Continuous quantum measurement for general Gaussian unravelings can exist

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Quantum measurements and the associated state changes are properly described in the language of instruments. We investigate the properties of a time continuous family of instruments associated with the recently introduced class of general Gaussian non-Markovian stochastic Schrödinger equations. In this article we find that when the covariance matrix for the Gaussian noise satisfies a particular δ-function constraint, the measurement interpretation is possible for a class of models with self-adjoint coupling operator. This class contains, for example the spin-boson and quantum Brownian motion models with colored bath correlation functions. Remarkably, due to quantum memory effects the reduced state, in general, does not obey a closed form master equation while the unraveling has a time continuous measurement interpretation.

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I. INTRODUCTION

Open quantum system dynamics and quantum measurement have a lot in common: both are described in terms of completely positive maps when the system and the environment or the system and measurement apparatus are initially prepared into a product state [1]. Since practically any quantum system is coupled to an environment [2], detailed understanding of the intricate connections between the two physical processes is necessary for both fundamental and applied research. For example, proper understanding of relaxation of a driven atom inside a leaky cavity is provided by the theory of open quantum systems [3], whereas the homodyning of the emitted light can be understood on the level of microscopical physical processes in terms of a time continuous measurement [4]. Averaging over all possible measurement records in the latter case provides the correct relaxation dynamics of the driven open system, thus reconciling the two approaches.

More recently, the role of quantum measurements on the thermodynamical properties of open quantum systems have been theoretically investigated [5–7]. In current experiments, the individual conditional trajectories of continuously measured weakly coupled open quantum systems can be tracked [8,9]. With the eye on possible future experiments and quantum technologies, it is important to understand whether a time continuous measurement interpretation is possible and how relevant conditional trajectories should be constructed when the open system is strongly coupled to its environment and possible memory effects are at play [10,11].

An open quantum system can be studied in many different ways. One of the most widespread is the usage of two types of master equations for the reduced state [2,12,13]: of the time local form [2] and time nonlocal form containing memory integrals [14,15]. In this article we focus on a different description, namely on unravelings of the reduced state evolution in terms of time continuous stochastic Schrödinger equations (SSEs). Applicability of such descriptions range from Markov evolutions [16], i.e., fulfilling the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation [17,18], to highly non-Markovian dynamics in terms of non-Markovian quantum state diffusion [19,20]. The SSE formalism is well developed in the Markov case, where a complete parametrization of diffusive SSEs has been known already for some years [21,22]. Similar progress in the non-Markovian regime has been made only recently by the introduction of the general Gaussian non-Markovian SSEs as well as the microscopically derived generalized Gaussian non-Markovian SSEs [10,11,23–25].

One advantage of stochastic descriptions of the dynamics in the Markov regime lies in their physical interpretation. A single trajectory corresponds to an evolution that is conditioned on time continuous monitoring of the environment of the open system [26]. For non-Markovian diffusive trajectories driven by complex-valued colored Gaussian noise, so far only a single-shot measurement interpretation has been established [27–32]. Interesting proposals for time continuous quantum measurements in the presence of memory effects were given in [28,30], where initially entangled observables were measured. However, it was pointed out subsequently in [29] that such an approach cannot lead to pure state trajectories, thus strengthening the commonly held viewpoint that in the presence of memory effects such a continuous measurement interpretation is not possible.

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Remarkably, in [33], using an approach complementary to ours, the authors show that linear diffusive SSE driven by a real-valued, *nonwhite* Ornstein-Uhlenbeck process leads to a random unitary type dynamics which does have a time continuous measurement interpretation. On the other hand, our approach, which will be elaborated on later, has a clear microscopic origin, and the noise driving the process must only satisfy a δ-function constraint which we will give later in this article.

The general Gaussian SSEs contain two types of correlation functions. Only the Hermitian (bath) correlation function $\alpha(t,s)$, occurring also in the standard SSEs, affects the reduced density operator evolution. The non-Hermitian correlation function $\eta(t,s)$ influences solely the properties of the stochastic trajectories. The new freedom in the description of the open quantum system dynamics introduced by the correlation $\eta(t,s)$ has already turned out to be beneficial for tasks where optimization over different pure state decompositions is needed, such as entanglement detection, entanglement bounds, and entanglement protection [11,34,35]. Consequently, it is a natural idea to examine whether the stochastic trajectories in this general description have a time continuous measurement interpretation beyond the white noise limit.

II. GENERAL GAUSSIAN NON-MARKOVIAN SSES

The general Gaussian non-Markovian SSEs investigated in [10,11,23] reads

$$\begin{aligned}\frac{d}{dt}|\psi_t(z^*)\rangle &= -iH_S|\psi_t(z^*)\rangle + Lz_t^*|\psi_t(z^*)\rangle \\ &\quad - \int_0^t ds[\alpha(t,s)L^\dagger + \eta(t,s)L]O(t,s,z^*)|\psi_t(z^*)\rangle, \quad (1)\end{aligned}$$

where the stochastic states $|\psi_t(z^*)\rangle$ are not normalized and the initial state is the same for all trajectories: $|\psi_0(z^*)\rangle = |\psi_0\rangle$. H_S is a Hamiltonian of the open system, and L is an operator describing the coupling of the open quantum system to its environment. We have already assumed that the assignment $\frac{\delta}{\delta z_s^*}|\psi_t(z^*)\rangle = O(t,s,z^*)|\psi_t(z^*)\rangle$ is possible. $O(t,s,z^*)$ can be calculated exactly for many relevant systems, otherwise one has to turn to some approximation scheme [36].

