Spreading nonlocality in a quantum network

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Starting from several copies of bipartite noisy entangled states, we design a global and optimal local measurement-based protocol in one- and two-dimensional lattices by which any two or more prefix sites can be connected via entanglement. Production of bipartite as well as multipartite entangled states in a network is verified in a device-independent way through the violation of Bell inequalities with two settings per site and with continuous range of settings. We also note that if the parties refuse to perform local measurements, the entanglement distribution scheme fails. We obtain critical values of noise allowed in the initial state so that the resulting output state show nonlocal correlation in different networks with arbitrary number of connections. We report that by employing our method, it is possible to create a multipartite entangled state, violating Bell inequality and having a continuous range of settings, from bipartite states which do not violate Clauser-Horne-Shimony-Halt Bell inequalities in an one-dimensional lattice with the minimal coordination number being six. Such a feature of superadditivity in violation can also be observed in a triangular two-dimensional lattice but not in a square lattice.

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I. INTRODUCTION

In the age of the internet, the ability to share information among an arbitrary number of parties situated in different locations is the basic building block for a communication network [1–4]. The information distributed can both be classical and quantum in nature. In both cases, it was shown that entangled states shared between the senders and the receivers can enhance the performance of the protocol, which cannot be achieved by unentangled states [5]. Therefore, the first step toward establishing a communication network is to generate highly entangled bipartite and multipartite states connecting different sites. To achieve this goal, several protocols have been developed, which include quantum repeaters [6,7] based on entanglement distillation from noisy shared states followed by entanglement swapping [8,9], entanglement percolation [10–13], which originated from the notion borrowed from statistical mechanics, and k-pair communication problem [3].

A network can be defined in one- and two-dimensional lattices with different geometrical structures—the edges of the lattice are covered by bipartite states and, depending on the lattice, several edges are connected through each vertex or node, determining the coordination number of the lattice (see, e.g., Refs. [10,14] and references therein). Initial states covering the lattice can also be ground states, a canonical equilibrium, or evolved states at certain times of a suitable Hamiltonian. A prominent example is the resonating valence bond states in which all nodes are connected by a singlet [15]. Typically, suitable joint measurements performed at each node can create a multipartite quantum correlated state between prefix sites [10] which can be used later for quantum information processing tasks. Note that such a notion has also been used to build a measurement-based quantum computer [16]. Performance of all these tasks are measured, for example, by localizable entanglement [17], singlet conversion probability [10], and gate fidelity [16].

One of the most counterintuitive features of entangled state is that it exhibits a kind of "nonlocal" effect. Specifically, it means no local realistic model can account for all the correlations emerging from local measurements on entangled states [18–20]. Although not all entangled states violate Bell inequality [21], the Bell test [22,23] turns out to be the determiner of device-independent certification of entangled states. Violation of Bell inequalities are shown to be useful in quantum cryptography [24], random number generation [25], etc. Moreover, several preprocessing protocols were prescribed to probe violation of Bell inequality for states which do not respond to the Bell test. In this direction, in the seminal paper Popescu [26] showed that local filtering can help to reveal the nonlocality known as hidden nonlocality (see also Refs. [27,28]). Other activation protocols involving the Bell test with multiple copies were proposed [29,30]. In a similar spirit, violation of local realism was demonstrated in a multisite domain employing entanglement swapping—the initial seven or more copies of Werner states which do not violate Clauser-Horne-Shimony-Halt (CHSH) Bell inequality [19], forming a star network, leads to the final multipartite state that violates functional Bell inequality [31]. It is known as superadditivity or superactivation in nonlocality [32] (cf. Ref. [33]).

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It was shown that if two copies of a quantum state or two different quantum states do not individually violate CHSH inequalities, the joint state after a suitable measurement can violate local realism [32–35]. Moreover, the quantitative analysis [35] shows that starting from quantum states which do not violate Bell inequalities, one can obtain violation of Bell inequalities when several copies of them are provided. Such a superactivation scenario can be used to connect violation of Bell inequalities with the teleportation protocol, specifically with the networks [36]. In particular, it was proved that all states that are useful for quantum teleportation deterministically violate the Bell inequality if suitable measurements are performed on a large number of copies of a shared state.

In this work, we propose a framework of a quantum network based on global and local measurement, in which bipartite and multipartite entangled states are generated in two or more arbitrary prefixed sites in one- and two-dimensional lattices. In contrast to the previous works [32–35], we consider here local measurements along with global measurement, such that a few prefixed parties become entangled. Specifically, the entire lattice is covered by the Werner states, and depending on the sites which we want to connect, a minimum number of joint and local measurements are implemented resulting in an entangled state which can be certified through the Bell test [18,19,31,37,38]. Notice that after global measurements, if instead of performing local measurements the parties does not collaborate, the resulting state turns out to be separable, thereby having a local realistic model. For the bipartite case, we consider the CHSH inequality [19], while for multipartite states, we employ Mermin-Belinskii-Klyshko (MBK) [37,38] and functional Bell (FB) inequalities [31].

In a one-dimensional (1D) lattice, for a fixed number of nodes, the value of critical noise allowed in the initial state, leading to a nonlocal multiparty state, is determined for different coordination numbers. A similar analysis has also been performed with varying numbers of nodes for a fixed coordination number. The analysis also reveals that for exhibiting superadditivity, the minimum coordination number required is six in a given lattice. Note here that if the Werner states are furnished in a linear chain, instead of a star network, and only Bell measurements are allowed, such superadditivity cannot be observed [32]. We also report the maximum amount of noise accepted in the initial state resulting in nonlocal correlations in the output state in square and triangular two-dimensional (2D) lattices. Moreover, we observe that the superadditivity in violation can only be shown in a two-dimensional lattice having lowest coordination number of six, e.g., in a triangular lattice but not in a square lattice. Specifically, we show that for a fixed number of joint and local measurements it is always possible to find a minimal number of nodes for which superadditivity in nonlocality can be exhibited.

We organize the paper in the following way. In Sec. II, we first introduce the notion of violation of Bell inequality in a multipartite domain when some of the parties collaborate, which we call as localizable nonlocality. We then consider localizable nonlocality in a star network in Sec. III and find the critical noise required to obtain violation in the output state. Section IV is devoted to the results obtained for 1D and 2D lattices and we conclude with discussion in Sec. V.

II. VIOLATION OF BELL INEQUALITY WITH COLLABORATION

Let us consider that N parties share a multipartite state, ρ_N, and among N parties, m number of parties collaborate by performing local projective measurements $\{M_i\}$ in their respective part of the state, which leads to an ensemble $\{p_i, \rho^i_{N-m}\}$. Here p_i is the probability pertaining to a specific outcome combination obtained by m parties who want to collaborate with other $N-m$ parties to perform Bell test. We define the average value of Bell expression of the postmeasurement ensemble consisting of $N-m$ party state, $\{p_i, \rho^i_{N-m}\}$ as *localizable nonlocality* (LNL):

$$\mathcal{L}_{NL} = \max_{\{M_i\}} \sum_i p_i \, BV\left(\rho^i_{N-m}\right), \tag{1}$$

where BV indicates the amount of violation of appropriate Bell inequality and maximization is performed over the set of all local measurements, $\{M_i\}$, by m number of parties. For example, when $N-2$ parties measure locally on their respective subsystems, the violation of CHSH inequality of the resulting bipartite state is studied, thereby certifying the entanglement of the bipartite state in a device-independent manner. On the other hand, when $N-m>2$, the output is a multipartite state and we analyze the violation of MBK [37,38] and FB [31] inequalities. Before investigating $\mathcal{L}_{NL}$ in network, let us briefly discuss the Bell operators that we will use in this paper. It is important to stress here that in both Ref. [32] and our study, the N-partite state is obtained via global measurement, as we will discuss in detail in the succeeding section, although the local measurements on m parties of the multipartite states are not considered in the previous study [32].

