

## Universal scattering phase shift in the presence of spin-orbit coupling

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Scattering phase shift, as a key parameter in scattering theory, plays an important role in characterizing low-energy collisions between ultracold atoms. In this work, we theoretically investigate the universal low-energy behavior of the scattering phase shifts for cold atoms in the presence of spin-orbit coupling. We first construct the asymptotic form of the two-body wave function when two fermions get as close as the interaction range, and consider perturbatively the correction of the spin-orbit coupling up to the second order, in which new scattering parameters are introduced. Then, for elastic collisions, the scattering phase shifts are defined according to the unitary scattering  $S$  matrix. We show how the low-energy behavior of the scattering phase shifts is modified by these new scattering parameters introduced by spin-orbit coupling. The universality of the scattering phase shifts is manifested as the independence of the specific form of the interatomic potential. The explicit forms of the new scattering parameters are analytically derived within a model of the spherical-square-well potential. Our method provides a unified description of the low-energy properties of scattering phase shifts in the presence of spin-orbit coupling.

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### I. INTRODUCTION

Owing to their versatility, ultracold atomic gases provide ideal platform on which to study fascinating quantum many-body phenomena in a highly controllable and tunable way [1–3]. As a building block of interacting many-body systems, the two-body problem is of fundamental importance in ultracold atomic physics [4–6]. On the one hand, two-body solutions determine the essential interaction parameter in the many-body Hamiltonian. On the other hand, the two-body physics even gives rise to a set of universal relations that characterize various properties of many-body systems, ranging from macroscopic thermodynamics to microscopic correlation functions [7–10]. It then opens up a new direction of studying many-body problems based on the two-body physics [11–20]. An important feature of ultracold atomic systems is that the mean distance between atoms is usually much larger than the length scale associated with interatomic potentials. Therefore, the two-body scattering properties outside the interatomic potential become independent of the short-range detail of

the potential between atoms and are universally characterized by the so-called scattering phase shift [21]. Moreover, microscopic two-body scattering parameters, such as the scattering length and effective range, can be defined based on the low-energy expansion of the scattering phase shift [22].

Recently, the successful attainment of the spin-orbit (SO) coupling in cold atomic gases has been a remarkable breakthrough [23–30]. Owing to the high controllability of cold atoms in interatomic interaction, purity, and geometry [3, 31–34], SO-coupled cold atoms have become a manifold platform for further researching and understanding of novel phenomena in condensed-matter physics, such as topological insulators and superconductors [35–37]. However, the introducing of SO coupling brings new challenges to the few-body physics as well as many-body physics. In the presence of SO coupling, the center-of-mass (c.m.) motions of pairs are coupled to their relative motions [38]. Besides, all the scattering partial waves are mixed, since the orbital angular momentum of the relative motion of two atoms is no longer conserved because of SO coupling [39]. These complications dramatically affect the theoretical description of interactions between cold atoms, and especially modify the asymptotic behavior of the many-body wave function when two atoms get close [40]. Then new scattering parameters need to be introduced by SO coupling besides the well-known scattering length and effective range [41,42]. Consequently, Tan’s universal relations, governed by the short-range behavior of the two-body physics, are also

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amended by SO-coupling. New contacts are then introduced and additional universal relations related to SO-coupling appear [41–46].

Two-body scattering problems in the presence of SO coupling have comprehensively been studied in the past few years [39,47–51]. However, a unified description of the low-energy behavior of the scattering phase shift is still elusive. Since new scattering parameters need to be introduced by SO coupling [41,42], it is of special interest how these new scattering parameters characterize the low-energy behavior of the scattering phase shift, and whether it is universal and independent of the specific form of interatomic potentials. It is the target of the present work to address these issues. In this work, we consider the two-body scattering problem of Fermi gases in the presence of three-dimensional (3D) isotropic SO coupling. We perturbatively take into account of SO coupling up to the second-order correction and construct the short-range asymptotic form of the two-body wave function. We show that additional scattering parameters need to be introduced by SO coupling, besides the scattering length (volume) and effective range. Utilizing the unitarity of the elastic collision between two fermions, we define the scattering phase shifts according to the  $S$  matrix, and then represent the form of the two-body wave function outside the interatomic potential by the scattering phase shifts. By expanding the two-body wave function at short distance, and comparing it with that obtained perturbatively by introducing new scattering parameters, we acquire the low-energy behavior of the scattering phase shifts. The low-energy expansion of the scattering phase shifts, characterizing by new scattering parameters, is found universal and independent of the specific form of interatomic potentials. As a simple example, we verify our results within a model of the spherical-square-well potential. The explicit forms of new scattering parameters introduced by SO coupling are analytically derived, which show excellent agreement with previous numerical calculations near the  $s$ -wave resonance [39].