The functions $\alpha(t,s)$ and $\eta(t,s)$ are, respectively, the Hermitian and non-Hermitian correlation functions of the Gaussian complex noise z_t^* completely specified by its mean $\mathcal{M}[z_t^*] = 0$ and correlations

$$\mathcal{M}[z_t z_s^*] = \alpha(t,s), \quad \mathcal{M}[z_t^* z_s^*] = \eta(t,s), \qquad (2)$$

where the averages are taken with respect to the corresponding Gaussian probability density.

By construction, the reduced density operator ρ_t evolves according to a completely positive and trace preserving map Φ_t obtained by averaging over the trajectories obeying (1): $\rho_t = \mathcal{M}[|\psi_t(z^*)\rangle\langle\psi_t(z^*)|] = \Phi_t(|\psi_0\rangle\langle\psi_0|)$. Indeed, the family of stochastic pure states $\{|\psi_t(z^*)\rangle\}_{z^*}$ unravel the reduced evolution. Rather surprisingly, the averaged reduced dynamics depends only on the Hermitian correlation $\alpha(t,s)$, whereas the dependency on the non-Hermitian correlation $\eta(t,s)$ occurs only in the trajectories $|\psi_t(z^*)\rangle$.

The most general form for the Gaussian non-Markovian SSE is obtained when the functions $\alpha(t,s)$ and $\eta(t,s)$ are only constrained by a general positivity condition, which guarantees that they are correlation functions for some complex Gaussian noise z_t^* [10,23].

III. MEASUREMENT INTERPRETATION

To investigate if the measurement interpretation of some stochastic process is possible, one introduces a notion of instrument and the measurement record [37,38]; see also Appendix A.

Let $(\Omega_u, \mathcal{F}_u)$ be a family of measurable spaces parametrized by time $u \geqslant 0$. Here Ω_u is the set of all possible measurement records and $\mathcal{F}_u$ is a family of increasing σ-algebras ($\mathcal{F}_{u'} \subset \mathcal{F}_u$, when $u' < u$) containing all possible events verifiable by a time continuous measurements up to time u. Any measurement scheme is then described by a family of instruments $\mathcal{Y} = \{Y_u(\cdot)[\cdot] : \mathcal{F}_u \times \mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_S); u \geqslant 0\}$, where each $Y_u(F_u)$ is an instrument, that is, a linear, trace nonincreasing, normalized, and completely positive map from trace class operators to trace class operators. Furthermore, the normalization is given by $\mathrm{tr}\{Y_u(\Omega_u)X\} = \mathrm{tr}\{X\}$, where X is an arbitrary trace class operator. Causality sets an additional constraint on $\mathcal{Y}$: The probability of verifying an event F_s has to be invariant with respect to measuring up to some later time $t > s$ ($\mathcal{F}_s \subset \mathcal{F}_t$) and discarding the gained information. Consequently, the following "compatibility demand" has to be satisfied for all s, $0 < s < t$:

$$\forall F_s \in \mathcal{F}_s : \ (Y_s)^\dagger(F_s)[\mathbb{1}] = (Y_t)^\dagger(F_s \times \Omega_s^t)[\mathbb{1}], \qquad (3)$$

where $(Y_s)^\dagger(F_s)[\cdot]$ refers to the dual map of $Y_s(F_s)[\cdot]$ and $F_s \times \Omega_s^t$ denotes all elements of Ω_t which coincide with F_s. As discussed in [31] such an instrument $Y_u(\cdot)[\cdot]$ can be seen as a sequence of arbitrary many instruments, consequently justifying the name continuous measurement interpretation.

In this article, we choose that the complex noise z_t^* up to time t is itself a measurement signal. In general, this is the case when the operator $O(t,s,z^*)$ occurring in the general Gaussian SSE (1) has at most linear dependence on z_t^* [31]. However, we will see later that this choice can always be made under exactly the same conditions when a time continuous measurement interpretation exists. We denote by $G_t(z_t^*)$ the solution to Eq. (1) with initial condition $G_0(z_0^*) = \mathbb{1}$. We can construct an instrument $Y_t(F_t)$ corresponding to general Gaussian SSE (1) by setting

$$Y_t(F_t)(|\psi_0\rangle\langle\psi_0|) = \int_{F_t} G_t(z_t^*)|\psi_0\rangle\langle\psi_0|G_t^\dagger(z_t^*)\mu(dz_t^*), \qquad (4)$$

which describes how the initial state $|\psi_0\rangle$ is mapped when measurement outcomes $z_t^* \in F_t$ are obtained. Here, $\mu(dz_t^*)$ is the Gaussian probability measure for the stochastic process z_t^*. Compatibility demand can be written in terms of the following two-times propagator

$$A_s^t(z_t^*) = G_t(z_t^*)G_s^{-1}(z_t^*), \qquad (5)$$

and with the help of Radon-Nikodym theorem [31,39] as follows:

$$\mathbb{1} = \int_{\Omega_s^t} A_s^{t\dagger}(z_t^*) A_s^t(z_t^*) \nu(dz_\tau^* | z_\sigma^* = \zeta_\sigma^*)$$
$$\equiv \mathcal{M}\big[A_s^{t\dagger}(z_t^*) A_s^t(z_t^*) \big| \zeta_\sigma^*\big], \quad \nu_0^s - \text{a.s.}, \tag{6}$$

where a.s. denotes almost surely and the $\mathcal{M}[\cdot|\zeta_\sigma^*] := \mathcal{M}[\cdot|\{z_\sigma^*|z_\sigma^* = \zeta_\sigma,\ \sigma \in (0, s]\}]$ is a shorthand notation for an expectation value conditioned on the history of a single noise realization z_σ^* taking values ζ_σ from time 0 till time s. Clearly, the condition (6) which guarantees that the probability of measuring a stochastic state is invariant with respect to measuring up to some later time $t > s$, and discarding the gained information (causality) is equivalent to the martingale condition

$$\mathcal{M}[||\psi_t(z_t^*)||^2 | \zeta_\sigma] = ||\psi_s(\zeta_s^*)||^2. \tag{7}$$

Accordingly, if the martingale condition (7) is satisfied, the measurement interpretation is in principle possible.