A. Condition for violation of CHSH inequality

As stated earlier, let us first describe the CHSH inequality and its violation for two spin-half particles [19]. Suppose a bipartite state ρ_{AB} is shared between two spatially separated observers, say, Alice and Bob. They both can choose to perform a dichotomic measurement at a time from a different set of two observables. CHSH inequality puts a restriction on a particular algebraic expression imposing locality and reality assumptions. It involves correlation between local measurement statistics of Alice and Bob, i.e.,

$$\mathcal{B} \equiv |\langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_2B_1\rangle - \langle A_2B_2\rangle| \leqslant 2. \tag{2}$$

Here $\langle A_iB_j\rangle = \mathrm{Tr}(A_i \otimes B_j \rho_{AB})$ is the correlation between measurement outcomes a_i and b_j for the measurement A_i and B_j performed by Alice and Bob respectively. For an arbitrary two-qubit state, the maximal violation of the CHSH inequality in terms of state parameters were derived [39]. In particular, maximal violation of local realism in this case can be written in terms of correlation matrix, T, whose elements are defined as

$$T_{ij} = \mathrm{Tr}[\rho_{AB}\sigma_i \otimes \sigma_j], \tag{3}$$

where σ_i $(i = x, y, z)$ are Pauli matrices. A state is considered to violate Bell inequality if

$$M(\rho_{AB}) > 1, \tag{4}$$

where M is the sum of the two maximum eigenvalues of $T^\dagger T$. Hence, maximal Bell violation is quantified as

$$BV(\rho_{AB}) = M(\rho_{AB}) - 1, \tag{5}$$

and finally $\mathcal{L}_{NL}$ reads as

$$\mathcal{L}_{NL}^{\text{CHSH}} = \max_{\{M_i\}} \sum_{i=1}^{2^{N-2}} p_i BV\left(\rho_{AB}^i\right), \tag{6}$$

where ρ_{AB}^i is obtained after performing local measurements on $N-2$ parties of an initial state ρ_N.

B. Violation of Mermin-Belinskii-Klyshko inequality

It is a multiparty correlation function Bell inequality in which each party can choose to measure from a set of two observables [38]. In an N-partite state, when m number of parties perform local measurements, the violation of MBK inequality for the rest of the $(N-m)$-qubit state is given by the expectation value of the MBK operator [40,41],

$$B_k = \tfrac{1}{2} B_{k-1} \otimes \left(\sigma_{a_k} + \sigma_{a'_k}\right) + \tfrac{1}{2} B'_{k-1} \otimes \left(\sigma_{a_k} - \sigma_{a'_k}\right), \tag{7}$$

where a_k and a'_ks are the two vectors on the unit sphere that indicate possible measurement directions of the corresponding party. B_k is obtained recursively from B_{k-1} and B'_k is obtained from B_k by interchanging all the a_ks by a'_ks. A state ρ_N is said to violate the MBK inequality if the average value of this operator becomes greater than 1, i.e.,

$$|\text{Tr}[B_N \rho_N]| > 1, \tag{8}$$

and the corresponding LNL can be computed as

$$\mathcal{L}_{NL}^{\text{MBK}} = \max_{\{M_i\}} \sum_{i=1}^{2^m} p_i |\text{Tr}[B_{N-m} \rho_{N-m}]|_i - 1. \tag{9}$$

C. Violation of functional Bell inequality

Like the MBK inequality, this multisite inequality is based on the Schwartz inequality. Instead of two settings at each site like CHSH and MBK inequalities, it considers a set-up involving a continuous range of settings [31]. Let G_n be an local observable at the nth party ($n = 1, \dots, N$), and each of them depends on some parameter η_n. Based on these measurements, G_ns, we define the correlation function as

$$C_{QM}(\eta_1, \dots, \eta_N) = Tr(\rho_N G_1, \dots, G_N), \tag{10}$$

and the corresponding correlation admitting the local hidden variable model, with the distribution of local variable, denoted by $\upsilon(\lambda)$, reads as

$$C_{\text{LHV}}(\eta_1, \dots, \eta_N) = \int d\lambda \upsilon(\lambda) \prod_{n=1}^{N} \mathbb{I}_n(\eta_n, \lambda). \tag{11}$$

Here $I_n(\eta_n, \lambda)$ is the predetermined measurement result of G_n for λ. To show $C_{QM} \neq C_{\text{LHV}}$, one can use the basic principle of Schwartz inequality, and so we compute

$$\langle C_{QM} | C_{\text{LHV}} \rangle = \int d\eta_1 \dots d\eta_N C_{QM}(\eta_1, \dots, \eta_N) C_{\text{LHV}}(\eta_1, \dots, \eta_N)$$

and the

$$\|C_{QM}\|^2 = \int d\eta_1 \dots d\eta_N (C_{QM}(\eta_1, \dots, \eta_N))^2, \tag{12}$$

which finally leads to $C_{QM} \neq C_{\text{LHV}}$.

In this case, to calculate localized nonlocality, we first consider the average overlap of C_{QM} and C_{LHV} over 2^m outcomes after m parties perform local measurements. Suppose we can show

$$\sum_{i=1}^{2^m} p_i \langle C_{QM} | C_{\text{LHV}} \rangle_i \leqslant H, \tag{13}$$

where H depends on local measurement parameters of m parties. On the other hand, for a given outcome i, we can get $\|C_{QM}\|_i^2$ calculated for a postmeasurement state of $(N-m)$ parties. Finally, we have

$$\mathcal{L}_{NL}^{FB} = \max_{\{M_i\}} \sum_{i=1}^{2^m} p_i \|C_{QM}\|_i^2 - H.$$

Here $\langle C_{QM} | C_{\text{LHV}} \rangle_i$ (overlap of C_{QM} and C_{LHV} for the ith outcome) and $\|C_{QM}\|_i^2$ (the norm of C_{QM} also with measurement result i) are defined as

$$\langle C_{QM} | C_{\text{LHV}} \rangle_i = \int d\eta_1 \dots d\eta_{N-m} C_{QM}(\eta_1, \dots, \eta_{N-m})_i \times C_{\text{LHV}}(\eta_1, \dots, \eta_{N-m})_i, \tag{14}$$

$$\|C_{QM}\|_i^2 = \int d\eta_1 \dots d\eta_{N-m} (C_{QM}(\eta_1, \dots, \eta_{N-m})_i)^2. \tag{15}$$

III. LOCALIZABLE NONLOCALITY IN A STAR NETWORK: A BUILDING BLOCK

Before detecting entanglement via the Bell test in a lattice, let us first fix the operations that we are going to perform for establishing a connection between any two or more nodes in a network. In this section, we study the violation of Bell inequality in a geometry which turned out to be a building block (unit) of an entire network.