The paper is arranged as follows: To warm up, we briefly review our method in Sec. II to construct the short-range asymptotic form of the two-body wave function in the presence of SO coupling and take into account of the second-order corrections of SO coupling. In Sec. III, the scattering phase shifts are defined according to the unitary  $S$  matrix for elastic collisions based on the exact two-body solution outside the interatomic potential. Then the asymptotic form of the two-body wave function represented by the scattering phase shifts is obtained. By comparing this short-range form of the two-body wave function with that obtained perturbatively by introducing new scattering parameters, the low-energy behavior of the scattering phase shifts is then acquired in Sec. IV. The explicit forms of the new scattering parameters introduced by SO coupling are derived and verified by using the model of a spherical-square-well potential in Sec. V. Finally, the remarks and conclusions are summarized in Sec. VI.

## II. MODEL AND TWO-BODY WAVE FUNCTION

To generalize our previous results [41,42], we briefly review the introducing of new parameters in handling two-body scattering problems with SO coupling and construct the short-range form of the two-body wave function up to

the second-order corrections of SO coupling. Let us consider two spin-1/2 fermions in the presence of 3D isotropic SO coupling, and the single-particle Hamiltonian takes the form of Refs. [39,41,51]

$$\hat{\mathcal{H}}_1 = \frac{\hbar^2 \hat{\mathbf{k}}^2}{2M} + \frac{\hbar^2 \lambda}{M} \hat{\mathbf{k}} \cdot \hat{\boldsymbol{\sigma}} + \frac{\hbar^2 \lambda^2}{2M}, \quad (1)$$

where  $\hat{\mathbf{k}} = -i\nabla$  and  $\hat{\boldsymbol{\sigma}}$  are respectively the single-particle momentum and spin operators,  $\lambda > 0$  denotes the strength of SO coupling,  $M$  is the atomic mass, and  $\hbar$  is Planck's constant divided by  $2\pi$ . For two fermions, by introducing the c.m. and relative coordinates  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$  and  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$  as usual, the two-body Hamiltonian can formally be written as  $\hat{\mathcal{H}}_2 = \hat{\mathcal{H}}_{cm} + \hat{\mathcal{H}}_r$  with

$$\hat{\mathcal{H}}_{cm} = \frac{\hbar^2 \hat{\mathbf{K}}^2}{4M} + \frac{\hbar^2 \lambda}{2M} \hat{\mathbf{K}} \cdot (\hat{\boldsymbol{\sigma}}_1 + \hat{\boldsymbol{\sigma}}_2), \quad (2)$$

$$\hat{\mathcal{H}}_r = \frac{\hbar^2 \hat{\mathbf{k}}^2}{M} + \frac{\hbar^2 \lambda}{M} \hat{\mathbf{k}} \cdot (\hat{\boldsymbol{\sigma}}_1 - \hat{\boldsymbol{\sigma}}_2) + \frac{\hbar^2 \lambda^2}{M} + V(r), \quad (3)$$

which describe the c.m. motion with total momentum  $\hat{\mathbf{K}}$  and relative motion with momentum  $\hat{\mathbf{k}} = (\hat{\mathbf{k}}_1 - \hat{\mathbf{k}}_2)/2$ , respectively. Here,  $V(r)$  is the short-range interaction potential between two fermions. As discussed in Ref. [39], the total angular momentum  $\mathbf{J}$  of two fermions as well as their total momentum  $\mathbf{K}$  is conserved. We may conveniently focus on the scattering problem in the subspace of  $\mathbf{K} = \mathbf{0}$  and  $\mathbf{J} = 0$ . In this case, the two-body Hamiltonian  $\hat{\mathcal{H}}_2$  is simply reduced to  $\hat{\mathcal{H}}_r$ , and only  $s$ - and  $p$ -wave scatterings are involved. Moreover, the subspace of  $\mathbf{K} = \mathbf{0}$  and  $\mathbf{J} = 0$  is spanned by two angular orthogonal basis  $\{\Omega_0(\hat{\mathbf{r}}), \Omega_1(\hat{\mathbf{r}})\}$  [39,41,42]:

$$\Omega_0(\hat{\mathbf{r}}) = Y_{00}(\hat{\mathbf{r}}|S), \quad (4)$$

$$\Omega_1(\hat{\mathbf{r}}) = -\frac{i}{\sqrt{3}}[Y_{1-1}(\hat{\mathbf{r}}|\uparrow\uparrow) + Y_{11}(\hat{\mathbf{r}}|\downarrow\downarrow) - Y_{10}(\hat{\mathbf{r}}|T)], \quad (5)$$

where  $Y_{lm}(\hat{\mathbf{r}})$  with angular variable  $\hat{\mathbf{r}} = (\theta, \varphi)$  for the relative motion of two fermions are the spherical harmonics, and  $|S\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$  and  $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |T\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}\}$  are the singlet and triplet spin states for two fermions, respectively. Then the two-body wave function in this subspace can generally be written in the form of

$$\Psi(\mathbf{r}) = \psi_0(r)\Omega_0(\hat{\mathbf{r}}) + \psi_1(r)\Omega_1(\hat{\mathbf{r}}), \quad (6)$$

and  $\psi_i(r)$  ( $i = 0, 1$ ) denotes the radial part of the wave function.