With the preliminaries elaborated earlier, we can now state the main result of this article in the form of the following theorem.

Theorem 1. If the correlations satisfy the following δ-function constraint

$$\alpha(t,s) + \eta(t,s) = \kappa\delta(t-s), \tag{8}$$

and if the coupling operator L is self-adjoint, a time continuous measurement interpretation is possible.

If condition (8) holds and $L = L^\dagger$, the SSE takes the simpler time-convolutionless form

$$\partial_t|\psi_t(z^*)\rangle = -iH_S|\psi_t(z^*)\rangle + z_t^* L - \frac{\kappa}{2}L^2|\psi_t(z^*)\rangle, \tag{9}$$

since $O(t,t,z^*) = L$ [36]. Clearly, z_t^* can be taken as the measurement signal since how the operator $O(t,s,z^*)$ would in general depend on z_t^* has no influence on the stochastic dynamics when the δ-function constraint (8) holds true. Remarkably, under that constraint (8) the corresponding master equation for the reduced state is in general not closed:

$$\partial_t\rho_t = -i[H_S, \rho_t] + (\mathcal{M}[\overline{O}(t,z^*)|\psi_t(z^*)\rangle\langle\psi_t(z^*)|]L$$
$$- L\mathcal{M}[\overline{O}(t,z^*)|\psi_t(z^*)\rangle\langle\psi_t(z^*)|] + \text{H.c.}), \tag{10}$$

with $\overline{O}(t,z^*) = \int_0^t ds\, \alpha(t,s) O(t,s,z^*)$; it can be seen as a signature of correlations between the open system and its environment which have a non-negligible effect on the timescale of the open system evolution. Significantly, the operator $O(t,s,z^*)$ occurs in the master equation (10), as opposed to the case of SSE (9).

The master equation retains the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form when the bath correlation function is singular $\alpha(t,s) \propto \delta(t-s)$ [17,18]. In this case, the future of the stochastic process is independent of the past and conditional mean values can be replaced by unconditional mean values; see Appendix C 2. Accordingly, in the white noise limit the martingale condition (7) and the δ-function constraint (8) are always satisfied.[1] Our δ-function constraint given by Eq. (8) allows, however, also for other forms of Hermitian correlation $\alpha(t,s)$, which, most notably, can also have finite correlation time.

With this, we can prove Theorem 1 as follows.

Proof. When the δ-function constraint (8) holds, the conditional mean value $\hat{\mu}_c(\tau)$ and the conditional covariance $\Sigma_c(\tau,\tau')$ of $(z_\tau, z_\tau^*)^T$ with $\tau, \tau' \in (s,t]$ satisfy

$$\hat{r}^\dagger\hat{\mu}_c(\tau) = 0, \qquad \hat{r}^\dagger\Sigma_c(\tau,\tau') = r^\dagger(\kappa,\kappa^*)\delta(\tau-\tau'),$$

where $\hat{r}^\dagger = (r^\dagger, r^\dagger)$ and r is an arbitrary operator acting on the Hilbert space of the system. If we additionally assume that $L = L^\dagger$, the conditional norm fulfills

$$\partial_\tau\mathcal{M}[||\psi_\tau(z^*)||^2|\zeta_\sigma] = 0,$$

with initial condition $\mathcal{M}[||\psi_s(z^*)||^2|\zeta_\sigma] = ||\psi_s(\zeta_s^*)||^2$. Thus the norm squared is a martingale. ■

We have presented the details of the proof in the Appendix D.

We want once again stress that the only conditions for the existence of a time continuous measurement interpretation of solutions of SSE (1) are the self-adjointness of the coupling operator $L = L^\dagger$ and fulfillment of the δ-function constraint (8). Consequently, the measurement interpretation is possible for important paradigms such as, among others, the spin-boson model, where $H_S = \frac{\omega}{2}\sigma_z$ and $L = g\sigma_x$, and quantum Brownian motion, where $H_S = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$ and $L = q$ even when the system-environment correlation function $\alpha(t,s)$ has a finite correlation time.

An important issue remains. Namely, the explicit construction of a nontrivial realization of condition (8).