Suppose $2N$ number of parties, A_i and B_i ($i = 1, 2 \dots N$), sharing N identical copies of arbitrary bipartite state among them as shown in the Fig. 1. Let us consider a scenario in which all the A_is are assumed to be situated in one place and hence we can replace them by an observer, say, Alice (A), while B_is are located in distant positions forming a star network [32]. Alice performs a projective joint measurement on the N parties in her possession and consequently a multipartite entangled state ρ_N is created among other distant N sites. Note that the creation of ρ_N in this way is similar to the one considered in previous works [32,33].

We consider a scenario in which all the A_i and B_i share N identical copies of the Werner state, given by

$$\rho_W = p|\psi^-\rangle\langle\psi^-| + (1-p)\frac{I}{4}, \tag{16}$$

where $p \in (0, 1)$. We know that it is entangled for $p > 1/3$, while it violates CHSH inequality having two settings when $p > 1/\sqrt{2} = 0.7071$ [21]. In the original paper [21], Werner proposed a local hidden variable model in the region $p \leqslant \frac{1}{2}$

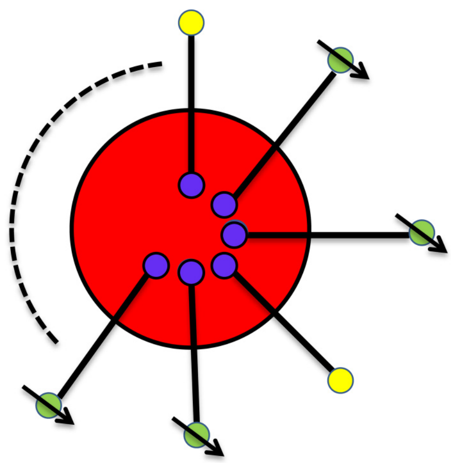

FIG. 1. Schematic diagram of a single block. It consists of N copies of ρ_W states. An N-party GHZ measurement is performed on the N qubits situated in the center marked as purple circles [32,33] while local measurements are carried out on the (green) qubits, marked with arrow. Note that the last step involving local measurement was not considered in previous works. Our aim is to produce an entangled state between the yellow qubits.

with projective measurements, while for general positive operator valued measurements (POVMs), the LHV model was given in a smaller region, namely $p \leqslant \frac{5}{12}$ [42]. Depending on the Grothendieck constant $K_G(3)$, the LHV model was later improved and was known for $p \leqslant 0.6829$ with projective measurements, $p \leqslant 0.4553$ with POVMs [43,44], and for the entire range, i.e., for $p < 1/\sqrt{2}$, when one of the parties is restricted to the planar measurements [45]. On the other hand, it was shown that the Werner state violates a different set of Bell inequalities when $0.7056 \leqslant p \leqslant 1$ [46]. In our paper, when the final state is bipartite, we will evaluate CHSH inequalities.

As previously mentioned, A is measured in an N-qubit basis consisting of a Greenberger- Horne-Zeilinger state (GHZ) [47] in the center of the star. If one of the outcomes, say, $\frac{1}{\sqrt{2}}(|00..0\rangle + |11\ldots 1\rangle)$ occurs, the state shared between $B_1, \ldots, B_N$ is projected to an entangled N-qubit state. For example, if the initial state is $\rho_W^{\otimes 3}$, A performs measurements in the $\{\frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle), \frac{1}{\sqrt{2}}(|001\rangle \pm |110\rangle), \frac{1}{\sqrt{2}}(|010\rangle \pm |101\rangle), \frac{1}{\sqrt{2}}(|100\rangle \pm |011\rangle)\}$ basis, which we call the GHZ basis, easily extended to an arbitrary number of qubits. It results in an output state of the form

$$q_1|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + q_2\frac{I}{8} + (1-q_1-q_2)(|000\rangle\langle 000| + |111\rangle\langle 111|). \quad (17)$$

Note that the probability of obtaining any of the outcomes by the joint measurement of A is equal for the initial state, $\rho_W^{\otimes 3}$. In general, when the input state is $\rho_W^{\otimes N}$, the form of the output state after the joint measurement by A remains same as in Eq. (17), which is known as an X-type state having nonvanishing diagonal and two cross-diagonal terms. In particular, if one writes the final state in the computational basis, we find nonvanishing coefficients appearing in all the diagonal and in two off-diagonal terms, which are given by

$$|00\ldots 0\rangle\langle 11\ldots 1| \text{ and } |11\ldots 1\rangle\langle 00\ldots 0|. \quad (18)$$

Interestingly, we notice that in the evaluations of the Bell-CHSH, MBK, and FB inequalities, the above off-diagonal terms play a crucial role which we investigate carefully.

In order to establish nonclassical correlation in the network in terms of violation of CHSH, MBK, or FB inequalities, we invoke two different strategies. In both the scenarios, we assume N parties perform joint measurement and the output state is shared between N parties, which may face two situations. Our aim is to produce a state, ρ_{N-m}, for which Bell violation is maximized over all local projective measurements. Note that we here consider only rank-1 measurement. In particular, we divide the entire protocol into the following three steps:

Step 1: We start with N copies of a given bipartite states forming a star network. Alice performs a global measurement on N parties in the center, consequently creating a N-partite entangled state.

Step 2: Any m number of parties either can perform local measurements or leave the protocol. It leads to the following cases:

(1) *Noncollaborative strategy.* m number of parties leave the protocol without any measurements; i.e., they do not collaborate with $N-m$ parties. Mathematically, $\rho_{N-m} = Tr_{1\ldots m}\rho_N$.

(2) *Collaborative strategy.* Among N parties, m parties collaborate in a sense that they perform local projective measurements on their subsystems, so that an output state of $N-m$ party is produced.

Step 3: We finally evaluate the violation of Bell inequality of the $(N-m)$-party state. If the final state is bipartite, we consider the CHSH inequality, while for a multipartite state, we check the violation of MBK as well as functional Bell inequality.

In the second step, if strategy 1 is followed, we call the associated nonclassical correlation certified via violation of local realism the *$N-m$ reduced nonlocality*, while if strategy 2 is followed, we call it *localisable nonlocality*. As we will see, the above scenario, especially the collaborative strategy, can help to spread entanglement over a large distance in a network from initial bipartite noisy entangled states.

No reduced nonlocality. If the initial state shared between $2N$ parties is the Werner state, we can easily find that noncollaborative strategy leads to an $(N-m)$-party state which is separable. Therefore, the noncollaborative strategy is not suitable for spreading nonclassicality. In the rest of the paper, we only concentrate on the collaborative strategy.

Chain scenario. Let us now suppose that there are three copies of Werner states shared between AB, BC, and CD pairs, and B and C perform measurements in the Bell basis. It can be shown that, after measurement, the resulting state between A and D is again a Werner state with p^3 and hence it violates Bell CHSH inequality for a smaller range of p than that of the initial state. With the increase of the number of copies, the situation deteriorates more, i.e., there always exists a range of p where the initial states violate CHSH inequalities while the final state does not.