The existence of SO coupling dramatically changes the short-range behavior of the two-body wave function [40]. However, for a realistic interaction potential  $V(r)$  with a short range  $\epsilon$ , the SO-coupling strength  $\lambda$  as well as the relative momentum  $k$  between two fermions is usually much smaller than  $\epsilon^{-1}$  in current experiments of cold atoms [24,25]. In this case, we may perturbatively construct the asymptotic form of the two-body wave function when two fermions approach as close as  $\epsilon$ , i.e.,

$$\begin{aligned} \Psi(\mathbf{r}) \approx & \phi(\mathbf{r}) + k^2 F(\mathbf{r}) - \lambda G(\mathbf{r}) + k^4 X(\mathbf{r}) \\ & - \lambda k^2 Y(\mathbf{r}) - \lambda^2 Z(\mathbf{r}). \end{aligned} \quad (7)$$

To generalize our previous results [41,42], here we take into account of the second-order corrections of  $(k^2, \lambda)$ . Later we will see that these second-order corrections modify the low-energy expansion of the scattering phase shifts.

$$\begin{aligned} \Psi(\mathbf{r}) = & \alpha_0 \left[ \frac{1}{r} + \left( -\frac{1}{a_0} + \frac{b_0 k^2}{2} + \frac{\alpha_1}{\alpha_0} u \lambda - \frac{\alpha_1}{\alpha_0} h \lambda k^2 - c_0 \lambda^2 \right) - \left( \frac{k^2}{2} + \frac{\alpha_1}{\alpha_0} \lambda k^2 + \frac{3\lambda^2}{2} \right) r \right] \Omega_0(\hat{\mathbf{r}}) \\ & + \alpha_1 \left[ \frac{1}{r^2} + \frac{1}{2} \left( k^2 + 2\lambda \frac{\alpha_0}{\alpha_1} - \lambda^2 \right) + \left( -\frac{1}{3a_1} + \frac{b_1 k^2}{6} + \frac{\alpha_0}{\alpha_1} v \lambda - \frac{\alpha_0}{\alpha_1} q \lambda k^2 - c_1 \lambda^2 \right) r \right] \Omega_1(\hat{\mathbf{r}}) \\ & + O(r^2), \end{aligned} \quad (8)$$

for  $r \gtrsim \epsilon$ , where  $a_0, b_0$  ( $a_1, b_1$ ) are the well-known  $s$ -wave ( $p$ -wave) scattering length (volume) and effective range,  $u, v$  are the new scattering parameters resulting from the first-order correction of SO coupling, and  $h, q, c_0, c_1$  are those introduced by the second-order correction of SO coupling. Here,  $\alpha_0$  and  $\alpha_1$  are two complex superposition coefficients. Near the  $s$ -wave resonance, the contribution of the  $p$ -wave scattering could be ignored and we have  $\alpha_1 \approx 0$ . The short-range form of the two-body wave function reduces to (up to a constant  $\alpha_0$ )

$$\begin{aligned} \Phi_0(\mathbf{r}) = & \left[ \frac{1}{r} + \left( -\frac{1}{a_0} + \frac{b_0 k^2}{2} - c_0 \lambda^2 \right) - \frac{1}{2} (k^2 + 3\lambda^2) r \right] \\ & \times \Omega_0(\hat{\mathbf{r}}) + \lambda [1 + (v - qk^2)r] \Omega_1(\hat{\mathbf{r}}) + O(r^2), \end{aligned} \quad (9)$$

which recovers the modified Bethe-Peierls boundary condition of Ref. [40] by noticing that we expand the wave function up to order  $r$  at short distances and the second-order terms of  $\lambda$  are retained. We can see that a considerable  $p$ -wave component is involved because of SO coupling, even near the  $s$ -wave resonance. In like manner, near the  $p$ -wave resonance, the contribution of the  $s$ -wave scattering is small and could be ignored ( $\alpha_0 \approx 0$ ). Subsequently, the form of the two-body wave function at short distance reduces to (up to a constant  $\alpha_1$ )

$$\begin{aligned} \Phi_1(\mathbf{r}) = & \lambda [(u - hk^2) - k^2 r] \Omega_0(\hat{\mathbf{r}}) \\ & + \left[ \frac{1}{r^2} + \frac{1}{2} (k^2 - \lambda^2) + \left( -\frac{1}{3a_1} + \frac{b_1 k^2}{6} - c_1 \lambda^2 \right) r \right] \\ & \times \Omega_1(\hat{\mathbf{r}}) + O(r^2). \end{aligned} \quad (10)$$

Similarly, an  $s$ -wave component is induced by SO coupling even near the  $p$ -wave resonance.

