



## Berry phase in the composite Fermi liquid

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We derive the definition of the Berry phase for the adiabatic transport of a composite fermion (CF) in a half-filled composite Fermi liquid (CFL). It is found to be different from that adopted in previous investigations by Geraedts *et al.* For the standard CFL wave function, we analytically show that the Berry curvature is uniformly distributed in the momentum space. For the Jain-Kamilla wave function, we numerically show that its Berry curvature has a continuous distribution inside the Fermi sea and vanishes outside. We conclude that the CF with respect to both the microscopic wave functions is not a massless Dirac particle.

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### I. INTRODUCTION

The ubiquitous presence of the Berry phase is notable in recent theoretical investigations of condensed matter physics. For noninteracting systems, it becomes a unifying concept for characterizing the orbital effects of the spin or other internal degrees of freedom [1], and plays central roles in systems such as topological insulators [2], Dirac and Weyl semimetals [3] and valleytronic materials [4]. Recently, it becomes clear that the Berry phase also plays a role in the theory of composite fermions (CFs) [5]. CFs can be regarded as weakly interacting particles residing in a hidden Hilbert space [6]. A wave function of noninteracting CFs in the hidden Hilbert space can be mapped into a wave function appropriate for describing the physical state of a strongly correlated, fractionally filled Landau level. The theory of CFs has achieved tremendous successes in illuminating the fractional quantum Hall effect and related phenomena [5].

However, although the wave functions prescribed by the CF theory are shown to be very accurate and widely accepted [7], the effective theory of CFs interpreting the wave functions is still open to debate. The conventional interpretation, as explicated in Halperin-Lee-Read (HLR) theory of the composite Fermi-liquid (CFL) [8–10], treats the CF as an ordinary Newtonian particle. In its pristine form, it suffers from an apparent difficulty: it cannot correctly predict the CF Hall conductance of a half-filled Landau level [11]. The difficulty motivated Son to propose that the CF should be a massless Dirac particle [12]. An alternative interpretation, *i.e.*, the CF is neither a Newtonian particle nor a Dirac particle, but a particle subject to a uniformly distributed Berry curvature in

the momentum space and the Sundaram-Niu dynamics [13], was also put forward [14–16]. It was also shown that the picture is equivalent to the dipole picture of CFs [17]. The three pictures imply three different distributions of the Berry curvature, *i.e.*, zero, singularly distributed, and uniformly distributed, respectively. The clarification of the issue then hinges on the determination of the Berry curvature for CFs.

A “first principles” approach for determining the Berry curvature of CFs should be based on the microscopic CF wave functions prescribed by the CF theory. To this end, several attempts have been made. In Ref. [16], the dynamics of the CF Wigner crystal is derived. It shows that the CF is subject to a uniformly distributed Berry curvature in the momentum space. For the half-filled CFL phase, the Berry curvature distribution is found to be uniform by determining the dynamics of a test (distinguishable) CF added to the CF Fermi sea [15]. A heuristic argument based on the dipole picture of CFs also suggests the same [14,15]. These works may draw criticism for neglecting the particle exchange symmetry in their treatments. It is in this context that the recent works by Geraedts *et al.* stand out [18–20]. Their calculations are based on a microscopic CFL wave function in its full antisymmetric form. However, a close scrutiny of the works reveals a number of difficulties. First, the definition of the Berry phase is a prescribed one and is not fully justified. Second, the evaluation of the Berry phase based on the definition seems to be not numerically robust, sensitive to the choices of paths, and prone to statistical errors. Moreover, there exist extraneous  $\pm\pi/2$  phases preventing direct interpretations of numerical results. Finally, the microscopic CFL wave function adopted for the calculation is of the Jain-Kamilla (JK) type [21], which is numerically efficient in implementing the projection to the lowest Landau level (LLL). However, it is unclear whether or not it yields the same result as that from the standard CFL wave function prescribed by the theory of CFs [5].

In this paper, we solve these issues and determine the distribution of the Berry curvature for CFs. First, we derive the definition of the Berry phase directly from the original definition of the Berry phase. It is found to be different from

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the prescribed one adopted by Geraedts *et al.* [18,19]. Then, we analytically show that the Berry curvature distribution is uniform in the momentum space for the standard CFL wave function. On the other hand, to compare with the results in Refs. [18,19], we also evaluate the Berry curvature of the JK wave function. With our definition, the numerical evaluation of the Berry phase becomes robust and free of the extraneous phases observed in Refs. [18,19]. It enables us to numerically determine the distribution of the Berry curvature in the whole momentum space. We find that the Berry curvature has a continuous distribution inside the Fermi sea and vanishes outside, and is different from the uniform distribution of the standard CFL wave function. We analytically show that the difference originates from the different quasiperiodicities of the two wave functions in the reciprocal space.

The reminder is organized as follows. In Sec. II, we derive the definition of the Berry phase of the CFL. In Sec. III C, we determine the Berry phase and Berry curvature of the standard CFL wave function analytically. In Sec. IV, we evaluate the Berry curvature of the JK wave function numerically. In Sec. V, we analyze the quasiperiodicities of the two wave functions in the reciprocal space, and determine the uniform background of the Berry curvature. In Sec. VI, we summarize and discuss our results.

## II. DEFINITION OF THE BERRY PHASE OF CFL

In this section, we derive the definition of the Berry phase for CFL systems. We first introduce the generic definition of the Berry phase. Next, we derive the definition of the Berry phase for CFL systems from the generic definition. Then, we discuss different representations of the Berry phases. Finally, we discuss and interpret Geraedts *et al.*'s definition and results.

### A. Berry phase

A quantum system acquires a geometric phase, i.e., the Berry phase, when it is adiabatically transported along a path  $C$  by varying parameters  $\alpha$  in its Hamiltonian  $\hat{H}(\alpha)$  [22]. The Berry phase is determined by a line integral in the parameter space

$$\gamma(C) = \int_C i \langle \Psi_\alpha | \nabla_\alpha \Psi_\alpha \rangle \cdot d\alpha, \quad (1)$$

where  $|\Psi_\alpha\rangle$  is the eigenstate of  $\hat{H}(\alpha)$ , and the integrand is called the Berry connection. The phase is independent of how the path is traversed as long as it is slow enough for the adiabaticity to hold. For a closed path, the phase is gauge invariant, i.e., independent of the choice of the phase factor of the wave function.

The Berry phase formula Eq. (1) can be recast into an alternative form as a time integral:

$$\gamma(C) = i \int_{t_0}^{t_1} \left\langle \Psi_{\alpha(t)} \left| \frac{d\Psi_{\alpha(t)}}{dt} \right. \right\rangle dt, \quad (2)$$

where the wave function evolves with time via the time dependence of its parameters, and  $\alpha(t)$  is an arbitrary time-dependent function that traverses the path  $C$  with  $t_0$  ( $t_1$ ) being the beginning (ending) time of the evolution. The integrand is

actually a part of the Schrödinger Lagrangian [13,23]:

$$L = \left\langle \Psi(t) \left| i\hbar \frac{d}{dt} - \hat{H} \right| \Psi(t) \right\rangle, \quad (3)$$

which governs the time evolution of a quantum system. This is why one sees the ubiquitous presence of the Berry phase in various contexts such as effective dynamics [13,23] and path-integral formalisms.

### B. Definition of the Berry phase of CFL

From the generic definition of the Berry phase, we can infer a definition of the Berry phase appropriate for CFL systems. For the purpose, it is more convenient to first consider a much simpler system, i.e., a set of non-interacting electrons residing in a Bloch band. The definition of the Berry phase for such a system is well known in the single-particle form [13]. Here, we will treat the system as a many-particle system and find a many-body generalization of the Berry phase definition. The generalization turns out to be general enough for applying to CFL systems.