IV. EXAMPLE: ORNSTEIN-UHLENBECK PROCESS

A noise process satisfying condition (8) and having a finite correlation time can be constructed using Ornstein-Uhlenbeck (O-U) processes. Suppose that the noise $z_t^* = x_t - iy_t$ driving the dynamics is decomposed into two independent real-valued O-U processes x_t, y_t. The noises n_t satisfy $\dot{n}_t = -a_n n_t + \sqrt{D_n}\xi_t^n$, where $a_n > 0$ are the drift coefficients and $D_n > 0$ the diffusion coefficients for $n \in \{x, y\}$. ξ_t^x and ξ_t^y are uncorrelated standard real-valued Gaussian white noises.[2]

Both real processes are of zero mean, and the covariances are

$$\langle x_t x_s\rangle = \frac{D_x}{2a_x}e^{-a_x|t-s|}, \quad \langle y_t y_s\rangle = \frac{D_y}{2a_y}e^{-a_y|t-s|}.$$

One can easily show that the Hermitian and non-Hermitian correlation functions read

$$\alpha(t,s) = \langle x_t x_s\rangle + \langle y_t y_s\rangle, \quad \eta(t,s) = \langle x_t x_s\rangle - \langle y_t y_s\rangle.$$

When we take the limit $a_x \to \infty$ while $D_x/a_x^2 = \frac{\kappa}{2}$, the complex process z_t satisfies the δ-function constraint (8). However, the Hermitian correlation function $\alpha(t,s)$ has a finite correlation time since it takes the form

$$\alpha(\tau) = \frac{\kappa}{2}\delta(\tau) + \frac{D_y}{2a_y}e^{-a_y|\tau|}, \tag{11}$$

[1] Here by white noise we mean that both $\eta(t,s)$ and $\alpha(t,s)$ are proportional to $\delta(t-s)$.

[2] $\mathcal{M}[\xi_t^\nu \xi_s^\sigma] = \delta_{\nu\sigma}\delta(t-s)$.

which describes exponentially damped environment correlations.

V. DISCUSSION

The question of the existence of a measurement interpretation for non-Markovian quantum trajectories has a long and vivid history. After the introduction of the general Gaussian non-Markovian stochastic Schrödinger equations, new potential to explore this question has opened up. In this paper we have investigated the possibility to have a time continuous measurement interpretation for the non-Markovian trajectories satisfying the relevant martingale condition (7).

We have shown that the stochastic trajectories of the general Gaussian non-Markovian SSE can have a time continuous measurement interpretation beyond the usual white noise limit. In fact, the stochastic trajectories of the general Gaussian SSE for models with self-adjoint coupling operator and correlations satisfying the δ-function constraint (8) possess a time continuous measurement interpretation. Moreover, we also showed that the set of processes satisfying that constraint is not empty.

Note that when rigorously derived from a microscopic open quantum system model, the imaginary part of α is nonzero. Then the δ-function constraint can only be met in an approximate sense (for instance, in the usual Born-Markov derivation of the GKSL master equation). Nevertheless, a real-valued α may appear in connection with quantum-classical hybrids [40–46], which have been actively studied in recent decades. A prominent example of a such system would be a quantum system which is coupled to a classical measurement apparatus. A well known exception to the typical case where the bath correlation function is complex valued would be a resonant coupling of a two-level atom to a zero temperature quantum environment with Lorentzian spectral density within the rotating wave approximation [4]. In this case, the bath correlation function is purely exponential.

The framework we use is abstract and therefore it has the advantage of describing the measurement process without making any reference to a particular physical measurement setting. However, an interesting question, which still has to be answered, is how such a time continuous measurement can be implemented experimentally. Accordingly, we hope that our work inspires further investigations on suitable physical interpretations of the stochastic trajectories described by (1) and the associated explicit experimental setup for time continuous measurement.

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APPENDIX A: MEASUREMENT INTERPRETATION

We introduce the quantum stochastic formalism of time continuous measurement and apply it to the stochastic open system states $|\psi_t(z^*)\rangle$ [31,47,48]. To be self-contained, we briefly sketch the method (essentially reproducing the results of [31]), leading to a compatibility demand given in Eq. (6) for the family of instruments describing the time continuous measurement.

We begin by recalling the notion of an instrument [37,38]. Let $(\Omega, \mathcal{F})$ be a measurable space, where Ω is a set and $\mathcal{F}$ is a σ-algebra on Ω. An instrument $Y(\cdot)[\cdot] : \mathcal{F} \times \mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_S)$ is a linear, normalized ($\mathrm{Tr}(Y(\Omega)[A]) = \mathrm{Tr}(A)$), trace nonincreasing and completely positive map. $\mathcal{T}(\mathcal{H}_S)$ denotes all trace class operators on the system Hilbert space $\mathcal{H}_S$. With a given instrument $Y(\cdot)[\cdot]$ and a normalized pre-measurement state ρ, the probability of verifying an event $F \in \mathcal{F}$ is

$$P(F|\rho) = \mathrm{Tr}(Y(F)[\rho]), \tag{A1}$$

and the corresponding post-measurement state reads

$$\rho_F \equiv \frac{Y(F)[\rho]}{\mathrm{Tr}(Y(F)[\rho])}. \tag{A2}$$

As shown by Ozawa in [49], the related indirect single-shot measurement interpretation exists.

By transition to time continuous measurement, one introduces a family of measurable spaces $(\Omega_t, \mathcal{F}_t)$, parametrized by the time $t \geqslant 0$. Here, Ω_t is a set of all possible measurement records, which in our case is an appropriate space of functions. The measurement scheme is then described by the corresponding family of instruments $\mathcal{Y} = \{Y_t(\cdot)[\cdot] : \mathcal{F}_t \times \mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_S); t \geqslant 0\}$. Causality sets an additional constraint on $\mathcal{Y}$: Probability of verifying an event F_s has to be invariant with respect to measuring up to some later time $t > s$ ($\mathcal{F}_s \subset \mathcal{F}_t$) and discarding the gained information. Consequently, the following "compatibility demand" has to be satisfied for all s, $0 < s < t$:

$$\forall F_s \in \mathcal{F}_s : (Y_s)^\dagger(F_s)[\mathbb{1}] = (Y_t)^\dagger\left(F_s \times \Omega_s^t\right)[\mathbb{1}], \tag{A3}$$

where $(Y_s)^\dagger(F_s)[\cdot]$ refers to the dual map of $Y_s(F_s)[\cdot]$ and $F_s \times \Omega_s^t$ denotes all elements of Ω_t which coincide with F_s if confined to $(0, s]$.