### III. THE S MATRIX AND SCATTERING PHASE SHIFTS

The range  $\epsilon$  of interaction potentials between neutral atoms is usually much smaller than the inverse of the relative momentum  $k$  as well as that of the SO-coupling strength  $\lambda$ . The two-body wave function outside the potential in the presence of SO coupling can easily be obtained by solving the Schrödinger equation with  $V(r) = 0$  for  $r > \epsilon$  [51],

$$\Psi(\mathbf{r}) = A \Psi_-^{(in)}(\mathbf{r}) + B \Psi_+^{(in)}(\mathbf{r}) + C \Psi_-^{(out)}(\mathbf{r}) + D \Psi_+^{(out)}(\mathbf{r}), \quad (11)$$

Substituting Eq. (7) into the two-body Schrödinger equation  $\hat{\mathcal{H}}_2 \Psi(\mathbf{r}) = E \Psi(\mathbf{r})$ , following the similar route as that in Ref. [41,42], and after some straightforward algebra, we obtain the general form of the two-body wave function at short distance (see Appendix for detail):

with

$$\Psi_{\pm}^{(in)}(\mathbf{r}) = k_{\pm} [h_0^{(2)}(k_{\pm} r) \Omega_0(\hat{\mathbf{r}}) \pm h_1^{(2)}(k_{\pm} r) \Omega_1(\hat{\mathbf{r}})], \quad (12)$$

$$\Psi_{\pm}^{(out)}(\mathbf{r}) = k_{\pm} [h_0^{(1)}(k_{\pm} r) \Omega_0(\hat{\mathbf{r}}) \pm h_1^{(1)}(k_{\pm} r) \Omega_1(\hat{\mathbf{r}})], \quad (13)$$

$k_{\pm} = k \pm \lambda$ , and  $k = (ME/\hbar^2)^{1/2}$ . Here,  $h_v^{(1)}$  and  $h_v^{(2)}$  denote the  $v$ th-order spherical Hankel functions of the first and second kinds, respectively, and  $A, B, C, D$  are superposition coefficients. The physical meaning of the solution (11) is apparent: due to unique properties of the single-particle dispersion relation [39,51], the incident wave with energy  $E$  is an arbitrary superposition of two spherical waves with two different magnitudes of momenta  $k_{\pm}$  (corresponding to the spherical Hankel function of the second kind  $h_v^{(2)}$ ); scattered elastically by the interatomic potential  $V(r)$ , the outgoing wave becomes a different superposition of the same two spherical waves (corresponding to the spherical Hankel function of the first kind  $h_v^{(1)}$ ).

Using the time-dependent Schrödinger equation  $i\hbar \partial \Psi(\mathbf{r}, t) / \partial t = \hat{\mathcal{H}}_2 \Psi(\mathbf{r}, t)$ , and noticing  $\Psi(\mathbf{r}, t) = \psi_0(r, t) \Omega_0(\hat{\mathbf{r}}) + \psi_1(r, t) \Omega_1(\hat{\mathbf{r}})$ , we easily obtain the following continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{dj}{dr} = 0, \quad (14)$$

where  $\rho = r^2(|\psi_0|^2 + |\psi_1|^2)$  is the radial probability density, and

$$\begin{aligned} j(r) = & \frac{i\hbar}{M} r^2 \left[ \sum_{i=0}^1 \left( \psi_i \frac{d\psi_i^*}{dr} - \psi_i^* \frac{d\psi_i}{dr} \right) \right. \\ & \left. - 2\lambda (\psi_0^* \psi_1 - \psi_0 \psi_1^*) \right] \end{aligned} \quad (15)$$

is the radial probability current density. For the stationary state of the system, the radial probability density  $\rho$  is obviously independent of time, and thus the radial probability current density  $j(r)$  is a constant. Moreover, we have  $j(r) = 0$  at  $r = 0$ , which in turn indicates that  $j(r)$  should be zero everywhere. Inserting the scattering solution Eq. (11) into Eq. (15) and using  $j(r) = 0$ , we easily obtain

$$|A|^2 + |B|^2 = |C|^2 + |D|^2. \quad (16)$$

This means that the outgoing wave is different from the incident wave only up to a unitary transformation, i.e.,

$$\begin{bmatrix} C \\ D \end{bmatrix} = S \begin{bmatrix} A \\ B \end{bmatrix}, \quad (17)$$

and  $S$  is a unitary  $2 \times 2$  matrix determined by the specific form of the short-range interaction potential. The unitarity of the matrix  $S$  was confirmed by using a spherical-square-well model of Ref. [51]. Here, we emphasize that this property is obviously universal for any short-range interaction potential in the presence of SO coupling, which is a natural consequence of the probability conservation during elastic collisions.

The unitary  $S$  matrix can formally be diagonalized as

$$W^\dagger S W = \begin{bmatrix} e^{i2\delta_0} & 0 \\ 0 & e^{i2\delta_1} \end{bmatrix} \quad (18)$$

by

$$W = \begin{bmatrix} \omega_{0-} & \omega_{1-} \\ \omega_{0+} & \omega_{1+} \end{bmatrix}, \quad (19)$$

where  $[\omega_{0-}, \omega_{0+}]^T$  and  $[\omega_{1-}, \omega_{1+}]^T$  are two eigenvectors of  $S$  corresponding to different eigenvalues  $e^{i2\delta_0}$  and  $e^{i2\delta_1}$ . We can see that  $\delta_{0,1}$  are the new scattering phase shifts characterizing the scattering effect in the presence of SO coupling. Inserting Eq. (17) into Eq. (11), we may represent the solution  $\Psi(\mathbf{r})$  outside the interatomic potential by using eigenvectors and eigenvalues of the  $S$  matrix, i.e.,