The many-body wave function of a set of non-interacting Bloch electrons is a Slater determinant of Bloch states:

$$\Psi_{\mathbf{k}}(\mathbf{z}) = \det[\psi_{k_j}(\mathbf{z}_i)], \quad (4)$$

where  $\mathbf{k} \equiv \{\mathbf{k}_1, \mathbf{k}_2, \dots\}$  denotes the list of the quasi-wave-vectors of the Bloch states occupied by electrons, and  $\psi_{k_j}(\mathbf{z}_i) = \exp(i\mathbf{k}_j \cdot \mathbf{z}_i) u_{k_j}(\mathbf{z}_i)$  is the Bloch wave function with  $u_{k_j}(\mathbf{z}_i)$  being its periodic part.

The wave function has a number of general properties which are actually shared by the much more complicated CFL wave functions: (a) it is parametrized by a set of wave vectors  $\mathbf{k}$ ; (b) it is an eigenstate of the (magnetic) center-of-mass translation operator  $\hat{T}(\mathbf{a})$  such that  $\hat{T}(\mathbf{a})\Psi_{\mathbf{k}}(\mathbf{z}) = \exp(i\sum_i \mathbf{k}_i \cdot \mathbf{a})\Psi_{\mathbf{k}}(\mathbf{z})$ , where  $\mathbf{a}$  is one of the vectors of the Bravais lattice with respect to the periodicity of the system. As a result, two states with different total wave vectors are orthogonal to each other. With proper normalizations of wave functions, we have

$$\langle \Psi_{\mathbf{k}} | \Psi_{\mathbf{k}'} \rangle = \delta\left(\sum_i \mathbf{k}_i - \sum_i \mathbf{k}'_i\right) f(\mathbf{k}, \mathbf{k}'), \quad (5)$$

where  $f(\mathbf{k}, \mathbf{k}')$  is a function with the property  $f(\mathbf{k}, \mathbf{k}) = 1$ , and the wave vectors in the Dirac delta function are regarded as equal if they are only different by a reciprocal lattice vector; (c) the wave function has the Fermionic exchange symmetry and can be obtained from an unsymmetrized wave function  $\varphi_{\mathbf{k}}$  by applying the anti-symmetrization operator  $\hat{P}$ :

$$\hat{P} = \frac{1}{N!} \sum_P (-1)^P \hat{P}, \quad (6)$$

$$\Psi_{\mathbf{k}}(\mathbf{z}) = \hat{P} \varphi_{\mathbf{k}}(\mathbf{z}) \equiv \frac{1}{N!} \sum_P (-1)^P \varphi_{\mathbf{k}}(\hat{P}\mathbf{z}), \quad (7)$$

where  $\hat{P}\mathbf{z}$  denotes a permutation of electron coordinates,  $N$  is the total number of electrons, and  $\varphi_{\mathbf{k}}(\mathbf{z}) = \sqrt{N!} \prod_i \psi_{k_i}(\mathbf{z}_i)$  for the Bloch system. In the unsymmetrized form, an electron is associated with a particular wave vector. The association is lost in the antisymmetrized form.

With the wave function in hand, one may be tempted to directly apply Eq. (1) to determine the Berry phase. However, a difficulty immediately arises. To see that, we treat  $\mathbf{k}_1$  as the parameters  $\alpha$ , substitute Eq. (4) into Eq. (1), apply the identity  $\hat{\mathcal{P}}^2 = \hat{\mathcal{P}}$ , and obtain  $\mathbf{A}_{\mathbf{k}_1} \equiv i \langle \Psi_{\mathbf{k}} | \partial_{\mathbf{k}_1} \Psi_{\mathbf{k}} \rangle = - \langle \Psi_{\mathbf{k}} | \mathbf{r}_1 | \varphi_{\mathbf{k}} \rangle + i \langle \Psi_{\mathbf{k}} | e^{i\mathbf{k}_1 \cdot \mathbf{r}_1} | \partial_{\mathbf{k}_1} u_{\mathbf{k}_1}(\mathbf{r}_1) \prod_{i \geq 2} \psi_{\mathbf{k}_i}(\mathbf{r}_i) \rangle$ . Unfortunately, the resulting Berry connection  $\mathbf{A}_{\mathbf{k}_1}$  is not a legitimate one because  $\langle \Psi_{\mathbf{k}} | \mathbf{r}_1 | \varphi_{\mathbf{k}} \rangle$  does not have a deterministic value since the Bloch states have definite momenta and therefore infinite position uncertainty. This is the difficulty we have to address before the generic definition can be applied to wave functions like Eq. (4).

The most straightforward approach to address the issue is to introduce a unitary transformation to the Hamiltonian:  $\hat{H}(\mathbf{k}_1) = e^{-i\mathbf{k}_1 \cdot \mathbf{r}_1} \hat{H} e^{i\mathbf{k}_1 \cdot \mathbf{r}_1}$ . The resulting Hamiltonian acquires dependence on the parameters  $\mathbf{k}_1$ , and the corresponding eigenstate wave function becomes  $e^{-i\mathbf{k}_1 \cdot \mathbf{r}_1} \varphi_{\mathbf{k}}(\mathbf{r}) = u_{\mathbf{k}_1}(\mathbf{r}_1) \prod_{i \geq 2} \psi_{\mathbf{k}_i}(\mathbf{r}_i)$ . One can then apply Eq. (1) to obtain the well-known result  $\mathbf{A}_{\mathbf{k}_1} = i \langle u_{\mathbf{k}_1} | \partial_{\mathbf{k}_1} u_{\mathbf{k}_1} \rangle$ . Such an approach is adopted and generalized in Ref. [15] to show that the standard CFL wave function yields a uniform Berry curvature  $\Omega(\mathbf{k}_1) = 1/qB$ , where  $q$  is the unit charge of carriers and  $B$  is the perpendicular component of the external magnetic field. However, the approach is not compatible with the exchange symmetry because  $\hat{H}(\mathbf{k}_1)$  obviously breaks the symmetry of exchanging the first particle (the particle being transported) with others. Adopting such an approach means that we have to ignore the exchange symmetry. This is what we want to avoid here.

We therefore adopt and generalize the approach presented in Ref. [13]. The basic idea is that, since the difficulty is due to the fact that a Bloch state does not have a deterministic position expectation value, we replace it with a wave packet state which has a central wave vector  $\mathbf{k}_c$  and gives rise to a deterministic position expectation value  $\mathbf{z}_c$ :

$$|\tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c}\rangle = \int d\mathbf{k}_1 a(\mathbf{k}_1, t) |\Psi_{\mathbf{k}}\rangle, \quad (8)$$

where we assume that  $|a(\mathbf{k}_1, t)|^2$  is narrowly distributed around  $\mathbf{k}_c$  and satisfies

$$\int d\mathbf{k}_1 |a(\mathbf{k}_1, t)|^2 = \langle \tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c} | \tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c} \rangle = 1, \quad (9)$$

$$\int d\mathbf{k}_1 |a(\mathbf{k}_1, t)|^2 \mathbf{k}_1 = \mathbf{k}_c. \quad (10)$$

We choose the time dependence of  $a(\mathbf{k}_1, t)$  to make  $\mathbf{k}_c$  traverses a path  $C$  while keeping  $\mathbf{z}_c$  fixed. By applying Eq. (2), we can then determine the Berry phase acquired by the wave-packet state. In the end, the width of the distribution  $|a(\mathbf{k}_1, t)|^2$  will be set to zero so that the wave-packet state approaches to the Bloch state. We will show that it yields a well-defined limit.

We still need to define  $\mathbf{z}_c$ . It is easy to see that  $\langle \tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c} | \mathbf{z}_1 | \tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c} \rangle$  does not yield a deterministic expectation value. This is because  $\mathbf{z}_1$  loses its association with  $\mathbf{k}_1$  in the antisymmetrized wave function, and  $\langle \tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c} | \mathbf{z}_i | \tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c} \rangle = \langle \tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c} | \mathbf{z}_i | \tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c} \rangle$  is nothing but the center-of-mass position. Since electrons, all but one, have definite wave vectors in  $|\tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c}\rangle$ , the center-of-mass position has infinite uncertainty.