A. Locating bipartite nonlocality

Suppose among m parties, the lth party performs a local measurement in the $\{|\pm\rangle_l\}$ basis, given by

$$|\pm\rangle_l = \cos\theta_l|0\rangle \pm e^{-i\phi_l}\sin\theta_l|1\rangle, \tag{19}$$

where $l = 1, 2, ..., m$. There are 2^m possible outcomes of the measurements. For each outcome, we evaluate the maximum possible violation of Bell-type inequality for the remaining $(N-m)$-partite state. If the state violates any kind of Bell inequality, we call the state *nonlocal*. The postselected state, ρ_{N-m}, takes the same form as the N-partite state with modified coefficients given in Eq. (17), provided the initial Werner states are projected by the joint measurement.

As mentioned earlier, in all the Bell expressions considered here, the off-diagonal terms of the density matrix, ρ_{N-m}, are important and are explicitly given by

$$\langle 00\dots0|\rho_{N-m}|11\dots1\rangle_i = \pm\frac{p^N\left[e^{+i\sum_1^m\phi_l}\prod_1^m\sin\theta_l\right]}{2[1-f_i(\theta_1,\theta_2,\dots,\theta_m)]}, \tag{20}$$

$$\langle 11\dots1|\rho_{N-m}|00\dots0\rangle_i = \pm\frac{p^N\left[e^{-i\sum_1^m\phi_l}\prod_1^m\sin\theta_l\right]}{2[1-f_i(\theta_1,\theta_2,\dots,\theta_m)]}. \tag{21}$$

The probability of getting a particular outcome is a function of local measurement parameters, namely

$$p_i = \frac{[1-f_i(\theta_1,\theta_2,\dots\theta_m)]}{2^m}. \tag{22}$$

First assume that after the N-qubit GHZ-basis measurement we are able to create N-party entangled state, and let us also focus on bipartite nonlocality. To consider the Bell-CHSH inequalities, $(N-2)$ parties perform local measurements having 2^{N-2} outcomes and the bipartite state ρ_2 is obtained. For each measurement result, we find that two maximum eigenvalues of $T^\dagger T$ of ρ_{12} are p^4 and $\frac{p^{2N}\prod_{l=1}^{N-2}\sin^2(\theta_l)}{[1-f_i(\theta_1,\dots,\theta_{N-2})]^2}$. The average Bell-CHSH violation can be calculated as

$$\mathcal{L}_{NL}^{\text{CHSH}} \equiv \sum p_i BV(\rho_i) = p^4 + \max_{\theta_l,\phi_l}\sum_{i=1}^{2^{N-2}}\frac{p^{2N}\prod_{l=1}^{N-2}\sin^2(\theta_l)}{2^{N-2}[1-f_i(\theta_1,\dots,\theta_{N-2})]} - 1. \tag{23}$$

For a moderate number of parties i.e., for $N = 4, 5, 6$, we check that maximization in $\mathcal{L}_{NL}^{\text{CHSH}}$ is obtained when all the parties perform same measurements, $\theta_i = \pi/2$, and it does not depend on ϕ_i. We also notice that

$$\max_{\{\theta_i\}}\sum_{i=1}^{2^{N-2}}\frac{\prod_{k=1}^{N-2}\sin(\theta_k)}{2^{N-2}[1-f_i(\theta_1,\dots,\theta_{N-2})]}|_{\theta_i=\pi/2} = 1. \tag{24}$$

If we assume that maximum is attained at $\theta_i = \pi/2$ for higher values of N as well, we obtain the condition for the critical value of noise allowed in the initial state so that the violation of the resulting state after steps 1 and 2 occurs. Specifically, we obtain p_{cr} by solving the equation given by

$$p^4 + p^{2N} - 1 = 0. \tag{25}$$

It implies that the initial Werner state should possess the mixing parameter, $p > p_{cr}$, which produces an output state violating the Bell-CHSH inequality. For example, if $N = 3$, we find that the critical value of the parent Werner state has to be greater than 0.869 to obtain the violation of Bell inequality of the resulting state, the value of p_{cr} increases with the number of parties in the network.

B. Mermin-Belinskii-Klyshko nonlocality in star network: Critical noise

Let us now discuss the prescription by which multipartite nonlocality among any prefix set of points in the star network can be to established. For N-qubit multipartite state, the MBK operator can be written as [41]

$$B_N = 2^{(N-1)/2}[e^{i\beta_N}|0\rangle\langle 1|^{\otimes N} + e^{-i\beta_N}|1\rangle\langle 0|^{\otimes N}], \tag{26}$$

where $\beta_N = \frac{\pi}{4N-4}$ and is obtained by putting $\sigma_{a_k} = \sigma_x$ and $\sigma_{a'_k} = \sigma_y$ for all values of k. Using this operator, we find the condition for violation of MBK inequality by the N-party state in Eq. (17) as

$$2^{(N-1)/2}P^N\cos\beta_N > 1, \tag{27}$$

which leads to

$$p > p_{cr} = \frac{1}{2^{\frac{N-1}{2N}}(\cos\beta_N)^{\frac{1}{N}}}. \tag{28}$$

On the other hand, in a collaborative network, m parties perform local measurements and leave the network. Violation of the MBK inequality can be calculated on the remaining $(N-m)$-partite state, after performing optimization over local projective measurements by m parties. The average violation of MBK inequality reduces to

$$\begin{aligned}\mathcal{L}_{NL}^{\text{MBK}} &= \sum_1^{2^m} p_i|\text{Tr}[B_{N-m}\rho_{N-m}]|_i - 1\\ &= \sum_1^{2^m}\frac{2^{(N-m-1)/2}p^N\left[\cos\left(\beta_{N-m}-\sum_1^m\phi_i\right)\prod_1^m\sin\theta_i\right]}{2^m} - 1\\ &= 2^{(N-m-1)/2}p^N\left[\cos\left(\beta_{N-m}-\sum_1^m\phi_i\right)\prod_1^m\sin\theta_i\right] - 1.\end{aligned}$$

From the analysis of small N, it can again be shown to reach maximal value when $\theta_i = \pi/2$ and $\beta_{N-m} = \sum_1^m\phi_i$. Therefore, the violation of MBK inequality of the resulting state leads to a maximum amount of noise permissible in the initial state. The condition reads as

$$2^{(N-m-1)/2}p^N > 1, \tag{29}$$

implying

$$p_{cr} = \frac{1}{\left(2^{\frac{N-m-1}{2}}\right)^{\frac{1}{N}}}. \tag{30}$$

C. Noise threshold from functional Bell inequality: Superadditivity

Let us move to a scenario where violation of FB inequality of the output state of $N-m$ parties in a star network is investigated. As before, we are also interested to find out the critical noise value of the initial Werner state leading to the violation of FB inequality in the multipartite state created after executing the protocol. We can choose the measurement operators in the x-y plane of the Bloch sphere to calculate the violation of FB inequalities for states shared between B_is, i.e.,

$$G_n(\eta_n) = |+, \eta_n\rangle\langle +, \eta_n| - |-, \eta_n\rangle\langle -, \eta_n|, \quad (31)$$

where $|\pm, \eta_n\rangle = |0\rangle \pm e^{i\eta_n}|1\rangle$. Since the state has only two off-diagonal terms and diagonal terms, quantum mechanical prediction in this case reads as [32]

$$C_{QM}(\eta_1, \dots, \eta_N) = \mathrm{Tr}(G_1 \dots G_N \rho_N) = p^N \cos\left(\sum_{i=1}^{N} \eta_i\right)$$

and

$$\begin{aligned}\|C_{QM}\|^2 &= \int d\eta_1 \dots d\eta_N (C_{QM}(\eta_1 \dots . \eta_n))^2 \\ &= p^{2N} \int_0^{2\pi} d\eta_1 \dots d\eta_N \left(1 + \cos\left(2\sum_{i=1}^{N} \eta_i\right)\right)/2 \\ &= p^{2N} \frac{(2\pi)^N}{2}.\end{aligned}$$

