states of quasiparticles after reflection by the boundaries. Then the U(1) symmetry of the system is broken and the nested algebraic Bethe ansatz does not work because the reference state, (24), is no longer valid. In this case, we expect that the off-diagonal Bethe ansatz [51,52] could be applied.

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