To obtain a deterministic position, we define  $\mathbf{z}_c$  as the position of the electron associated with the wave vector  $\mathbf{k}_1$  by

$$\mathbf{z}_c = \text{Re} \langle \tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c} | \mathbf{z}_1 | \tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c} \rangle, \quad (11)$$

where  $|\tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c}\rangle$  is the unsymmetrized form of  $|\tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c}\rangle$ , i.e.,  $|\tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c}\rangle \equiv \hat{\mathcal{P}} |\tilde{\Psi}_{\mathbf{k}_c, \mathbf{z}_c}\rangle$ .

We can show that  $\mathbf{z}_c$  does have a deterministic value. To see this, we substitute Eq. (8) into Eq. (11), and have

$$\begin{aligned} \mathbf{z}_c &= \text{Re} \int d\mathbf{k}_1 \int d\mathbf{k}'_1 a^*(\mathbf{k}'_1, t) a(\mathbf{k}_1, t) \\ &\quad \times [-i \partial_{\mathbf{k}'_1} \langle \Psi_{\mathbf{k}'} | \varphi_{\mathbf{k}} \rangle + i \langle \Psi_{\mathbf{k}'} | e^{i\mathbf{k}'_1 \cdot \mathbf{z}_1} | \partial_{\mathbf{k}'_1} u_{\mathbf{k}'} \rangle] \\ &= \text{Re} \int d\mathbf{k}_1 [a^*(\mathbf{k}_1, t) i \partial_{\mathbf{k}_1} a(\mathbf{k}_1, t) \\ &\quad + i |a(\mathbf{k}_1, t)|^2 \langle \Psi_{\mathbf{k}} | e^{i\mathbf{k}_1 \cdot \mathbf{z}_1} | \partial_{\mathbf{k}_1} u_{\mathbf{k}} \rangle], \end{aligned} \quad (12)$$

where we define  $u_{\mathbf{k}}(\mathbf{z}) \equiv e^{-i\mathbf{k}_1 \cdot \mathbf{z}_1} \varphi_{\mathbf{k}}(\mathbf{z})$ . To obtain the last expression, we make use of Eq. (5) which reduces to  $\langle \Psi_{\mathbf{k}'} | \varphi_{\mathbf{k}} \rangle = \langle \Psi_{\mathbf{k}'} | \Psi_{\mathbf{k}} \rangle = \delta(\mathbf{k}_1 - \mathbf{k}'_1)$  for the current case. Moreover, one can show that  $e^{i\mathbf{k}'_1 \cdot \mathbf{z}_1} | \partial_{\mathbf{k}'_1} u_{\mathbf{k}'} \rangle$  has the same center-of-mass periodicity as  $|\Psi_{\mathbf{k}}\rangle$ , thus  $\langle \Psi_{\mathbf{k}'} | e^{i\mathbf{k}'_1 \cdot \mathbf{z}_1} | \partial_{\mathbf{k}'_1} u_{\mathbf{k}'} \rangle \propto \delta(\mathbf{k}_1 - \mathbf{k}'_1)$ .

Writing the amplitude  $a(\mathbf{k}_1, t)$  as the form  $a(\mathbf{k}_1, t) = |a(\mathbf{k}_1, t)| e^{-i\gamma(\mathbf{k}_1, t)}$ , and setting the width of the distribution  $|a(\mathbf{k}_1, t)|^2$  to zero, we obtain

$$\mathbf{z}_c = \frac{\partial \gamma(\tilde{\mathbf{k}}, t)}{\partial \mathbf{k}_c} - \text{Im} \langle \Psi_{\tilde{\mathbf{k}}} | e^{i\mathbf{k}_c \cdot \mathbf{z}_1} | \partial_{\mathbf{k}_c} u_{\tilde{\mathbf{k}}} \rangle \quad (13)$$

with  $\tilde{\mathbf{k}} \equiv \{\mathbf{k}_c, \mathbf{k}_2, \dots\}$ . Equation (13) is the many-body generalization of Eq. (2.8) of Ref. [13].

We then apply Eq. (2) to determine the Berry phase of transporting  $\mathbf{k}_c$ . Following a procedure similar to that for determining  $\mathbf{z}_c$ , we obtain

$$i \left\langle \tilde{\Psi}_{\mathbf{k}_c(t), \mathbf{z}_c} \left| \frac{d\tilde{\Psi}_{\mathbf{k}_c(t), \mathbf{z}_c}}{dt} \right. \right\rangle = \frac{\partial \gamma(\tilde{\mathbf{k}}, t)}{\partial t}. \quad (14)$$

Using the relation

$$\frac{\partial \gamma(\tilde{\mathbf{k}}, t)}{\partial t} = \frac{d\gamma(\tilde{\mathbf{k}}, t)}{dt} - \dot{\mathbf{k}}_c \cdot \frac{\partial \gamma(\tilde{\mathbf{k}}, t)}{\partial \mathbf{k}_c} \quad (15)$$

and Eq. (13), we determine the phase acquired by transporting  $\mathbf{k}_c$  along  $C$ :

$$\begin{aligned} \gamma(C) &= \int_{t_0}^{t_1} dt \left[ \frac{d\gamma(\tilde{\mathbf{k}}, t)}{dt} - \dot{\mathbf{k}}_c \cdot \mathbf{z}_c \right] \\ &\quad - \int_C \text{Im} \langle \Psi_{\tilde{\mathbf{k}}} | e^{i\mathbf{k}_c \cdot \mathbf{z}_1} | \partial_{\mathbf{k}_c} u_{\tilde{\mathbf{k}}} \rangle \cdot d\mathbf{k}_c. \end{aligned} \quad (16)$$

The first term is integrable (provided  $\mathbf{z}_c$  is fixed) and vanishes when  $C$  is a closed path. We thus exclude it from the definition of the Berry phase, and define the Berry connection as

$$\mathbf{A}_{\mathbf{k}_1} = -\text{Im} \langle \Psi_{\tilde{\mathbf{k}}} | e^{i\mathbf{k}_1 \cdot \mathbf{z}_1} | \partial_{\mathbf{k}_1} u_{\tilde{\mathbf{k}}} \rangle, \quad (17)$$

where we relabel  $\mathbf{k}_c$  as  $\mathbf{k}_1$ . The Berry phase is just a line integral of the Berry connection.

For numerical calculations, it is more convenient to calculate the Berry phase induced by a small but discrete change of the wave vector. By using the trapezoidal rule to estimate the

line integral and approximating  $\partial_{k_i} u_k$  as a first-order divided difference, we determine the Berry phase for  $\mathbf{k}_1 \rightarrow \mathbf{k}'_1$ :

$$\phi_{\mathbf{k}'\mathbf{k}} = \frac{1}{2}[\arg\langle \varphi_{\mathbf{k}'} | e^{iq \cdot \mathbf{z}_1} | \Psi_{\mathbf{k}} \rangle - (\mathbf{k} \Rightarrow \mathbf{k}')], \quad (18)$$

with  $\mathbf{q} \equiv \mathbf{k}'_1 - \mathbf{k}_1$ .

Equations (17) and (18) are the definitions of the Berry connection and Berry phase for a many-body system, respectively. The definitions are directly descended from the original definition Eq. (1). In our derivation, we only make use of the aforementioned three properties of the wave function. As we will show later, CFL wave functions considered in this paper do have these properties. Therefore, the definitions are also applicable for CFL systems.