Similarly, the inner product of C_{QM} and C_{LHV} takes the form

$$\begin{aligned}\langle C_{QM}|C_{\mathrm{LHV}}\rangle &= \int_0^{2\pi} d\eta_1 \dots d\eta_N C_{QM} C_{\mathrm{LHV}} \\ &= \int_0^{2\pi} d\eta_1 \dots d\eta_N \int d\lambda \rho(\lambda) \prod_{n=1}^{N} \mathbb{I}_n(\eta_n, \lambda) p^N \\ &\quad \times \cos\left(\sum_{j=1}^{N} \eta_j\right) \leqslant p^N 4^N, \quad (32)\end{aligned}$$

where we have used the fact that [31]

$$\int_0^{2\pi} d\eta_1 ... d\eta_N \int d\lambda \rho(\lambda) \prod_{n=1}^{N} \mathbb{I}_n(\eta_n, \lambda) \cos\left(\sum_{j=1}^{N} \eta_j\right) \leqslant 4^N. \quad (33)$$

When $\|C_{QM}\|^2$ is greater than $p^N 4^N$, the state violates the FB inequality, which leads to the threshold noise of the initial state, given by

$$\begin{aligned} p^N &\geqslant 2(2/\pi)^N. \\ p_{cr} &= 2^{\frac{1}{N}}(2/\pi). \quad (34)\end{aligned}$$

It was shown in Ref. [32] that the resulting multipartite state after the central GHZ measurement by Alice exhibits nonlocality by violating functional Bell inequality even if the initial state does not violate CHSH inequality; the feature was called *superadditivity,* which is revealed when $N \geqslant 7$.

We will now show that such a superadditivity of nonlocality can also be obtained in a collaborative star network. After local measurements by m number of B_is, the quantum mechanical correlation among $N - m$ parties postselecting upon ρ_{N-m} is given by

$$\begin{aligned} &C_{QM}(\eta_1 \dots \eta_{N-m})_i \\ &= Tr(G_1 \dots G_{N-m}\rho_{N-m}) \\ &= \frac{p^N\left[\prod_1^m \sin\theta_i\right] \cos\left(\sum_{i=1}^m \phi_i - \sum_{i=1}^{N-m} \eta_i\right)}{[1 - f_i(\theta_1, \theta_2, \dots, \theta_m)]}, \quad (35)\end{aligned}$$

and its norm can be found as

$$\begin{aligned}\|C_{QM}\|_i^2 &= \int d\eta_1, \dots, d\eta_{N-m} (C_{QM}(\eta_1, \dots, \eta_{N-m})_i)^2 \\ &= \frac{p^{2N}\left[\prod_1^m \sin^2\theta_i\right] \int d\eta_1 \dots \eta_n \left(1 + \cos\left(2\sum_{j=1}^m \phi_j - 2\sum_{k=1}^{N-m} \eta_k\right)\right)}{2[1 - f_i(\theta_1, \theta_2, \dots, \theta_m)]^2} = \frac{p^{2N}\left[\prod_1^m \sin^2\theta_i\right](2\pi)^{N-m}}{2[1 - f_i(\theta_1, \theta_2, \dots, \theta_m))^2}. \quad (36)\end{aligned}$$

The averaged QM predictions, where averaging is done over 2^m outcomes, can then be written as

$$\sum_{i=1}^{2^m} p_i \|C_{QM}\|_i^2 = p^{2N} \left[\prod_1^m \sin^2\theta_i\right] \frac{(2\pi)^{N-m}}{2} \sum_{i=1}^{2^m} \frac{1}{2^m[1 - f_i(\theta_1, \theta_2, \dots, \theta_m)]^2}.$$

Similarly, the individual inner product of C_{QM} and C_{LHV} at the ith measurement outcome is

$$\begin{aligned}\langle C_{QM}|C_{\mathrm{LHV}}\rangle_i &= \int d\eta_1, \dots, d\eta_{N-m} \int d\lambda \rho(\lambda) \prod_{n=1}^{N-m} \mathbb{I}_n(\eta_n, \lambda) \frac{p^N\left[\prod_1^m \sin\theta_j\right] \cos\left(\sum_{k=1}^m \phi_k - \sum_{l=1}^{N-m} \eta_l\right)}{[1 - f_i(\theta_1, \theta_2, ..., \theta_m)]} \\ &= \frac{p^N\left[\prod_1^m \sin\theta_j\right]}{[1 - f_i(\theta_1, \theta_2, ..., \theta_m)]} \int d\eta_1 \dots d\eta_{N-m} \int d\lambda \rho(\lambda) \prod_{n=1}^{N-m} \mathbb{I}_n(\eta_n, \lambda) \\ &\quad \times \left[\cos\left(\sum_{k=1}^m \phi_k\right) \cos\left(\sum_{l=1}^{N-m} \eta_l\right) + \sin\left(\sum_{i=1}^m \phi_i\right) \sin\left(\sum_{i=1}^{N-m} \eta_i\right)\right] \\ &\leqslant \frac{p^N\left[\prod_1^m \sin\theta_i\right]}{[1 - f_i(\theta_1, \theta_2, ..., \theta_m)]} 4^{N-m} \left[\cos\left(\sum_{k=1}^m \phi_k\right) + \sin\left(\sum_{l=1}^m \phi_l\right)\right], \quad (37)\end{aligned}$$

and its averged value can be written as

$$\sum_{i=1}^{2^m} p_i \langle C_{QM}|C_{\rm LHV}\rangle_i \leqslant p^N\left[\prod_1^m \sin\theta_i\right]4^{N-m}\left[\cos\left(\sum_{k=1}^m\phi_k\right)+\sin\left(\sum_{l=1}^m\phi_l\right)\right]\leqslant p^N\left[\prod_1^m\sin\theta_i\right]4^{N-m}\sqrt{2}. \qquad (38)$$

Again, for small N, we find that the measurement gives the optimal violation when all the θ_is take the value $\pi/2$ and ϕ_is do not play a role. Therefore, from the violation of the localized FB inequality, $\mathcal{L}_{NL}^{FB}$, we obtain the critical value of the initial noise parameter, satisfying the condition, given by

$$p^N > 2\sqrt{2}\left(\frac{2}{\pi}\right)^{N-m}, \qquad (39)$$

which gives

$$p_{cr} = 2^{\frac{3}{2N}}\left(\frac{2}{\pi}\right)^{\frac{N-m}{N}}. \qquad (40)$$

Comparing Eqs. (30) and (40), we conclude that if we use FB inequality to detect multipartite nonlocality in the network, the amount of noise allowed for obtaining violation of the output state is higher than that of the MBK inequality, implying high robustness of localizable FB inequality against noise. We also show that for a fixed N, p_{cr} increases with the increase of number of measurements, m, while for fixed m, it decreases with N, as shown in Fig. 2. Moreover, we again report that the nonvanishing value of $\mathcal{L}_{NL}^{FB}$ can be obtained even when the initial shared state cannot violate CHSH inequality, as depicted in Fig. 2, thereby also giving rise to superaditivity in violation of FB inequality in a localized scenario (see Ref. [32] for nonlocalized case).