It was shown in [47] that a general continuous quantum measurement can be described in an integral form containing stochastic evolution operators $\{V_t^i\}_i$, satisfying the orthonormality relation. As both the initial and the final states of our stochastic trajectories are rank 1, the continuous measurement is described by a single stochastic operator V_t for a fixed time t (an efficient measurement [50]). More than that, any instrument corresponding to an efficient measurement can be decomposed in terms of a positive scalar measure $\mu_t : \mathcal{F}_t \to I \in [0, \infty)$ and stochastic evolution in the following form [47,48]:

$$Y_t(F_t)[\rho] = \int_{F_t} V_t(x_t)\rho(V_t(x_t))^\dagger \mu_t(dx_t), \tag{A4}$$

where x_t is a measurement record till time t. The associated probability for measuring a record in the vicinity of x_t and the post-measurement state read, respectively,

$$P_t(dx_t|\psi_0) = ||V_t(x_t)\psi_0||^2\mu_t(dx_t), \tag{A5}$$

$$|\psi(x_t)\rangle = \frac{V_t(x_t)|\psi_0\rangle}{||V_t(x_t)\psi_0||}, \tag{A6}$$

where $|\psi_0\rangle$ is the initial state. In [48], additional conditions on stochastic evolution operators were set, which are, however, equivalent to the compatibility demand (A3).

APPENDIX B: MEASUREMENT INTERPRETATION OF THE GENERAL STOCHASTIC SCHRÖDINGER EQUATIONS

By investigating the general SSEs in terms of existence of a time continuous measurement interpretation, the following three demands on our construction are plausible [31]:

1. The compatibility demand (A3) should be fulfilled,
2. the obtained set of the pure post-measurement states should equal the set of all normalized solutions of the general Gaussian non-Markovian SSE (1) in the main text, and
3. the unconditional post-measurement state should be equal to the reduced density operator.

Note, that 3 does not follow from 2, as 2 contains no statement about the measurement probabilities.

As elaborated in [31], the stochastic evolution operator $V_s^t(z_t^*)$ can be set to be equal to a two-time propagator $A_s^t(z_t^*)$, fulfilling

$$A_s^t(z_t^*) = G_t(z_t^*)[G_s(z_s^*)]^{-1}, \tag{B1}$$

where $G_t(z_t^*)$ is the stochastic propagator corresponding to Eq. (1) in the main text: $|\psi_t(z^*)\rangle = G_t(z_t^*)|\psi(0)\rangle$. Further, the stochastic propagator $G_t(z_t^*)$ is a functional of the noise process $z_{t'}^*$, where $t' \in (0, t]$. Here, the complex noise z_t^* up to time t is itself a measurement signal, an assignment which can be done without loss of generality when $L = L^\dagger$ and δ-constraint (8) is satisfied. The condition 3 is then satisfied only when the scalar measure $\mu_t(dz_t^*)$ from Eq. (A4) is the Gaussian probability measure $\nu_t(dz_t^*)$ of the complex noise z_t^* in Eq. (1). Consequently, by applying the Radon-Nikodym theorem [39], the compatibility demand (A3) takes the form [31]

$$\begin{aligned}\mathbb{1} &= \int_{\Omega_s^t} A_s^{t\dagger}(z_t^*)A_s^t(z_t^*)\nu(dz_\tau^*|z_\sigma^* = \zeta_\sigma^*)\\ &\equiv \mathcal{M}[A_s^{t\dagger}(z_t^*)A_s^t(z_t^*)|\zeta_\sigma^*], \quad \nu_0^s - \text{a.s.},\end{aligned} \tag{B2}$$

where the $\mathcal{M}[\cdot|\zeta_\sigma^*] := \mathcal{M}[\cdot|\{z_\sigma^*|z_\sigma^* = \zeta_\sigma,\ \sigma \in (0, s]\}]$ is a shorthand notation for an expectation value conditioned on the history of a single noise realization z_σ^* taking values ζ_σ from time 0 till time s. From now on, we denote the "past" with $\sigma \in I_P(s) = (0, s]$ and "future" with $\tau \in I_F(t) = (s, t]$.

The compatibility demand, Eq. (B2), is equivalent to

$$\begin{aligned}\mathcal{M}[||\psi_t(z_t^*)||^2|\zeta_\sigma] &= \mathcal{M}[\langle\psi_s(z_s^*)|A_s^{t\dagger}(z_t^*)A_s^t(z_t^*)|\psi_s(z_s^*)\rangle|\zeta_\sigma]\\ &= ||\psi_s(\zeta_s^*)||^2,\end{aligned} \tag{B3}$$

which states that the norm of the stochastic trajectories is a martingale. Equivalence is obtained by simply by sandwiching Eq. (B2) with $|\psi_s(\zeta_s^*)\rangle$ and using the properties of the two-times propagator.