### C. Representations of the Berry phase

Motivated by the dipole picture of CFs [17], we also define another Berry phase (connection). According to the dipole picture, a CF is a bound state of an electron and quantum vortices, and the position of the quantum vortices  $\mathbf{z}_c^v$  is displaced from that of the electron by [15,16]

$$\mathbf{z}_c^v = \mathbf{z}_c^e + \mathbf{n} \times \mathbf{k}_c, \quad (19)$$

where  $\mathbf{z}_c^e \equiv \mathbf{z}_c$  is the position of the electron, and  $\mathbf{n}$  denotes the unit normal vector of the system plane. Obviously, the phase determined by Eq. (16) depends on which position is fixed when  $\mathbf{k}_c$  is transported. The Berry connection Eq. (17) and the Berry phase Eq. (18) are defined by assuming that  $\mathbf{z}_c^e$  is fixed. Hereafter, we will label  $A_{\mathbf{k}_1}$  ( $\phi_{\mathbf{k}'\mathbf{k}}$ ) as  $A_{\mathbf{k}_1}^e$  ( $\phi_{\mathbf{k}'\mathbf{k}}^e$ ) to explicitly indicate the assumption. On the other hand, if we assume that  $\mathbf{z}_c^v$  is fixed, we should replace  $\mathbf{z}_c$  in Eq. (16) with  $\mathbf{z}_c^v - \mathbf{n} \times \mathbf{k}_c$ , and obtain another Berry connection  $A_{\mathbf{k}_1}^v$ :

$$A_{\mathbf{k}_1}^v = A_{\mathbf{k}_1}^e - \mathbf{k}_1 \times \mathbf{n}. \quad (20)$$

The corresponding Berry phase for  $\mathbf{k}_1 \rightarrow \mathbf{k}'_1$  is

$$\phi_{\mathbf{k}'\mathbf{k}}^v = \phi_{\mathbf{k}'\mathbf{k}}^e + (\mathbf{k}_1 \times \mathbf{k}'_1) \cdot \mathbf{n}. \quad (21)$$

Then, which one is the Berry connection (phase) of the CF? The answer depends on how we define the position of a CF. By definition, the  $\mathbf{k}$ -space Berry connection is the connection of transporting  $\mathbf{k}_1$  while keeping the position fixed. If one chooses to define the CF position as the position of the quantum vortices (electron), then  $A_{\mathbf{k}_1}^v$  ( $A_{\mathbf{k}_1}^e$ ) should be regarded as the CF Berry connection. One can even adopt other definitions of the CF position, and obtain other definitions of the Berry connections. Mathematically, all these definitions are equivalent. They are just different representations of the same physics.

Nevertheless, for a reason to be discussed in the next subsection, we will refer to the  $v$ -representation as the CF representation, and interpret  $\phi_{\mathbf{k}'\mathbf{k}}^v$  and  $A_{\mathbf{k}_1}^v$  as the Berry phase and Berry connection of CFs.

### D. Interpretation of Geraedts *et al.*'s result

Finally, we would like to comment on the definition of the Berry phase introduced by Geraedts *et al.* in Refs. [18,19]. It is obviously not a definition descended from the original definition of the Berry phase. It should be more appropriately called a scattering phase. Indeed, the transition amplitude for

$\mathbf{k} \rightarrow \mathbf{k}'$  induced by a single-body scalar potential  $V(\mathbf{z}_i) = e^{iq \cdot \mathbf{z}_i}$  is proportional to

$$U_{\mathbf{k}'\mathbf{k}} = \langle \Psi_{\mathbf{k}'} | \sum_i e^{iq \cdot \mathbf{z}_i} | \Psi_{\mathbf{k}} \rangle \quad (22)$$

with  $\mathbf{k}'_1 = \mathbf{k}_1 + \mathbf{q}$ . It is exactly the matrix element calculated by Geraedts *et al.* Their results thus provide a ‘‘first-principles’’ determination of the scattering amplitude.

We can actually interpret Geraedts *et al.*'s results in light of the picture of independent CFs. They observe that the matrix element has an  $i$  ( $-i$ ) factor for a discrete wave-vector change in the clockwise (anti-clockwise) sense.<sup>1</sup> It means that the matrix element has a form like

$$U_{\mathbf{k}'\mathbf{k}} \propto \mathbf{in} \cdot (\mathbf{k}'_1 \times \mathbf{k}_1) e^{i\Phi_{\mathbf{k}'\mathbf{k}}}, \quad (23)$$

and  $\Phi_{\mathbf{k}'\mathbf{k}}$  is interpreted as the Berry phase. For a CF system, the potential will induce a density modulation, which in turn induces a modulation of the effective magnetic field. The scattering amplitude induced by the modulated effective magnetic field does have a factor  $\propto \mathbf{in} \cdot (\mathbf{k}'_1 \times \mathbf{k}_1)$ , as shown in Eq. (23) of Ref. [24]. Differently from what is assumed in Ref. [24], the scattering amplitude carries an extra phase  $\Phi_{\mathbf{k}'\mathbf{k}}$ , indicating the presence of a Berry phase when the transition  $\mathbf{k} \rightarrow \mathbf{k}'$  occurs.

It is then interesting to ask how the Berry phase inferred from the scattering amplitude is related to the Berry phases we have defined. Our numerical calculation (See Sec. IV C) suggests that

$$\Phi_{\mathbf{k}'\mathbf{k}} \approx \phi_{\mathbf{k}'\mathbf{k}}^v, \quad (24)$$

i.e., the Berry phase inferred from the scattering amplitude is actually the Berry phase defined in the  $v$ -representation. For the dipole picture it means that, as far as the impurity scattering is concerned, the position of a CF should be defined as the position of its quantum vortices.

On the other hand, one expects that CFs are scattered not only by the fluctuation of the effective magnetic field, but also by the scalar potential itself. As a result, the scattering amplitude should in general have the form [24]

$$U_{\mathbf{k}'\mathbf{k}} = [V_1(\mathbf{k}', \mathbf{k}) + \mathbf{in} \cdot (\mathbf{k}'_1 \times \mathbf{k}_1) V_2(\mathbf{k}', \mathbf{k})] e^{i\Phi_{\mathbf{k}'\mathbf{k}}}. \quad (25)$$

The presence of the first term will interfere the determination of  $\Phi_{\mathbf{k}'\mathbf{k}}$ . Indeed, we observe that the Berry phase inferred from the scattering amplitude by assuming a vanishing  $V_1(\mathbf{k}', \mathbf{k})$  always deviates from  $\pi$ , and the deviation becomes more pronounced when  $N$  is scaled up (see Table II). The approach becomes unreliable when the two terms in the prefactor of Eq. (25) are comparable in magnitude. In this case, the phase carried by the prefactor cannot be easily distinguished from the Berry phase to be determined. One encounters such a situation when studying the effect of the Landau level mixing [25]. In the study, Pu *et al.* adopt the wave function

$$\Psi_{\mathbf{k}}^{\text{mix}} = (1 - \beta)\Psi_{\mathbf{k}} + \beta\Psi_{\mathbf{k}}^{\text{unproj}}, \quad (26)$$

<sup>1</sup>The correspondence is determined from our own numerical evaluation of the matrix element.



TABLE I. The phases for  $N = 13$ . For Geraedts *et al.*'s definition (old), both the results presented in Ref. [19] and our own calculation are shown. The results with respect to our definition (new) are shown in the last row. The numerical uncertainties for paths a and b are not shown because they are too small.