$$\Psi(\mathbf{r}) = \alpha_0 \Phi_0(\mathbf{r}) + \alpha_1 \Phi_1(\mathbf{r}), \quad (20)$$

with

$$\Phi_i(\mathbf{r}) = \omega_{i-} \Psi_-^{(in)} + \omega_{i+} \Psi_+^{(in)} + e^{i2\delta_i} (\omega_{i-} \Psi_-^{(out)} + \omega_{i+} \Psi_+^{(out)}), \quad (21)$$

and  $\alpha_i = \omega_{i-}^* A + \omega_{i+}^* B$  for  $i = 0, 1$ . Substituting Eqs. (12) and (13) into Eq. (21), we arrive at

$$\begin{aligned} \Phi_i(\mathbf{r}) = & 2e^{i\delta_i} \cos \delta_i \sum_{\nu=0}^1 \sum_{\eta=\pm} \eta^\nu \omega_{i\eta} k_\eta [j_\nu(k_\eta r) \\ & - n_\nu(k_\eta r) \tan \delta_i] \Omega_\nu(\hat{\mathbf{r}}) \end{aligned} \quad (22)$$

for  $i = 0, 1$ , where  $j_\nu$  and  $n_\nu$  are the  $\nu$ th-order spherical Bessel functions of the first and second kind, respectively.

$$\begin{aligned} \Phi_1(\mathbf{r}) = & \left[ \frac{\omega_1(k^2 - \lambda^2)}{k - \omega_1\lambda} \frac{1}{r} + \frac{(k^2 - \lambda^2)(\omega_1 k + \lambda)}{(k - \omega_1\lambda) \tan \delta_1} - \frac{\omega_1(k^4 - \lambda^4) + 2\lambda k^3 - 2\lambda^3 k}{2(k - \omega_1\lambda)} r \right] \Omega_0(\hat{\mathbf{r}}) \\ & + \left[ \frac{1}{r^2} + \frac{(k^2 - \lambda^2)(k + \omega_1\lambda)}{2(k - \omega_1\lambda)} + \frac{k^2 - \lambda^2}{3(k - \omega_1\lambda)} \frac{k^2 + \lambda^2 + 2\omega_1\lambda k}{\tan \delta_1} r \right] \Omega_1(\hat{\mathbf{r}}) + O(r^2) \end{aligned} \quad (25)$$

for  $r \sim \epsilon^+$ , and  $\omega_1 \equiv (\omega_{1+} + \omega_{1-})/(\omega_{1+} - \omega_{1-})$ . In like manner, comparing the corresponding terms of Eq. (25) with Eq. (10), we easily obtain

$$k^3 \cot \delta_1 \approx \frac{k^4}{k^4 - \lambda^4} \left( -\frac{1}{a_1} + \frac{b_1}{2} k^2 - 3c_1 \lambda^2 \right). \quad (26)$$

#### IV. LOW-ENERGY EXPANSION OF SCATTERING PHASE SHIFTS

For ultracold atoms, the low-energy behavior of the scattering phase shift gives rise to some key microscopic scattering parameters, such as the scattering length (volume) and effective range. In the presence of SO coupling, the low-energy expansion of the scattering phase shift is expected to be modified. Then we may consider such modification perturbatively, since the energy scale of the SO-coupling strength is usually much smaller than that corresponding to the range of interatomic potentials. This can be done by simply comparing the short-range expansion of Eq. (22) with those of Eqs. (9) and (10).

Near  $s$ -wave resonances, we may expand Eq. (22) with  $i = 0$  at small  $r$  and obtain

$$\begin{aligned} \Phi_0(\mathbf{r}) = & \left( \frac{1}{r} + \frac{k + \omega_0\lambda}{\tan \delta_0} - \frac{k^2 + \lambda^2 + 2\omega_0\lambda k}{2} r \right) \Omega_0(\hat{\mathbf{r}}) \\ & + \left[ \frac{\omega_0 k - \lambda}{k^2 - \lambda^2} \frac{1}{r^2} + \frac{\omega_0 k + \lambda}{2} + \frac{\omega_0(k^2 + \lambda^2) + 2\lambda k}{3 \tan \delta_0} r \right] \\ & \times \Omega_1(\hat{\mathbf{r}}) + O(r^2) \end{aligned} \quad (23)$$

for  $r \sim \epsilon^+$ , where we have omitted an overall factor  $2(\omega_{0+} + \omega_{0-})e^{i\delta_0} \sin \delta_0$  and introduced  $\omega_0 \equiv (\omega_{0+} - \omega_{0-})/(\omega_{0+} + \omega_{0-})$ . Comparing the corresponding terms between Eq. (23) and Eq. (9), we find

$$k \cot \delta_0 \approx \frac{k^2}{k^2 + \lambda^2} \left( -\frac{1}{a_0} + \frac{b_0}{2} k^2 - c_0 \lambda^2 \right), \quad (24)$$

which recovers that of Ref. [39] if we keep the terms in  $\tan \delta_0$  up to  $\lambda^2$  and notice  $b_0 \approx 0$  near broad  $s$ -wave resonances. It is apparent that  $\delta_0$  is the counterpart of the  $s$ -wave scattering phase shift in the absence of SO coupling, and then Eq. (24) reduces to the well-known effective-range expansion of the  $s$ -wave scattering phase shift, i.e.,  $k \cot \delta_0 = -1/a_0 + b_0 k^2/2 + O(k^4)$ . In the presence of SO coupling, the new scattering parameter  $c_0$  is involved, which modifies the low-energy behavior of the  $s$ -wave scattering phase shift.