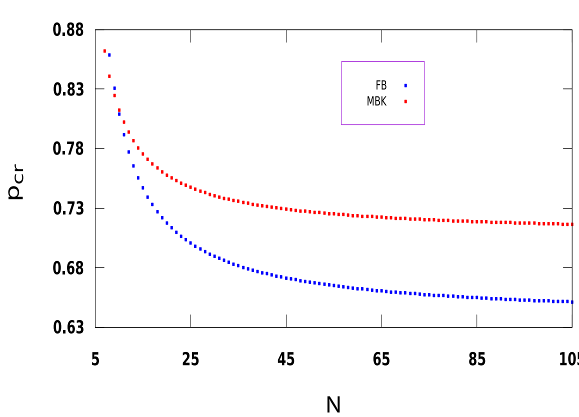

FIG. 2. Superadditivity in violation of local realism in a star network with local measurements. Variation of the threshold noise allowed in the initial state, p_{cr} (vertical axis), decreases with the number of copies of the initial state, N (horizontal axis). The number of local measurements is fixed to be 3. Violations of MBK and FB inequalities are calculated. Clearly, we see that to obtain violation via FB inequality of the final state, the ρ_W state can have $p < \frac{1}{\sqrt{2}}$.

IV. DISTRIBUTION AND DETECTION OF NONLOCALITY IN LATTICES

With the development of quantum communication protocols, establishing and detecting nonclassical correlations in one- and two-dimensional lattice networks play an important role. We also investigate the minimal amount of entanglement required to obtain the quantum correlation among any prefix sites, detectable through the violation of Bell inequality after the entire protocol is completed. In a one-dimensional network, we will also report that the output state obtained after the global and local measurement protocol can exhibit superadditivity in violation of Bell-type inequalities.

A. One-dimensional lattice

Consider a one-dimensional lattice consisting of z number of nodes with coordination number a (the coordination number is defined as the number of connection in each node, for example, in Fig. 3, $a = 4$). We call A_1, A_2, ... nodes. The entire lattice is covered by bipartite quantum states, namely, Werner state. First, A_ks perform joint measurements and all the sites except those parties whom we want to connect perform optimal local projective measurements. For example, in Fig. 3, suppose we want to create an entangled state among 1, 2, 3, and 4; the local measurements are performed by all the sites except these. In this situation, we are interested to find out whether the resulting multiparty state, shared among 1, 2, 3, and 4, violates Bell-type inequalities. After the measurement by A_is, we get a $[z(a-2)+2]$-party state, whose off-diagonal terms only contribute in the violation of Bell-CHSH, MBK, or FB inequalities and are given by

$$\langle 000\ldots|\rho_{z(a-2)+2}|111\ldots\rangle = p^{z(a-1)+1}/2, \qquad (41)$$

$$\langle 111\ldots|\rho_{z(a-2)+2}|000\ldots\rangle = p^{z(a-1)+1}/2. \qquad (42)$$

1. Spreading nonlocality

To obtain a two-party state in two distant locations, say, between 1 and 2 in Fig. 3, we perform measurements on remaining $z(a-2)$ parties having $2^{z(a-2)}$ outcomes. The average violation of Bell-CHSH inequalities can be calculated as

$$\sum p_i BV(\rho_i) = p^6 + \sum_{i=1}^{2^{z(a-2)}} \frac{p^{2[z(a-1)+1]}\prod_{k=1}^{2^{z(a-2)}}\sin^2(\theta_i)}{2^{z(a-2)}[1-f_i(\theta_1,\ldots,\theta_{z(a-2)})]} - 1. \qquad (43)$$

As in the previous cases, average violation of Bell-CHSH inequalities attains its maximal value for $\theta_i = \pi/2$ and does not depend on ϕ_i, which we check for a small lattice size. Therefore, the maximum amount of noise permissible for the initial state can be obtained from

$$p^6 + p^{2[z(a-1)+1]} - 1 > 0. \qquad (44)$$

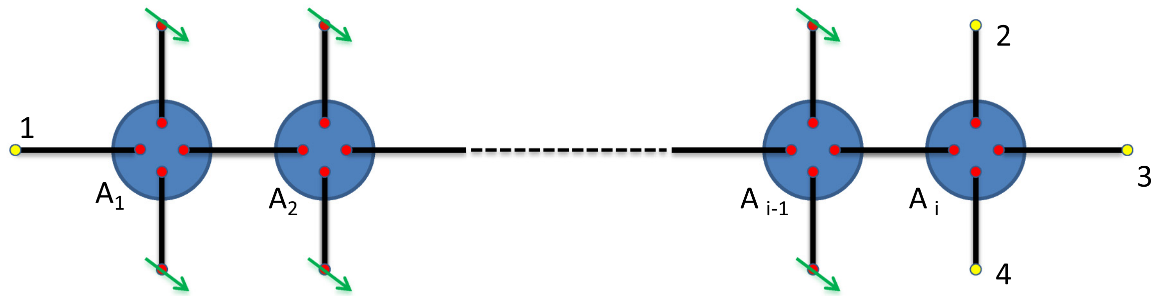

FIG. 3. Schematic diagram of an one-dimensional lattice composed of ρ_W states with a fixed coordination number of 4. Four-party GHZ basis and optimal local measurements are performed on the red qubits. We intend to produce a multipartite entangled state between the yellow qubits, marked as 1, 2, 3, and 4, whose entanglement can be verified by using the violation of local realism.

From the above equation, it is clear that p_{cr} depends on both z and a. For a fixed z, we observe that p_{cr} obtained from the violation of Bell-CHSH inequality increases with a and the same trends persist when a is fixed and z is varying, as depicted in Figs. 4 and 5.