Path		(a)	(b)	(c)
old	Ref. [19]	0.813(7)	0.965(6)	-0.058(6)
	This work	0.821(1)	0.964(2)	-0.050(1)
new		1.110	0.906	0.068(1)

where  $\Psi_k$  is a CFL wave function projected to the LLL [either Eq. (27) or Eq. (47)],  $\Psi_k^{\text{unproj.}}$  is the unprojected form of  $\Psi_k$  [i.e., Eq. (27) without applying  $\hat{P}_{\text{LLL}}$ ], and  $\beta$  controls the strength of the Landau level mixing. One expects that  $V_1$  dominates in the scattering amplitude of the unprojected wave function  $\Psi_k^{\text{unproj.}}$ , and  $V_1 > 0$  ( $V_1 < 0$ ) for particles (holes). From Eq. (25), the prefactor contributes a phase 0 ( $\pi$ ) for each step of transporting a particle (hole) [25]. On the other hand, for the projected wave function  $\Psi_k$ ,  $V_2$  dominates, and the prefactor contributes a phase  $\pm\pi/2$ . In between, for a mixed wave function, the phase depends on the relative strength of  $V_1$  and  $V_2$ . Without a reliable way of subtracting the phase from the scattering amplitude, the determined ‘‘Berry phase’’ may fluctuate widely. This is indeed observed in Ref. [25]. In contrast, our definition Eq. (18) does not suffer from the difficulty. It is easy to see that the Berry phase of  $\Psi_k^{\text{unproj.}}$  is always zero. For the mixed wave function, it is reasonable to expect that the Berry phase accumulated by transporting a CF around the Fermi circle is a value between zero and that yielded by  $\Psi_k$ , i.e.,  $\pi$ .

### III. BERRY PHASE OF THE STANDARD CFL WAVE FUNCTION

In this section, we evaluate the Berry phase of the standard CFL wave function. First, we introduce the explicit form of

the CFL wave function on the torus geometry. Next, we show its center-of-mass translational symmetry under the magnetic translation operator. Then, we analytically determine the Berry curvature of the standard CFL wave function.

To unify notations, we use the symbols  $a_i \equiv a_{ix} + ia_{iy}$ ,  $a_i^* \equiv a_{ix} - ia_{iy}$  and  $\mathbf{a}_i \equiv (a_{ix}, a_{iy})$  to denote an electron-related variable in its complex form, complex conjugate form, and vector form, respectively, with the subscript  $i$  indexing electrons. Symbols without a subscript (e.g.,  $a \equiv \{a_i\}$ ) denote a list of the variables for all electrons, and symbols in the upper case (e.g.,  $A \equiv \sum_i a_i$ ) denote sums of the variables over all electrons.  $a \cdot b \equiv \sum_i a_i b_i$  denotes the inner product of two lists of variables. The unit of length is set to the magnetic length  $l_B \equiv \sqrt{\hbar/e|B|}$ .

#### A. CFL wave function

The standard CFL wave function for a system on a torus with a filling fraction  $\nu = 1/m$  ( $m$  is an even integer) can be written as (omitting the Gaussian factor  $e^{-\sum_i |z_i|^2/4}$ ) [26]

$$\Psi_k^{\text{CF}}(z) = \hat{P}_{\text{LLL}} \det[e^{i(k_i z_j^* + k_j^* z_i)/2}] J(z), \quad (27)$$

$$J(z) = \tilde{\sigma}^m(Z) \prod_{i < j} \tilde{\sigma}^m(z_i - z_j), \quad (28)$$

TABLE II. CF Berry phases  $\phi^\nu$  and minimal overlaps  $|D|_{\text{min}}$  along different paths for the JK wave function. The paths are indicated by arrowed solid lines comprised of steps with minimal changes of the quantized wave vectors. Three kinds of paths are considered: (a) the Fermi circle; (b) a unit plaquette inside (b1) or outside (b2) the Fermi sea; (c) a closed path inside the Fermi sea. Both results for our definition (new) and Geraedts *et al.*'s definition (old) are shown.  $|D|_{\text{min}}$  is the minimum overlap among steps along a path. For the paths inside the Fermi sea, a hole is transported, and resulting Berry phases are shown with inverted signs. The values marked with \* have been scaled by a factor of  $N$ . U.D. indicates an undeterminable result due to a vanishing overlap.

Path		(a)	(b)	(c)			
$N$		13	38	110	36(b1)	38(b2)	36
$\phi^\nu/\pi$	old	0.82	0.72	0.57	U.D.	U.D.	0.93
	new	1.11	1.03	1.01	0.75*	0.05*	0.61
$ D _{\text{min}}$	old	0.65*	0.36*	0.18*	0.04*	0.01*	0.22*
	new	0.94	0.98	0.99	0.99	0.99	0.99

where  $\hat{P}_{\text{LLL}}$  denotes the projection to the LLL, which is effectively to replace  $z_i^*$  with an operator  $2\partial_{z_i}$  acting on all  $z_i$ 's [5], and  $J(z)$  is the Bijl-Jastrow factor [6] expressed in terms of the modified sigma function  $\tilde{\sigma}$  which has the quasiperiodicity [27]

$$\tilde{\sigma}(z_i + L) = \xi(L)e^{\frac{\pi L^*}{A}(z_i + \frac{1}{2}L)}\tilde{\sigma}(z_i), \quad (29)$$

where  $L$  is a period of the torus, and  $\xi(L) = 1$  if  $L/2$  is also a period and  $-1$  otherwise,  $A = 2\pi N_\phi \equiv 2\pi mN$  is the total area of the torus, and  $N_\phi$  is the total number of the magnetic fluxes passing through the torus. The wave function is parametrized in a set of wave vectors  $\mathbf{k}$  which are quantized as usual, i.e.,  $\mathbf{k}_i \in \{n_1\mathbf{b}_1 + n_2\mathbf{b}_2, n_1, n_2 \in Z\}$  with  $\mathbf{b}_\alpha = (2\pi/A)\mathbf{n} \times \mathbf{L}_\alpha$ ,  $\alpha = 1, 2$ , where  $\mathbf{L}_1$  and  $\mathbf{L}_2$  denote the two edges defining the torus. In the complex form, we have  $k_i \in \{(n_1L_1 + n_2L_2)/N_\phi, n_1, n_2 \in Z\}$ .

After applying the projection and expanding the determinant, we write the wave function explicitly as  $\Psi_{\mathbf{k}}^{\text{CF}}(z) = \hat{P}\varphi_{\mathbf{k}}^{\text{CF}}(z)$ , and

$$\begin{aligned} \varphi_{\mathbf{k}}^{\text{CF}}(z) &= N!e^{ik^* \cdot z/2}\tilde{\sigma}^m(Z + iK) \\ &\times \prod_{i < j} \tilde{\sigma}^m(z_i + ik_i - z_j - ik_j). \end{aligned} \quad (30)$$

$\varphi_{\mathbf{k}}^{\text{CF}}$  is the unsymmetrized form of  $\Psi_{\mathbf{k}}^{\text{CF}}$ .

### B. Translational symmetry

The wave function is an eigenstate of the magnetic center-of-mass translation operator defined by [28]

$$\begin{aligned} \hat{T}(\mathbf{a}) &= \prod_i e^{\mathbf{a} \cdot \partial_{z_i} + \frac{1}{2}i(\mathbf{n} \times z_i) \cdot \mathbf{a}} \\ &= e^{\frac{1}{4}(aZ^* - a^*Z)} \prod_i e^{\mathbf{a} \cdot \partial_{z_i}} \end{aligned} \quad (31)$$

and  $\mathbf{a} \in \{n\mathbf{L}/N, n \in Z\}$ , where  $\mathbf{L}$  is a period of the torus. To show that, we apply  $\hat{T}(\mathbf{a})$  to the wave function (30), and have

$$\begin{aligned} \hat{T}(\mathbf{a})\varphi_{\mathbf{k}}^{\text{CF}}(z)e^{-\frac{\sum_i |z_i|^2}{4}} &= e^{\frac{1}{2}i\mathbf{k}^* \cdot \mathbf{a} - \frac{a^*}{2}(Z + \frac{nL}{2})} e^{\frac{1}{2}i\mathbf{k}^* \cdot z} \tilde{\sigma}^m(Z + iK + nL) \\ &\times e^{-\frac{\sum_i |z_i|^2}{4}} \prod_{i < j} \tilde{\sigma}^m(z_i + ik_i - z_j - ik_j), \end{aligned} \quad (32)$$

where we assume  $\mathbf{a} = n\mathbf{L}/N$ ,  $n \in Z$ . Using Eq. (29), we have

$$\tilde{\sigma}^m(Z + iK + nL) = e^{\frac{1}{2}a^*(Z + iK + \frac{nL}{2})}\tilde{\sigma}^m(Z + iK). \quad (33)$$