Near  $p$ -wave resonances, we again expand Eq. (22) with  $i = 1$  at small  $r$  and obtain [up to an overall constant  $2(\omega_{1+}/k_+ - \omega_{1-}/k_-)e^{i\delta_1} \sin \delta_1$ ]

It recovers that of Ref. [39] near  $p$ -wave resonances [52]. We can see that the SO-coupling induced scattering parameter  $c_1$  is involved in characterizing the scattering phase shift  $\delta_1$ . Obviously, the scattering phase shift  $\delta_1$  is the counterpart of the  $p$ -wave scattering phase shift in the absence of SO coupling. It reduces to the

well-known effective-range expansion of the  $p$ -wave scattering phase shift without SO coupling, i.e.,  $k^3 \cot \delta_1 = -1/a_1 + b_1 k^2/2 + O(k^4)$ .

### V. A SPHERICAL-SQUARE-WELL POTENTIAL MODEL

Till now, we have discussed the introducing of new scattering parameters in the short-range asymptotic behavior of the two-body wave function and considered how these new scattering parameters characterize and modify the scattering phase shifts at low energy. To demonstrate that these new scattering parameters  $c_0$  and  $c_1$  are not artificially introduced

by our perturbation method and are in fact physically determined by real interatomic potentials, in the following, we consider a two-body scattering problem within a model of the spherical-square-well potential, i.e.,

$$V(r) = \begin{cases} -V_0, & 0 \leq r \leq \epsilon \\ 0, & r > \epsilon, \end{cases} \quad (27)$$

with the depth  $V_0 > 0$ . By solving the Schrödinger equation inside and outside the potential, respectively, and utilizing the continuity of the two-body wave function at  $r = \epsilon$  as well as its first-order derivative, we obtain (see Appendix)

$$\frac{c_0}{\epsilon} = -1 + \frac{\tilde{V}_0}{3[\sqrt{\tilde{V}_0} - \tan(\sqrt{\tilde{V}_0})]^2} + \frac{1}{\tilde{V}_0 - \sqrt{\tilde{V}_0} \tan(\sqrt{\tilde{V}_0})}, \quad (28)$$

$$c_1 \epsilon = -\frac{\tilde{V}_0 [(-15 + \tilde{V}_0)\sqrt{\tilde{V}_0} \cos(\sqrt{\tilde{V}_0}) + 3(5 - 2\tilde{V}_0) \sin(\sqrt{\tilde{V}_0})]}{5[-3 + \tilde{V}_0 + 3\sqrt{\tilde{V}_0} \cot(\sqrt{\tilde{V}_0})]^2 [\sqrt{\tilde{V}_0} \cos(\sqrt{\tilde{V}_0}) - \sin(\sqrt{\tilde{V}_0})]}, \quad (29)$$

and  $\tilde{V}_0 = M\epsilon^2 V_0 / \hbar^2$ .

We present  $c_0$  as a function of the reduced depth  $\tilde{V}_0$  of the potential near the  $s$ -wave resonance in Fig. 1. The new parameter  $c_0$  introduced by SO coupling characterizes the second-order correction of  $\lambda$  to the scattering phase shift  $\delta_0$  and has been numerically evaluated in Ref. [39] near the  $s$ -wave resonance. Our analytical result, i.e., Eq. (28), shows accurate agreement with the numerical calculation of Ref. [39]. The scattering phase shift  $\delta_0$  involves all short-range information of the interatomic potential and governs the  $s$ -wave scattering properties outside the potential  $r > \epsilon$ . The

resonance position is determined by  $\delta_0 = \pi/2$ , which yields the well depth  $\tilde{V}_0 = \pi^2/4$  at the resonance in the absence of SO coupling, as indicated by the red dashed line in Fig. 1. When the SO coupling is gradually turned on, the resonance position is dramatically shifted by SO coupling because of considerable value of  $c_0$ , as shown in the inset of Fig. 1. For the same consideration, we plot  $c_1$  as a function of the reduced depth  $\tilde{V}_0$  near the  $p$ -wave resonance in Fig. 2, which characterizes the second-order correction of  $\lambda$  to the  $p$ -wave scattering phase shift  $\delta_1$ . Unlike the case of the  $s$ -wave scattering, we can see that  $c_1$  is extremely small near the  $p$ -wave