2. Spreading multipartite nonlocality

Let us move to the violation of the averaged MBK inequality between the first site connected to the first node and $(a-1)$ parties of the last node of this chain. To establish such a connection, the rest sites, i.e., $(z-1)(a-2)$ number of parties, perform optimal local measurements, which results an **a**-party state. The localizable MBK violation in an **a**-party state reads as

$$\sum_1^l p_i|\mathrm{Tr}[B^a\rho_a]|_i - 1$$

$$= \sum_1^l \frac{2^{(a-1)/2}p^{z(a-1)+1}\left[\cos\left(\beta_a - \sum_1^l \phi_j\right)\prod_1^l \sin\theta_i\right]}{l} - 1$$

$$= 2^{(a-1)/2}p^{z(a-1)+1}\left[\cos\left(\beta_a - \sum_1^l \phi_j\right)\prod_1^l \sin\theta_i\right] - 1,$$

with

$$l = 2^{(z-1)(a-2)},$$

which can again be shown to be maximized when $\theta_i = \pi/2$ and $\beta_a = \sum_1^l \phi_i$. The state violates MBK inequality when p satisfies the condition, given by

$$2^{(a-1)/2}p^{z(a-1)+1} > 1, \tag{45}$$

and therefore, we get

$$p_{cr} = 2^{\frac{1-a}{2z(a-1)+2}}. \tag{46}$$

3. Violation of functional Bell inequality and superadditivity

Similar consideration also leads to p_{cr} using the FB inequality between the first site of the first node and $(a-1)$ parties of the last node. It reads as

$$p^{z(a-1)+1} > 2\sqrt{2}\left(\frac{2}{\pi}\right)^a$$

$$\Rightarrow p_{cr} = 2^{\frac{3}{2z(a-1)+2}}\left(\frac{2}{\pi}\right)^{\frac{a}{z(a-1)+1}}. \tag{47}$$

Interestingly, in the multipartite case, the threshold value of noise of the initial state decreases with **a** for fixed z. With the moderate value of the coordination number, p_{cr}, obtained from the violation of FB inequality, decreases much faster than that of the MBK inequality (see Fig. 4). However, for fixed **a**, p_{cr} increases with the increase of number of nodes z

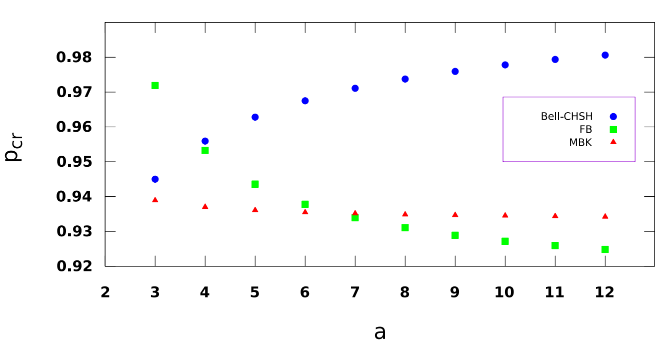

FIG. 4. Variation of the critical value of the noise parameter of the initial state against the co-ordination number, a, of a chain. The abscissa and ordinate respectively represent p_{cr} and a. We fix $z = 5$. For fixed number of nodes, p_{cr} increases with a for obtaining violation of Bell CHSH inequality (circle) while it decreases when MBK (square) and FB (triangle) inequalities are considered.

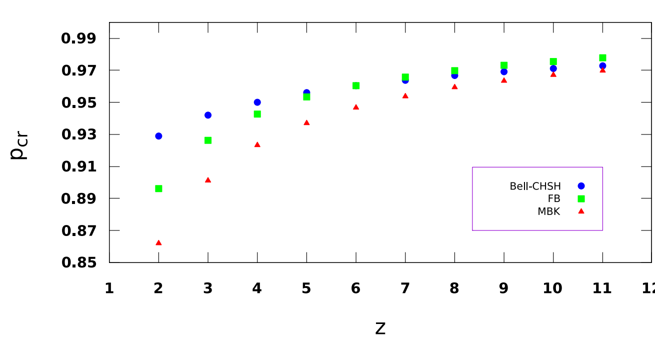

FIG. 5. p_{cr} (y axis) obtained by considering Bell-CHSH, MBK, and FB inequalities with the increase of the number of nodes, z (x axis). Here we have $a = 4$. Other specifications are the same as Fig. 4.

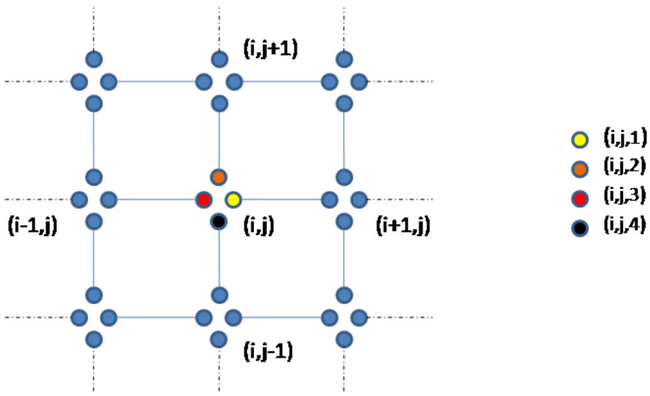

FIG. 6. Each node can be identified as (i, j). We require another coordinate to indicate sites in each node. Each site can be spotted by (i, j, q). Nearest nodes connected to (i, j) are shown.

(see Fig. 5). Without local measurements on sites, the multiparty states violates FB inequality when the initial state has

$$p > p_{cr} = 2^{\frac{1}{z(a-1)+1}} \left(\frac{2}{\pi}\right)^{\frac{z(a-2)+2}{z(a-1)+1}}. \tag{48}$$

Interestingly, superadditivity of nonlocality can be observed in this scenario with $a \geqslant 6$ for any arbitrary z. Specifically, if a one-dimensional lattice having a fixed number of nodes is considered, we find that we require the coordination number to be *six* to obtain an output state which can violate FB inequality starting with non-CHSH violating Werner states.

On the other hand, in a chain with $a = 6$ and for a fixed number of local measurements, we find the minimum number of nodes, z, required to exhibit the superadditivity in violation. Notice that with the increase of m, z increases to show superadditivity. With m number of local measurements, p_{cr} is modified as

$$p_{cr} = 2^{\frac{3}{2z(a-1)+2}} \left(\frac{2}{\pi}\right)^{\frac{z(a-2)+2-m}{z(a-1)+1}}. \tag{49}$$

For example, if we restrict ourselves to local measurements at 10 sites, we find that $z \geqslant 69$ leads to superadditivity.

B. Two-dimensional lattice

We consider two kinds of lattices having two different coordination numbers, namely square (with $a = 4$) and triangular (having $a = 6$) lattices (see Figs. 6–8). In both cases, we prescribe an algorithm to share an entangled state between any two or more distant points of the network.

1. Square lattice

Suppose in a square lattice, as in Fig. 7, our aim is to have a bipartite state between A and D. The prescription for establishing the connection is as follows:

(1) Let us first fix the notation used to describe nodes. First, the network is in a 2D plane and hence the position of any node can be described by using two numbers. Since $a = 4$, each node consists of four parties which are eventually connected to four different nodes, in four directions. After specifying the position of a node, another number is required

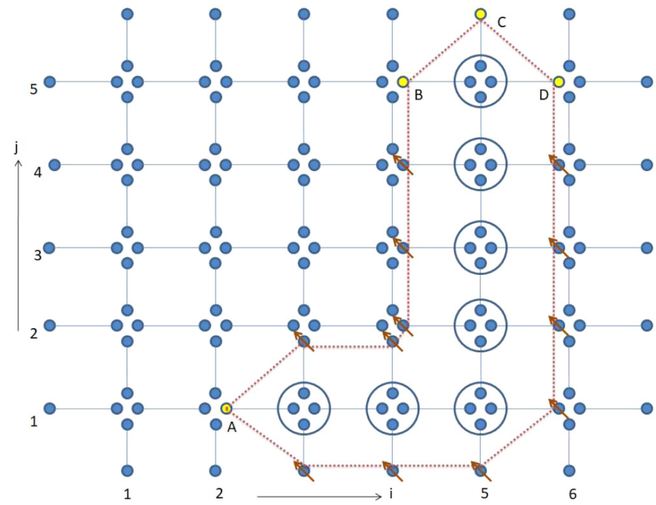

FIG. 7. A square lattice. To establish entanglement among A, B, C, and D, the joint and local measurements are marked by circle and arrow respectively.

to fix the position of the party within this node as shown in Fig. 6. For an example, the number (i, j, q) denotes the qth party in the (i, j)th node, where q can be 1, 2, 3, or 4.