Substituting the relation into Eq. (32), and noting that  $\hat{T}(\mathbf{a})$  commutes with  $\hat{P}$ , we obtain

$$\hat{T}(\mathbf{a})\Psi_{\mathbf{k}}^{\text{CF}}(z)e^{-\frac{\sum_i |z_i|^2}{4}} = e^{i\mathbf{k} \cdot \mathbf{a}}\Psi_{\mathbf{k}}^{\text{CF}}(z)e^{-\frac{\sum_i |z_i|^2}{4}}. \quad (34)$$

We can also show that  $\tilde{\varphi}_{\mathbf{k}}(z) \equiv e^{ik_1 \cdot z_1} \partial_{k_1} u_{\mathbf{k}}(z)e^{-\frac{\sum_i |z_i|^2}{4}}$  with  $u_{\mathbf{k}}(z) \equiv e^{-ik_1 \cdot z_1} \varphi_{\mathbf{k}}^{\text{CF}}(z)$  is an eigenstate of  $\hat{T}(\mathbf{a})$  of the same eigenvalue. It is easy to obtain

$$\hat{T}(\mathbf{a})u_{\mathbf{k}}(z)e^{-\frac{\sum_i |z_i|^2}{4}} = e^{i\sum_{i \geq 2} k_i \cdot \mathbf{a}} u_{\mathbf{k}}(z)e^{-\frac{\sum_i |z_i|^2}{4}}. \quad (35)$$

Using the relation, we obtain

$$\hat{T}(\mathbf{a})\tilde{\varphi}_{\mathbf{k}}(z) = e^{ik_1 \cdot (z_1 + \mathbf{a})} \partial_{k_1} \hat{T}(\mathbf{a})u_{\mathbf{k}}(z)e^{-\frac{\sum_i |z_i|^2}{4}} = e^{i\mathbf{k} \cdot \mathbf{a}} \tilde{\varphi}_{\mathbf{k}}(z). \quad (36)$$

It is indeed an eigenstate of  $\hat{T}(\mathbf{a})$  with an eigenvalue  $e^{i\mathbf{k} \cdot \mathbf{a}}$ .

### C. Berry phase and Berry curvature

It is obvious from the above discussions that the standard CFL wave function has all the properties outlined in Sec. II B. Therefore, the definition of the Berry phase Eq. (18) can be applied. It turns out that the standard CFL wave function has a simple structure which makes possible an analytic determination of the Berry phase.

To determine the Berry phase, we need to determine the matrix element

$$\begin{aligned} \langle \Psi_{\mathbf{k}'}^{\text{CF}} | e^{-i\mathbf{q} \cdot z_1} | \varphi_{\mathbf{k}'}^{\text{CF}} \rangle &= \langle \Psi_{\mathbf{k}'}^{\text{CF}} | \hat{P}_{\text{LLL}} e^{-i\mathbf{q} \cdot z_1} | \varphi_{\mathbf{k}'}^{\text{CF}} \rangle \\ &= e^{-\frac{|\mathbf{q}|^2}{4}} \langle \Psi_{\mathbf{k}'}^{\text{CF}} | \hat{t}_1(-\mathbf{q}) | \varphi_{\mathbf{k}'}^{\text{CF}} \rangle, \end{aligned} \quad (37)$$

where  $\mathbf{k}' \equiv \{\mathbf{k}_1 + \mathbf{q}, \mathbf{k}_2, \dots\}$ , and we define the operator

$$\hat{t}_i(\mathbf{k}_\alpha) \equiv \exp\left(ik_\alpha \partial_{z_i} + \frac{i}{2}k_\alpha^* z_i\right). \quad (38)$$

It is easy to verify the relation

$$\hat{t}_i(\mathbf{k}_\alpha)\hat{t}_i(\mathbf{k}_\beta) = e^{\frac{1}{2}i(\mathbf{k}_\alpha \times \mathbf{k}_\beta) \cdot \mathbf{n}} \hat{t}_i(\mathbf{k}_\alpha + \mathbf{k}_\beta). \quad (39)$$

On the other hand, apart from an unimportant coefficient, the unsymmetrized wave function  $\varphi_{\mathbf{k}}^{\text{CF}}$  can be written as

$$\varphi_{\mathbf{k}}^{\text{CF}}(z) = \hat{t}_1(\mathbf{k}_1 + \mathbf{q}) \prod_{i \geq 2} \hat{t}_i(\mathbf{k}_i)J(z). \quad (40)$$

Using Eq. (39), we have

$$\begin{aligned} \hat{t}_1(-\mathbf{q})\varphi_{\mathbf{k}'}^{\text{CF}}(z) &= e^{-\frac{i}{2}(\mathbf{q} \times \mathbf{k}_1) \cdot \mathbf{n}} \prod_i \hat{t}_i(\mathbf{k}_i)J(z) \\ &\equiv e^{-\frac{i}{2}(\mathbf{q} \times \mathbf{k}_1) \cdot \mathbf{n}} \varphi_{\mathbf{k}}^{\text{CF}}(z). \end{aligned} \quad (41)$$

Inserting the relation into Eq. (37), we obtain

$$\langle \Psi_{\mathbf{k}'}^{\text{CF}} | e^{-i\mathbf{q} \cdot z_1} | \varphi_{\mathbf{k}'}^{\text{CF}} \rangle = e^{\frac{1}{2}i(\mathbf{k}_1 \times \mathbf{q}) \cdot \mathbf{n} - \frac{1}{4}|\mathbf{q}|^2} \langle \Psi_{\mathbf{k}}^{\text{CF}} | \Psi_{\mathbf{k}}^{\text{CF}} \rangle. \quad (42)$$

Applying Eq. (18), we determine the Berry phase in the e-representation:

$$\phi_{\mathbf{k}'\mathbf{k}}^{\text{e}} = \frac{1}{2}(\mathbf{q} \times \mathbf{k}_1) \cdot \mathbf{n}. \quad (43)$$

The Berry phase in the CF (v)-representation can be determined by applying Eq. (21):

$$\phi_{\mathbf{k}'\mathbf{k}}^{\text{v}} = -\frac{1}{2}(\mathbf{q} \times \mathbf{k}_1) \cdot \mathbf{n}. \quad (44)$$

The Berry connections corresponding to the Berry phases are

$$\mathbf{A}_{\mathbf{k}_1}^{\text{e/v}} = \pm(\mathbf{k}_1 \times \mathbf{n})/2. \quad (45)$$

They give rise to the Berry curvatures (in the unit of  $1/qB$ )

$$\Omega_{\mathbf{k}_1}^{\text{e/v}} \equiv (\nabla_{\mathbf{k}_1} \times \mathbf{A}_{\mathbf{k}_1}^{\text{e/v}}) \cdot \mathbf{n} = \mp 1. \quad (46)$$

It indicates that the Berry curvature in the momentum space is a constant, exactly the one suggested in the uniform-Berry-curvature picture of CFs [14–16].

#### IV. BERRY PHASE OF THE JK WAVE FUNCTION

In this section, we evaluate the Berry phase and Berry curvature of the JK wave function of the CFL. First, we introduce the JK wave function on the torus geometry. Next, we describe the numerical implementation of the calculations of the Berry phase. Then, we numerically evaluate the Berry curvature and determine its distribution in the whole wave-vector space.