APPENDIX C: CONDITIONAL CUMULANTS OF GENERAL COMPLEX GAUSSIAN NOISE

From the definition of the conditional average, we immediately note that [51]

$$\mathcal{M}[\mathcal{M}[z_\tau^*|\zeta_\sigma^*]z_\sigma] = \mathcal{M}[z_\tau^* z_\sigma] = \alpha^*(\tau, \sigma), \tag{C1}$$

$$\mathcal{M}[\mathcal{M}[z_\tau^*|\zeta_\sigma^*]z_\sigma^*] = \mathcal{M}[z_\tau^* z_\sigma^*] = \eta(\tau, \sigma), \tag{C2}$$

where the external average is taken with respect to both the past and the future. Equations (C1) and (C2) imply that $\tilde{z}_\tau^* = z_\tau^* - \mathcal{M}[z_\tau^*|\zeta_\sigma^*]$ is orthogonal to z_σ and z_σ^* with respect to the inner product $\mathcal{M}[\tilde{z}_\tau^* z_\sigma] = \mathcal{M}[\tilde{z}_\tau^* z_\sigma^*] = 0$. A Gaussian process conditioned on its past is again a Gaussian process [52,53]. Conditional mean value $\mathcal{M}[z_\tau^*|\zeta_\sigma^*]$ for a Gaussian process is linear in both ζ_σ and ζ_σ^* [53], which leads to

$$\begin{aligned}\hat{\mu}_c(\tau) &= \mathcal{M}[\hat{z}_\tau|\hat{\zeta}_\sigma]\\ &\equiv \mathcal{M}\left[\begin{pmatrix} z_\tau \\ z_\tau^*\end{pmatrix}\middle|\begin{pmatrix}\zeta_\sigma \\ \zeta_\sigma^*\end{pmatrix}\right]\\ &= \int_{I_P(s)} d\sigma\, V(\tau, \sigma)\hat{\zeta}_\sigma.\end{aligned} \tag{C3}$$

$V(\tau, \sigma)$ is a linear operator with support on $I_F(t) \times I_P(s)$.

From the orthogonality conditions $\mathcal{M}[\tilde{z}_\tau^* z_\sigma] = 0 = \mathcal{M}[\tilde{z}_\tau^* z_\sigma^*]$ follows a connection between $V(\tau, \sigma)$ and the covariance matrix $\Sigma(\tau, \sigma) = \mathcal{M}[\hat{z}_\tau \hat{z}_\sigma^\dagger]$:

$$\Sigma(\tau, \sigma) = \int_{I_P(s)} d\sigma'\, V(\tau, \sigma')\Sigma(\sigma', \sigma). \tag{C4}$$

From the above equation, we can solve $V(\tau, \sigma)$ by multiplying from the right with $\Sigma^{-1}(\sigma, \sigma'')$ which is a linear operator with support on $I_P(s) \times I_P(s)$ satisfying

$$\int_{I_P(s)} d\sigma' \int_{I_P(s)} d\sigma\ \Sigma(\sigma', \sigma)\Sigma^{-1}(\sigma, \sigma'')f(\sigma') = f(\sigma'') \tag{C5}$$

for an arbitrary vector-valued function f. We thus find, for all $\tau \in I_F(t)$, $\sigma \in I_P(s)$,

$$V(\tau, \sigma) = \int_{I_P(s)} d\sigma'\ \Sigma(\tau, \sigma')\Sigma^{-1}(\sigma', \sigma). \tag{C6}$$

If $\eta(\tau, \sigma) = 0$ then $\mathcal{M}[\tilde{z}_\tau^* z_\sigma] = \mathcal{M}[\tilde{z}_\tau^* z_\sigma^*] = 0$ implies statistical independence [54]. Next, we show that with the conditional mean (C3), orthogonality of $\tilde{z}_\tau^*$ with z_σ and z_σ^* also implies statistical independence.

Let

$$\hat{z}_\nu = \begin{pmatrix} x_\nu + iy_\nu \\ x_\nu - iy_\nu\end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i\end{pmatrix}\begin{pmatrix} x_\nu \\ y_\nu\end{pmatrix} = T\begin{pmatrix} x_\nu \\ y_\nu\end{pmatrix} = T\hat{u}_\nu,$$

where x_ν and y_ν are zero mean real-valued Gaussian processes such that the covariance matrix $\Sigma'(\nu, \mu) = \mathcal{M}[\hat{u}_\nu \hat{u}_\mu^T]$ satisfies

$$\begin{aligned}\Sigma(\nu, \mu) &= \mathcal{M}\left[\begin{pmatrix} z_\nu \\ z_\nu^*\end{pmatrix}\begin{pmatrix} z_\mu^*, & z_\mu\end{pmatrix}\right]\\ &= \mathcal{M}[(T\hat{u}_\nu)(T\hat{u}_\mu)^\dagger]\\ &= T\Sigma'(\nu, \mu)T^\dagger.\end{aligned} \tag{C7}$$

For a real-valued Gaussian process, the conditional mean is given by

$$\mathcal{M}[\hat{u}_\tau|\hat{m}_\sigma] \equiv \mathcal{M}[\hat{u}_\tau|\hat{u}_\sigma = \hat{m}_\sigma] = \int_{I_P(s)} d\sigma A(\tau, \sigma)\hat{m}_\sigma, \tag{C8}$$

where the matrix $A(\tau, \sigma)$ satisfies $\Sigma'(\tau, \sigma) = \int_{I_P(s)} d\sigma'\, A(\tau, \sigma')\Sigma'(\sigma', \sigma)$. We define $\tilde{\hat{u}}_\tau = \hat{u}_\tau - \mathcal{M}[\hat{u}_\tau|\hat{u}_\sigma]$ and clearly $\tilde{\hat{u}}_\tau$ and $\hat{u}_\sigma$ are orthogonal. Since $\hat{u}_\tau$ is a real-valued and Gaussian process, it implies that $\tilde{\hat{u}}_\tau$ and $\hat{u}_\sigma$ are statistically independent. As the statistical independence is invariant under T, it follows that $\tilde{\hat{z}}_\tau = T\tilde{\hat{u}}_\tau$ (and thus $\tilde{z}_\tau$) and $\hat{z}_\tau$ are statistically independent.