(2) Suppose we choose two sites of different nodes, given by (i, j, q) and (i', j', q'). To connect these two parties, we can apply the following rule:

(a) First, we have to find the nearest node connected to (i, j, q) and (i', j', q'). Depending on the value of q, we can specify the nearest connected node to (i, j, q) as follows:

$$\begin{aligned} q &= 1 : (i + 1, j), \\ q &= 2 : (i, j + 1), \\ q &= 3 : (i - 1, j), \\ q &= 4 : (i, j - 1). \end{aligned} \tag{50}$$

Similarly, nearest nodes to any point can be identified (see Fig. 6).

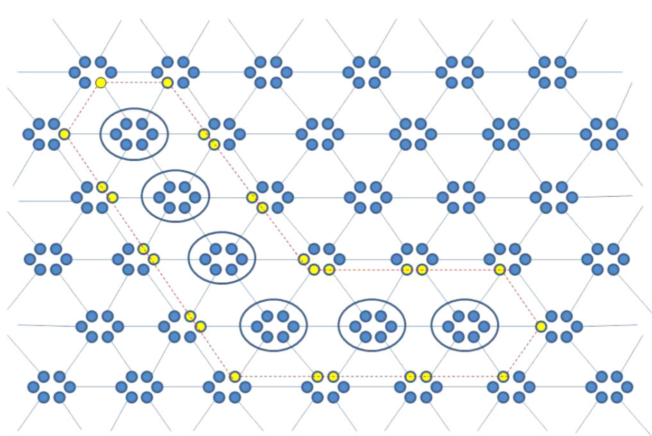

FIG. 8. A triangular lattice with coordination number 6. Creation of multipartite states after joint measurements are shown with the dotted line.

TABLE I. Example of p_{cr} for spreading nonlocality from a fixed point of a 2D network to three different points.

(i, j, q)	$(i\prime, j\prime, q\prime)$	(i_1, j_1)	(i_2, j_2)	z	a	p_{cr}
(2,1,1)	(6, 5, 3)	(2,2)	(5,5)	7	4	0.9658
(2,1,1)	(3, 3, 2)	(2,2)	(3,4)	4	4	0.9427
(2,1,1)	(3, 2, 4)	(2,2)	(3,2)	2	4	0.8963

(b) Suppose $q = 1$ and $q' = 3$. The nearest connected nodes are then $(i + 1, j)$ and $(i' - 1, j')$. Let us perform four-party GHZ-basis measurements (marked as a circle in Fig. 7) from the nodes denoted by $(i + 1, j)$ to $(i' - 1, j)$, denoted by $(i + 1, j) \to (i' - 1, j)$. Then measurements are performed in a direction given by $(i' - 1, j) \to (i' - 1, j')$. After these measurements, we get a chain which establishes multipartite entangled state between (i, j, q) and (i', j', q') and hence we can apply the results obtained in the previous subsection. To get the violation of local realism between A and D, one has to measure locally in an optimal basis at all the sites except A and D. Similarly we can also create a multiparty entangled state between them by performing GHZ-basis measurements in the direction $(i + 1, j) \to (i + 1, j') \to (i' - 1, j')$. To obtain the violation of Bell-CHSH inequality between A and D, the maximal noise allowed in the Werner state can be obtained by using Eq. (44). Specifically, after replacing z and a as

$$z = (i_2 - i_1) + (j_2 - j_1 + 1); \;\; a = 4, \tag{51}$$

where (i_1, j_1) and (i_2, j_2) are nearest connected nodes of (i, j, q) and (i', j', q') respectively.

The above prescription can also be used to generate multipartite entangled states when the entire lattice comprises of ρ_W. Critical noise of the initial state leading to a multipartite state among A, B, C, and D which violates FB inequality is listed in Table I.

2. Triangular lattice

In the triangular lattice, the coordination number is 6. This geometry is considered since we show a 1D lattice having coordination number 6 is special. Hence, such a lattice has potential to show superadditivity in violation of local realism. Suppose we want to create a multipartite state between sites, marked with yellow dots in Fig. 8. For generating such a multipartite entangled state, six joint measurements have to be performed in a six-qubit GHZ basis. From the results obtained for a 1D lattice, it is clear that even if the initial Werner state has $p < \frac{1}{\sqrt{2}}$, which does not violate CHSH inequality, the final state still violates FB inequality.

As previously stated, the LHV model was shown to exist in the whole range where CHSH inequality is not violated ($p \leqslant \frac{1}{\sqrt{2}}$) if any one of the party is restricted to perform planar projective measurements [44,45]. Our results show that even when the initial state possesses a local hidden variable model, we can obtain a resulting state which violates a multipartite Bell inequality with continuous range of settings, thereby showing a signature of superadditivity. Specifically, by using Eq. (49), we find that in a triangular lattice, the superadditivity can be observed for a multiparty state having 53 sites, if we perform local measurement on a single site of the lattice. Moreover, for a fixed number of local measurements, m, we can always provide a minimum number of nodes required to exhibit superadditivity by evaluating Eq. (49).

V. CONCLUSION

Establishing a connection between two or more parties by producing entanglement between them is essential to implement quantum information protocols in network. Generation of entanglement among an arbitrary prefix set of points has to be guaranteed by using certain detection procedures. In this paper, we employed bipartite as well as multipartite Bell tests to certify entanglement in a device-independent way in networks of one- and two-dimensional square and triangular lattices, which are initially covered by an arbitrary number of noisy entangled states. In a two-party scenario, we considered Clauser-Horne-Shimony-Halt (CHSH) inequality, while for multipartite states, we evaluated Mermin-Belinskii-Klyshko, a two-setting Bell inequality, as well as a functional Bell inequality having continuous settings. We proposed joint and local measurement-based method to establish entanglement in such a network with an arbitrary number of nodes. Note that the previous works which considered similar scenarios are only based on global measurements and do not involve local measurements. For a fixed number of nodes and a fixed coordination number, we found the entanglement content of the initial state so that the resulting state violates certain types of Bell inequalities. Our method shows that the number of nodes, coordination number, and joint and local measurements have a interplay in obtaining violation of Bell inequality.

We reported that a method presented here can produce a state that violates Bell inequality and has a continuous range of settings, although the initial state does not violate Bell inequalities with two settings and two outcomes. In particular, we found that the minimum coordination number required to activate such a superadditivity phenomena is six in a one-dimensional lattice with an arbitrary number of nodes. Based on this result, we designed a protocol on a triangular lattice in which there exists a final output state violating functional Bell inequality after joint and local measurements, although the initial states covering the lattice do not violate CHSH inequality. Such a phenomena is absent in a square lattice. Our proposed architecture of connecting any prefix sites in a lattice can be a step toward building the quantum internet.

ACKNOWLEDGMENTS

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