##### A. JK Wave function

It is numerically hard to implement the LLL projection in Eq. (27) since it expands the wave function to  $N!$  terms. To address the issue, Jain and Kamilla introduce an alternative projection method [21]. The projection method is adopted by Refs. [18–20] for numerically evaluating the Berry phase. The wave function (JK wave function) has the form [26]

$$\Psi_k^{\text{JK}}(z) = \det[\psi_i(\mathbf{k}_j)] \tilde{\sigma}^m(Z + iK) \prod_{i < j} \tilde{\sigma}^{m-2}(z_i - z_j), \quad (47)$$

$$\psi_i(\mathbf{k}_j) = e^{ik_j^* z_i / 2} \prod_{k \neq i} \tilde{\sigma}(z_i - z_k + imk_j - im\bar{k}), \quad (48)$$

with  $\bar{k} \equiv K/N$ . Its unsymmetrized form is

$$\varphi_k^{\text{JK}}(z) = \tilde{\sigma}^m(Z + iK) \prod_{i < j} \tilde{\sigma}^{m-2}(z_i - z_j) \prod_i \psi_i(\mathbf{k}_i). \quad (49)$$

The quantization of  $\mathbf{k}$  is the same as that in Eq. (27). To evaluate the wave function, one only needs to calculate a determinant. As a result, the computational complexity is greatly reduced compared to the standard CFL wave function. This enables us to deal with large systems numerically.

It is easy to see that the JK wave function has the same center-of-mass translational property as that of the standard CFL wave function (see Sec. III B). Therefore, the Berry phase definition Eq. (18) can be applied.

It is argued that the two wave functions are equivalent physically [5,21]. It turns out that this may not be true for evaluating the Berry phase, as we will show next.

##### B. Numerical implementation

For both Geraedts *et al.*'s definition and our definition, the calculation of the (Berry) phase involves the evaluation of a matrix element  $\langle \Psi_k | \tilde{\Psi}_{k'} \rangle$ . For Geraedts *et al.*'s definition,  $\tilde{\Psi}_{k'}$  is defined by

$$\tilde{\Psi}_{k'} = e^{-iq_1 \cdot z_1} \Psi_{k'}, \quad (50)$$

where we drop the superscript JK for brevity. We note that Geraedts *et al.*'s original definition uses the factor  $\rho_q = \sum_i e^{-iq \cdot z_i}$ . The two forms are equivalent except for an unimportant factor.

For our definition, after inserting  $\hat{P}$  into the matrix element of Eq. (18), we can write  $\tilde{\Psi}_{k'}$  as a form like Eq. (47) but with

the determinant modified to

$$\begin{vmatrix} e^{-iq_1 \cdot z_1} \psi_1(\mathbf{k}_1 + \mathbf{q}_1) & \psi_1(\mathbf{k}_2) & \dots & \psi_1(\mathbf{k}_N) \\ e^{-iq_1 \cdot z_2} \psi_2(\mathbf{k}_1 + \mathbf{q}_1) & \psi_2(\mathbf{k}_2) & \dots & \psi_2(\mathbf{k}_N) \\ \vdots & \vdots & \ddots & \vdots \\ e^{-iq_1 \cdot z_N} \psi_N(\mathbf{k}_1 + \mathbf{q}_1) & \psi_N(\mathbf{k}_2) & \dots & \psi_N(\mathbf{k}_N) \end{vmatrix}, \quad (51)$$

i.e., the column with respect to the transported wave-vector ( $\mathbf{k}_1$ ) is modified by the ‘‘momentum boost operator’’  $e^{-iq_1 \cdot z_i}$ .

We implement a Metropolis Monte Carlo algorithm similar to that detailed in Ref. [19]. Specifically, the overlap  $|D|$  and phase  $\phi$  of the matrix elements are evaluated by

$$\begin{aligned} |D| e^{i\phi} &\equiv \frac{\langle \Psi_k | \tilde{\Psi}_{k'} \rangle}{\sqrt{\langle \Psi_k | \Psi_k \rangle \langle \tilde{\Psi}_{k'} | \tilde{\Psi}_{k'} \rangle}} \\ &= \frac{\frac{1}{N} \sum' |\Psi_k|^2 \frac{\tilde{\Psi}_{k'}}{\Psi_k}}{\sqrt{\frac{1}{N} \sum' |\Psi_k|^2 \cdot \left| \frac{\tilde{\Psi}_{k'}}{\Psi_k} \right|^2}}, \end{aligned} \quad (52)$$

where  $N$  denotes the normalization factor of  $|\Psi_k|^2$ , and  $\sum'$  stands for lattice summation of  $z_i$ 's [14,19]. The Markov chains of our Monte Carlo simulation sample  $z_i$ 's by assuming a probability density  $\propto |\Psi_k(z)|^2$ . By using a Markov chain, we can determine the phases and overlaps with respect to both the definitions simultaneously, since the two definitions only differ by  $\tilde{\Psi}_{k'}$ .

##### C. Numerical results

To test our numerical implementation, we first calculate the Berry phases along the paths calculated in Ref. [19]. The numerical results are presented in Table I. For Geraedts *et al.*'s definition, the results presented in Ref. [19] and our own calculation results coincide well within numerical uncertainties. The results with respect to our definition are also shown.

Berry phases with respect to both definitions for a few representative paths are shown in Table II. An immediate observation is that the calculation with our definition is much more robust numerically, as is evident from the magnitudes of the overlap. With our definition, the overlap is always close to 1 and improves when  $N$  is scaled up. For Geraedts *et al.*'s definition, the overlap is nowhere close to 1 and further deteriorates for larger  $N$ , and even nearly vanishes for steps along directions perpendicular to the Fermi circle, resulting in poor statistics and undeterminable results. Moreover, our definition yields directly interpretable results, i.e., no subtraction of the extraneous  $\pm\pi/2$  phases noted in Ref. [18] is needed.

It is interesting to observe that the two different definitions actually lead to similar qualitative conclusions as long as  $\phi^v$  is interpreted as the CF Berry phase. With our definition, the Berry phase of adiabatic transport of a CF around the Fermi circle is converged to  $\pi$  (path a,  $N = 110$ ), whereas, with Geraedts *et al.*'s definition, it involves guesswork to reach the same conclusion. We also find that the Berry phase for transport around a unit plaquette outside the Fermi sea (path b2) nearly vanishes. This is consistent with Geraedts *et al.*'s observation that the phase is independent of the area of the

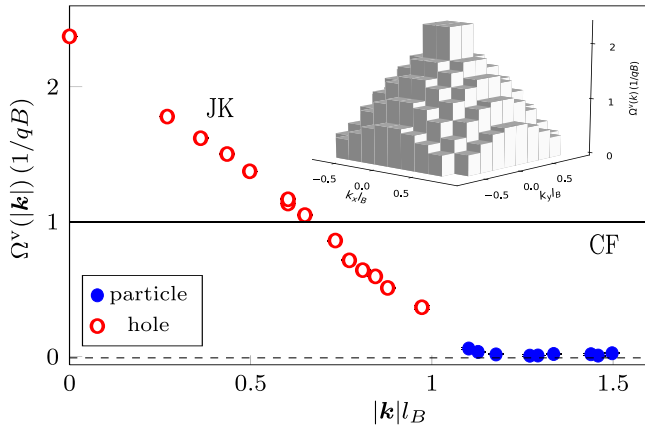


FIG. 1. The Berry curvature  $\Omega^\nu(|\mathbf{k}|)$  as a function of the CF wave number  $|\mathbf{k}|$  for the half-filled CFL ( $m = 2$ ). The Berry curvature for the JK wave function is numerically determined by transporting a CF (hole) outside (inside) the Fermi sea consist of 109 CFs, shown as filled (empty) dots. The inset bar plot shows its distribution on the 2D plane of the momentum space. The Berry curvature for the standard CFL wave function is equal to 1, shown as the solid line.

trajectory enclosing the Fermi sea. The consistencies may not be a coincidence. See Sec. IID for an interpretation.