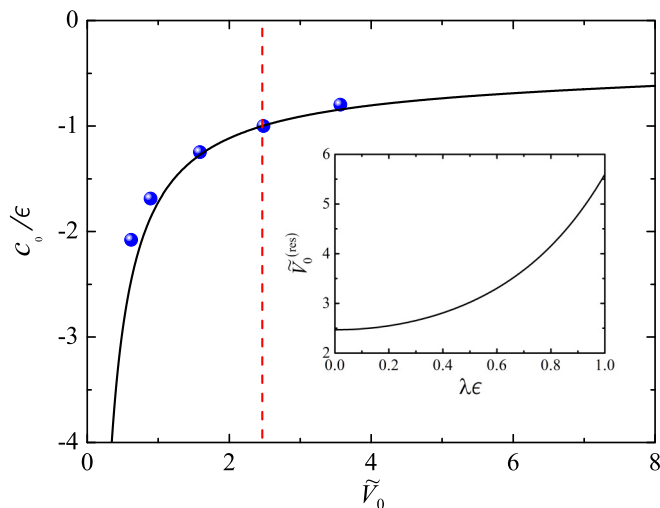


FIG. 1. The new scattering parameter  $c_0/\epsilon$ , characterizing the second-order correction of  $\lambda$  to the scattering phase shift  $\delta_0$ , evolving with the depth of the potential  $\tilde{V}_0 = M\epsilon^2 V_0 / \hbar^2$ . The red dashed line indicates the  $s$ -wave resonance in the absence of spin-orbit coupling, and the blue numerical data is from Ref. [39]. The inset shows the position of  $s$ -wave scattering resonance changing with the spin-orbit-coupling strength.

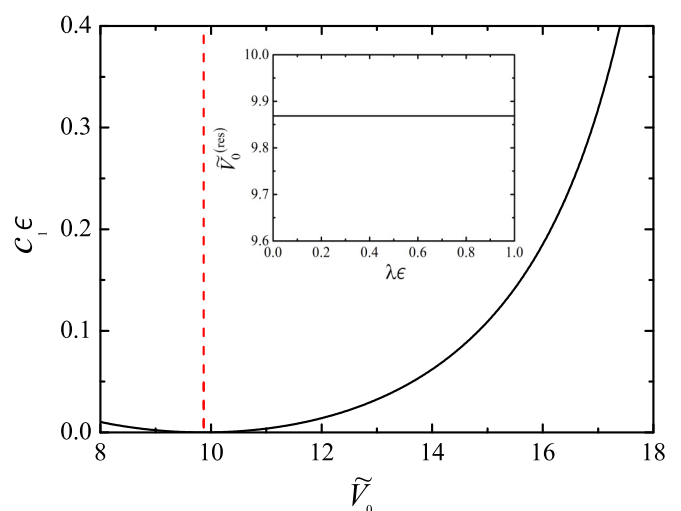


FIG. 2. The new scattering parameter  $c_1\epsilon$ , characterizing the second-order correction of  $\lambda$  to the scattering phase shift  $\delta_1$ , evolving with the depth potential  $\tilde{V}_0 = M\epsilon^2 V_0 / \hbar^2$ . The red dashed line indicates the  $p$ -wave resonance in the absence of spin-orbit coupling. The inset shows the position of  $p$ -wave scattering resonance changing with the spin-orbit-coupling strength.

resonance without SO coupling, which means the correction from the SO coupling to the  $p$ -wave scattering phase shift is negligibly small. Consequently, the resonance position is nearly unchanged when the SO coupling is gradually turned on as shown in the inset of Fig. 2.

## VI. CONCLUSIONS

In this work, we investigate the two-body scattering problems of Fermi gases in the presence of the three-dimensional isotropic spin-orbit coupling. Since the energy scale corresponding to the spin-orbit-coupling strength is much smaller than that corresponding to the range of interatomic potentials, we perturbatively construct the asymptotic behavior of two-body wave function when two fermions approach as close as the interaction range. To generalize our previous results [41,42], we consider up to the second-order correction of the spin-orbit-coupling strength to the two-body wave function, and additional new scattering parameters are introduced. Furthermore, the scattering phase shifts are defined based on the unitary  $S$  matrix for an elastic collision. The low-energy behavior of the scattering phase shifts is discussed, which is modified by the new scattering parameters in the presence of spin-orbit coupling. Our results naturally reduce to the well-known effective-range expansions of the scattering phase shifts without spin-orbit coupling. Within the model of a spherical-square-well potential, our analytical results are verified and agree well with previous numerical calculations near the  $s$ -wave resonance [39]. To simplify the presentation of this work, we focus our discussions on the subspace of zero center-of-mass momentum and zero total angular momentum of two fermions, and then only  $s$ - and  $p$ -wave scatterings are involved. The advantage of our method is that the correction of the spin-orbit coupling to the two-body wave function at short distance may perturbatively be considered order by order, as well as to the scattering phase shifts at low energy. Therefore, it is straightforward for the generalization to the case of nonzero center-of-mass momentum and nonzero total angular momentum. Then more scattering partial waves should be involved, and additional scattering parameters would be introduced to characterize the short-range behavior of the two-body wave function as well as the low-energy expansion of the scattering phase shifts. Our method provides a unified description of the low-energy properties of scattering phase shifts in the presence of spin-orbit coupling.