The conditional Hermitian covariance is

$$\alpha_c(\tau,\tau') = \mathcal{M}[(z_\tau - \mathcal{M}[z_\tau|\zeta_\sigma])(z^*_{\tau'} - \mathcal{M}[z^*_{\tau'}|\zeta^*_\sigma])|\zeta^*_\sigma],$$

$\tau' \in I_F(t)$, and analogously for non-Hermitian covariance $\eta_c(\tau,\tau')$. Statistical independence implies that we can replace the conditional mean by unconditional mean [52,53]. With the help of Eq. (C3) and using $\alpha(\nu,\mu) = \alpha^*(\mu,\nu)$ and $\eta(\nu,\mu) = \eta(\mu,\nu)$, we can then write

$$\Sigma_c(\tau,\tau') = \mathcal{M}[\tilde{\hat{z}}_\tau \tilde{\hat{z}}^\dagger_{\tau'}] = \Sigma(\tau,\tau') - \int_{I_P(s)} d\sigma V(\tau,\sigma)\Sigma(\sigma,\tau'). \tag{C9}$$

1. Case when the δ-function constraint is satisfied [$\alpha(t,s) + \eta(t,s) = \kappa\delta(t-s)$]

In this case we have the following conditional statistics. Let $\hat{r} = \binom{r}{r}$ be a column vector, where r is an operator acting on the Hilbert of the system. The conditional mean value with respect to a particular history ζ^*_σ is

$$\hat{r}^\dagger \hat{\mu}_c(\tau) = \int_{I_P(s)} d\sigma\, \hat{r}^\dagger V(\tau,\sigma)\hat{\zeta}_\sigma = \int_{I_P(s)} d\sigma \int_{I_P(s)} d\sigma'\, \hat{r}^\dagger \Sigma(\tau,\sigma')\Sigma^{-1}(\sigma',\sigma)\hat{\zeta}_\sigma. \tag{C10}$$

Now, we see that

$$\hat{r}^\dagger \Sigma(\tau,\sigma') = r^\dagger(\alpha(\tau,\sigma') + \eta(\tau,\sigma'),\ \alpha^*(\tau,\sigma') + \eta^*(\tau,\sigma')\ \ ) = r^\dagger(\kappa,\kappa^*)\delta(\tau-\sigma'). \tag{C11}$$

Since $\tau \in I_F(t)$ and $I_F(t) \cap I_P(s) = \emptyset$, we find

$$\hat{r}^\dagger \hat{\mu}_c(\tau) = 0. \tag{C12}$$

Similarly, for the conditional covariance we can obtain

$$\begin{aligned} \hat{r}^\dagger \Sigma_c(\tau,\tau') &= \hat{r}^\dagger \Sigma(\tau,\tau') - \int_{I_P(s)} d\sigma \int_{I_P(s)} d\sigma'\, \hat{r}^\dagger \Sigma(\tau,\sigma')\Sigma^{-1}(\sigma',\sigma)\Sigma(\sigma,\tau') \\ &= r^\dagger(\kappa,\kappa^*)\delta(\tau-\sigma') + \int_{I_P(s)} d\sigma \int_{I_P(s)} d\sigma'\, \hat{r}^\dagger \kappa\delta(\tau-\sigma')\Sigma^{-1}(\sigma',\sigma)\Sigma(\sigma,\tau') = r^\dagger(\kappa,\kappa^*)\delta(\tau-\sigma'), \end{aligned} \tag{C13}$$

since $\tau \notin I_P(s)$.

2. White noise

Let us assume that the Gaussian complex noise is white, which means that

$$\Sigma^{\mathrm{W}}(t,s) = \begin{pmatrix} \gamma & \eta^* \\ \eta & \gamma^* \end{pmatrix} \delta(t-s), \tag{C14}$$

such that $\Sigma^{\mathrm{W}}(t,s)$ is not negative. From Eq. (C4) it follows that $V^{\mathrm{W}}(\tau,\sigma) = \mathbb{1}\delta(\tau-\sigma)$. This, as it turns out, implies that the conditional mean value and covariance satisfy

$$\mu^{\mathrm{W}}_c(\tau) = 0, \quad \Sigma^{\mathrm{W}}_c(\tau,\tau') = \Sigma^{\mathrm{W}}(\tau,\tau'), \tag{C15}$$

when $\tau,\tau' \in (s,t]$ and $\sigma \in (0,s]$. The implication is that in the white noise limit the conditional mean can be replaced with the unconditional mean.