The distribution of the Berry curvature, both inside and outside the Fermi sea, can now be determined because of the improved numerical robustness. To determine the Berry curvature, we transport a CF or a hole along the edges of a unit plaquette (see path b in Table II), and the Berry curvature for the plaquette is determined by  $\Omega^\nu = \phi_B^\nu/S_0$ , where  $S_0 = 2\pi/N_\phi$  is the area of the unit plaquette. The result is shown in Fig. 1. We see that the Berry curvature has a continuous distribution inside the Fermi sea and vanishes outside. The distribution is obviously *not* the singular one implied by the Dirac interpretation [20,29].

## V. UNIFORM BACKGROUND OF THE BERRY CURVATURE

It is evident that, for both the wave functions, the Berry curvature is a constant in most of the region of the  $\mathbf{k}$ -space except the one occupied by the Fermi sea. In other words, the Berry curvature has a uniform background. The values of the background for the two wave functions are different: 1 for the standard CFL wave function and 0 for the JK wave function, respectively. In this section, we show that the values can be determined analytically by inspecting the quasiperiodicities of the wave functions in the  $\mathbf{k}$ -space [30]. It turns out that the two wave functions have different quasiperiodicities, resulting in the different values of the uniform background.

We first examine the standard CFL wave function (30). By using Eq. (29), it is easy to show that it has the quasiperiodicity in the  $\mathbf{k}$ -space:

$$\Psi_{\mathbf{k}}^{\text{CF}}|_{\mathbf{k}_1 \rightarrow \mathbf{k}_1 + \mathbf{L} \times \mathbf{n}} = \xi^{N_\phi} (L) e^{\frac{L^2}{2}(i\mathbf{k}_1 + \frac{1}{2}\mathbf{L})} \Psi_{\mathbf{k}}^{\text{CF}}, \quad (53)$$

where  $\mathbf{k}_1 \rightarrow \mathbf{k}_1 + \mathbf{L} \times \mathbf{n}$  corresponds to  $i\mathbf{k}_1 \rightarrow i\mathbf{k}_1 + \mathbf{L}$  in the complex form.

As a result, we can define a super-Brillouin zone (SBZ) spanned by  $\mathbf{K}_\alpha = \mathbf{L}_\alpha \times \mathbf{n}$ ,  $\alpha = 1, 2$  with  $\mathbf{L}_1$  and  $\mathbf{L}_2$  being the two edges of the torus. From Eq. (53), by applying either the original definition Eq. (1)<sup>2</sup> or the generalized definition Eq. (17), we find that the Berry connection has the quasiperiodicity

$$A_{\mathbf{k}_1 + \mathbf{K}}^e = A_{\mathbf{k}_1}^e + \frac{1}{2}(\mathbf{K} \times \mathbf{n}), \quad (54)$$

where  $\mathbf{K}$  is one of reciprocal lattice vectors of the SBZ. The total Chern number of the SBZ can be determined by  $C_{\text{tot}} = (2\pi)^{-1} \oint A_{\mathbf{k}_1}^e \cdot d\mathbf{k}_1$  with the circuit integral along the boundary of the SBZ [30]. Using Eq. (54), it is easy to show that the integral is equal to  $-A/2\pi = -N_\phi$ . Since the Berry curvature is a constant in most of region of the SBZ,  $C_{\text{tot}}$  is equal to  $\bar{\Omega}^e A/2\pi$  in the limit of  $A \rightarrow \infty$ , where  $\bar{\Omega}^e$  is the value of the uniform background of the Berry curvature. We thus obtain

$$\bar{\Omega}^e = -1 \quad (55)$$

for the standard CFL wave function. For the CF representation, we have

$$\bar{\Omega}^\nu = 1. \quad (56)$$

Similarly, it is easy to show that  $\Psi_{\mathbf{k}}^{\text{JK}}$  has an approximated quasiperiodicity:

$$\Psi_{\mathbf{k}}^{\text{JK}}|_{\mathbf{k}_1 \rightarrow \mathbf{k}_1 + \mathbf{L} \times \mathbf{n}} \propto \exp(im\mathbf{L}^* \mathbf{k}_1/2) \Psi_{\mathbf{k}}^{\text{JK}}, \quad (57)$$

where we ignore the small change of  $\bar{\mathbf{k}}$  in the limit of  $N \rightarrow \infty$ . The presence of  $m$  in the phase factor is notable. It originates from the  $im\mathbf{k}_j$  factor in the argument of  $\bar{\sigma}$  function in Eq. (48) [26,31]. The quasiperiodicity of the Berry connection is modified to

$$A_{\mathbf{k}_1 + \mathbf{K}}^e = A_{\mathbf{k}_1}^e + \frac{m}{2}(\mathbf{K} \times \mathbf{n}). \quad (58)$$

By applying the same analysis as that for the standard CFL wave function, we obtain that the uniform background of the Berry curvature for the JK wave function is  $-m$  in the  $e$ -representation and  $2 - m$  in the CF representation.

We summarize the values of the Berry curvature background as follows:

$$\bar{\Omega}^\nu = \begin{cases} 1 & \text{(CF)}, \\ 2 - m & \text{(JK)}. \end{cases} \quad (59)$$

Note that the result is solely determined by the quasiperiodicities of the wave functions. The fact that the two wave functions have different quasiperiodicities means that they must have different Berry curvatures.

## VI. DISCUSSION AND SUMMARY

In summary, we have (a) derived the definition of the Berry phase applicable for CFL systems; (b) analytically determined the Berry phase of the CFL with respect to the standard CF wave function, and found that it yields a uniform Berry

<sup>2</sup>Note that for unnormalized wave functions, the Berry connection is determined by the formula  $A_\alpha = -\text{Im} \langle \Psi_\alpha | \partial_\alpha \Psi_\alpha \rangle / \langle \Psi_\alpha | \Psi_\alpha \rangle$ .



curvature; (c) numerically calculated the Berry phase with respect to the JK wave function, and determined the distribution of the Berry curvature in the whole momentum space; (d) analytically shown that the Berry phases with respect to the two wave functions must be different because of their different quasiperiodicities.

For both the wave functions, we find that a CF adiabatically transported around the Fermi circle acquires a Berry phase  $\pi$  in the CF representation. Since the Berry phase can be interpreted as the intrinsic anomalous Hall conductance [32] (in units of  $-e^2/2\pi h$  for  $\sigma_{xy}$  [33]), both the wave functions can correctly predict the Hall conductance of CFs for a particle-hole symmetric half-filled Landau level [11], in both its magnitude and sign. The result is actually consistent with the Dirac theory. However, microscopically, both the Berry curvature distributions with respect to the two wave functions are *not* the singular one implied by the Dirac picture. We thus expect that the effective theories with respect to the two wave functions are not the Dirac theory when physics away from the Fermi level is concerned.

On the other hand, one may question the physical relevance of these subtle differences between different effective theories. Indeed, up to now, most predictions of different effective theories have focused on the effects of the  $\pi$ -Berry phase, and are indistinguishable. It does not help that the HLR theory, which has no  $\pi$ -Berry phase, can also correctly predict the Hall conductance of CFL by considering the effect

of scattering by the fluctuation of the effective magnetic field [24]. The situation is actually typical, as in other theories at a similar stage when different pictures compete and seem to provide equally good explanations for a limited set of observations. For the CF theory, we would like to argue that (a) A wave function must have one and only one correct effective theory, unless different effective theories can be shown to be equivalent. (b) Microscopic details of different effective theories are relevant because they may lead to different physical predictions. One such example is shown in Ref. [34], which indicates that different ways of modulating CF systems can induce different asymmetries in geometric resonance experiments [35] as a result of the “subatomic” dipole structure of the CF. Predictions like that could be tested experimentally and differentiate different effective theories. (c) Correct microscopic details may be the key of obtaining a consistent effective theory free of difficulties such as the effective mass divergence [10].

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