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## APPENDIX: SCATTERING UNDER A SPHERICAL-SQUARE-WELL POTENTIAL

In this Appendix, we present the derivation detail of the short-range behavior of the two-body wave function (8) and calculate the specific forms of the new scattering parameters

for a spherical-square-well potential. Substituting the ansatz Eq. (7) into the two-body Schrödinger equation  $\hat{\mathcal{H}}_2\Psi(\mathbf{r}) = E\Psi(\mathbf{r})$  and comparing the corresponding coefficients of the terms ( $k^2$ ,  $\lambda$ ,  $k^4$ ,  $\lambda k^2$ ,  $\lambda^2$ ) on both sides, we obtain the following coupled equations:

$$\left[-\nabla^2 + \frac{MV(\mathbf{r})}{\hbar^2}\right]\phi(\mathbf{r}) = 0, \quad (\text{A1})$$

$$\left[-\nabla^2 + \frac{MV(\mathbf{r})}{\hbar^2}\right]F(\mathbf{r}) = \phi(\mathbf{r}), \quad (\text{A2})$$

$$\left[-\nabla^2 + \frac{MV(\mathbf{r})}{\hbar^2}\right]G(\mathbf{r}) = \hat{Q}(\mathbf{r})\phi(\mathbf{r}), \quad (\text{A3})$$

$$\left[-\nabla^2 + \frac{MV(\mathbf{r})}{\hbar^2}\right]Y(\mathbf{r}) = \hat{Q}(\mathbf{r})F(\mathbf{r}) + G(\mathbf{r}), \quad (\text{A4})$$

$$\left[-\nabla^2 + \frac{MV(\mathbf{r})}{\hbar^2}\right]Z(\mathbf{r}) = \phi(\mathbf{r}) - \hat{Q}(\mathbf{r})G(\mathbf{r}), \quad (\text{A5})$$

with  $\hat{Q}(\mathbf{r}) = \hat{\mathbf{k}} \cdot (\hat{\boldsymbol{\sigma}}_1 - \hat{\boldsymbol{\sigma}}_2)$ . The corresponding expressions of functions  $\phi(\mathbf{r})$ ,  $F(\mathbf{r})$ ,  $G(\mathbf{r})$ ,  $Y(\mathbf{r})$ , and  $Z(\mathbf{r})$  outside the potential for  $r \gtrsim \epsilon$  can easily be obtained by solving these coupled equations:

$$\begin{aligned} \phi(\mathbf{r}) &= \alpha_0 \left( \frac{1}{r} - \frac{1}{a_0} \right) \Omega_0(\hat{\mathbf{r}}) + \alpha_1 \left( \frac{1}{r^2} - \frac{1}{3a_1} r \right) \Omega_1(\hat{\mathbf{r}}) \\ &\quad + O(r^2), \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} F(\mathbf{r}) &= \alpha_0 \left( \frac{1}{2} b_0 - \frac{1}{2} r \right) \Omega_0(\hat{\mathbf{r}}) + \alpha_1 \left( \frac{1}{2} + \frac{b_1}{6} r \right) \Omega_1(\hat{\mathbf{r}}) \\ &\quad + O(r^2), \end{aligned} \quad (\text{A7})$$

$$G(\mathbf{r}) = -\alpha_1 u \Omega_0(\hat{\mathbf{r}}) - \alpha_0 (1 + vr) \Omega_1(\hat{\mathbf{r}}) + O(r^2), \quad (\text{A8})$$

$$Y(\mathbf{r}) = \alpha_1 (h + r) \Omega_0(\hat{\mathbf{r}}) + \alpha_0 qr \Omega_1(\hat{\mathbf{r}}) + O(r^2), \quad (\text{A9})$$

$$\begin{aligned} Z(\mathbf{r}) &= \alpha_0 \left( c_0 + \frac{3}{2} r \right) \Omega_0(\hat{\mathbf{r}}) + \alpha_1 \left( \frac{1}{2} + c_1 r \right) \Omega_1(\hat{\mathbf{r}}) \\ &\quad + O(r^2). \end{aligned} \quad (\text{A10})$$

Inserting these functions into the two-body wave function (7), we arrive at Eq. (8). Obviously, the derivation here is independent of the specific form of the interatomic potential, and new scattering parameters are introduced in the short-range form of the two-body wave function.

To determine the new scattering parameters in the two-body wave function, let us consider a specific model of the spherical-square-well potential. Outside the potential, i.e.,  $V(r) = 0$ , we have already obtained the form of the two-body wave function as shown in Eqs. (A6) to (A10). While inside the potential, we have  $V(r) = -V_0$ , and the corresponding specific forms of these functions inside the potential are obtained by solving the Schrödinger equation. By using the continuity of the two-body wave function at  $r = \epsilon$  as well as its first-order derivative, all the scattering parameters are then determined. To simplify the presentation, we only show the specific forms of  $c_0$  and  $c_1$  in Eqs. (28) and (29), which characterize the second-order corrections of SO coupling to the scattering phase shifts.

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