APPENDIX D: MARTINGALE PROPERTY FOR $L = L^\dagger$ AND δ-FUNCTION CONSTRAINT

When the noise process satisfies the δ-function constraint $\alpha(t,s) + \eta(t,s) = \kappa\delta(t-s)$, we can compute the equation of motions for average norm $|\psi_\tau(z^*)\rangle$ conditioned on some history $z_\sigma = \zeta_\sigma$, where $\sigma \in I_P(s)$. Evolution of the square of the conditional norm from $\tau \in I_F(t)$ to $\tau + \epsilon$, with $\epsilon \geqslant 0$, is

$$\begin{aligned} \mathcal{M}[||\psi_{\tau+\epsilon}(z^*)||^2|\zeta_\sigma] &= \mathcal{M}[\langle\psi_\tau(z^*)|G^\dagger_{\tau+\epsilon,\tau}(z^*)G_{\tau+\epsilon,\tau}(z^*)|\psi_\tau(z^*)\rangle|\zeta_\sigma] \\ &= \mathcal{M}\left[\langle\psi_\tau(z^*)|\left\{\mathbb{1} + \epsilon\left(iH_S + z_\tau L - \frac{\kappa^*}{2}L^2\right) + \mathcal{O}(\epsilon^2)\right\}\right. \\ &\quad \times \left.\left\{\mathbb{1} + \epsilon\left(-iH_S + z^*_\tau L - \frac{\kappa}{2}L^2\right) + \mathcal{O}(\epsilon^2)\right\}|\psi_\tau(z^*)\rangle\middle|\zeta_\sigma\right] \\ &= \mathcal{M}[||\psi_\tau(z^*)||^2|\zeta_\sigma] + \epsilon\mathcal{M}[\langle\psi_\tau(z^*)|(z_\tau + z^*_\tau)L|\psi_\tau(z^*)\rangle|\zeta_\sigma] - \epsilon\mathrm{Re}(\kappa)\mathcal{M}[\langle\psi_\tau(z^*)|L^2|\psi_\tau(z^*)\rangle|\zeta_\sigma] + \mathcal{O}(\epsilon^2), \end{aligned}$$

where $\mathrm{Re}(\kappa)$ is the real part of κ.

We get thus

$$\partial_\tau \mathcal{M}[||\psi_\tau(z^*)||^2|\zeta_\sigma] = \lim_{\epsilon\to 0}\frac{\mathcal{M}[||\psi_{\tau+\epsilon}(z^*)||^2|\zeta_\sigma] - \mathcal{M}[||\psi_\tau(z^*)||^2|\zeta_\sigma]}{\epsilon}$$
$$= \mathcal{M}[\langle\psi_\tau(z^*)|(z_\tau + z_\tau^*)L|\psi_\tau(z^*)\rangle|\zeta_\sigma] - \mathrm{Re}(\kappa)\mathcal{M}[\langle\psi_\tau(z^*)|L^2|\psi_\tau(z^*)\rangle|\zeta_\sigma],$$

with initial condition $\mathcal{M}[||\psi_\tau(z^*)||^2|\zeta_\sigma]|_{\tau=s} = ||\psi_s(\zeta^*)||^2$. We set $(L,\ \ L) = \hat{\mathbf{L}}^T$, where L is the self-adjoint coupling operator. With the Novikov theorem [55] one obtains

$$\mathcal{M}[\mathrm{tr}\{\hat{z}_\tau^\dagger\hat{\mathbf{L}}|\psi_\tau(z^*)\rangle\langle\psi_\tau(z^*)|\}|\zeta_\sigma] = \mathrm{tr}\{\hat{\mu}_c^\dagger(\tau)\hat{\mathbf{L}}\mathcal{M}[|\psi_\tau(z^*)\rangle\langle\psi_\tau(z^*)||\zeta_\sigma]\}$$
$$+\,\mathrm{tr}\left\{\mathcal{M}\left[\int_s^\tau d\tau'\,\hat{\mathbf{L}}^T\Sigma_c(\tau,\tau')\begin{pmatrix}\frac{\delta}{\delta z_{\tau'}}\\ \frac{\delta}{\delta z_{\tau'}^*}\end{pmatrix}|\psi_\tau(z^*)\rangle\langle\psi_\tau(z^*)|\middle|\zeta_\sigma\right]\right\}.$$

Using the results of the Appendix C 1, we have $\mu_c^\dagger(\tau)\hat{\mathbf{L}} = 0$ and $\hat{\mathbf{L}}^T\Sigma_c(\tau,\tau') = L(\kappa,\kappa^*)\delta(\tau-\tau')$. Since $\frac{\delta}{\delta z_\tau}|\psi_\tau(z^*)\rangle\langle\psi_\tau(z^*)| = |\psi_\tau(z^*)\rangle\langle\psi_\tau(z^*)|L$ and $\frac{\delta}{\delta z_\tau^*}|\psi_\tau(z^*)\rangle\langle\psi_\tau(z^*)| = L|\psi_\tau(z^*)\rangle\langle\psi_\tau(z^*)|$, we finally have

$$\mathcal{M}[\mathrm{tr}\{\hat{z}_\tau^\dagger\hat{\mathbf{L}}|\psi_\tau(z^*)\rangle\langle\psi_\tau(z^*)|\}|\zeta_\sigma] = \mathrm{Re}(\kappa)\mathcal{M}[\langle\psi_\tau(z^*)|L^2|\psi_\tau(z^*)\rangle|\zeta_\sigma],$$

which gives

$$\partial_\tau\mathcal{M}[||\psi_\tau(z^*)||^2|\zeta_\sigma] = 0, \quad \mathcal{M}[||\psi_s(z^*)||^2|\zeta_\sigma] = ||\psi_s(\zeta^*)||^2. \tag{D1}$$

We have thus shown that the conditional mean value of the norm is a martingale. Therefore a measurement interpretation exists.

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