Formation of $\bar{H}^+$ via radiative attachment of e^+ to $\bar{H}$

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(Received 17 October 2019; published 30 January 2020; corrected 20 November 2020)

We consider the formation of positive ions of antihydrogen $\bar{H}^+$ via radiative attachment of incident positrons e^+ to antihydrogen atoms $\bar{H}$. The formation mechanisms include (i) spontaneous radiative attachment in which the ion is formed due to spontaneous emission of a photon by a positron incident on $\bar{H}$; (ii) induced radiative attachment where the formation proceeds in the presence of a laser field via induced photoemission; and (iii) two-center attachment which takes place in the presence of a neighboring atom B and in which an incident positron is attached to $\bar{H}$ via resonant transfer of energy to B with its subsequent relaxation through spontaneous radiative decay. We show that the mechanisms (ii) and (iii) can strongly dominate over the known mechanism (i). Besides, according to our estimates, in the range of positron energies ($\simeq$1 eV) where the radiative channels are most efficient, the mechanism of (nonradiative) three-body attachment, in which one of two positrons incident on $\bar{H}$ forms the ion whereas the other one carries away the energy excess, is much weaker than the channel (i). We also briefly discuss three-body attachment where, instead of two positrons, a positron and an electron are incident on $\bar{H}$.

DOI: 10.1103/PhysRevResearch.2.013105

I. INTRODUCTION

The asymmetry between the presence of matter and antimatter in the Universe (the baryon asymmetry), the reasons for which are not yet (fully) understood, makes the studies of antimatter in laboratories paramountly important first of all from the point of view of fundamental physics. Besides, already now such simplest pieces of antimatter as antiprotons, which, when slowly moving in medium, deposit energy quite differently compared to protons, are considered as a promising tool for applications in biology and medicine.

During the last two decades a tremendous progress was achieved in the production of substantial amounts of the simplest atom of antimatter, the antihydrogen $\bar{H}$ (see, e.g., [1–8]). They are (see, e.g., [9–11]) and will be used for high-precision experiments with antihydrogen and comparisons between the properties of antimatter and matter [12].

The species of antimatter, next in complexity to $\bar{H}$, are the positive ion of antihydrogen $\bar{H}^+$ and the antihydrogen molecular ion $\bar{H}_2^+$. In particular, $\bar{H}^+$ will be used as an intermediate particle in experiments on the behavior of $\bar{H}$ in the gravitational field of the Earth (GBAR experiment, see e.g., [13–15] and references therein) whereas $\bar{H}_2^+$ was suggested in [16] as allowing much more precise, compared to $\bar{H}$, tests of the fundamental interactions.

Concerning the formation of the $\bar{H}^+$ ion there are two main pathways to produce it directly from antihydrogen atoms. One of them is to use collisions between $\bar{H}$ and positronium Ps $(e^+ + e^-)_{\mathrm{bound}}$, in which the capture reaction $(e^+ + e^-)_{\mathrm{bound}} + \bar{H} \rightarrow \bar{H}^+ + e^-$ is possible (see, e.g., [14,17] and references therein). The second involves free positrons incident on $\bar{H}$ atoms in which the $\bar{H}^+$ ions can be formed either via radiative or three-body attachment. The former mechanism involves just one positron per $\bar{H}$ and proceeds via emission of a photon which carries away the energy excess. In the latter (at least) two positrons have to be in the vicinity of $\bar{H}$: one of them is attached forming $\bar{H}^+$ whereas the other one takes the energy excess.

In this paper we focus on the formation of $\bar{H}^+$ via radiative attachment, which involves photoemission as its key signature, and present a comparative consideration of three mechanisms for the radiative formation of $\bar{H}^+$. They include (i) (spontaneous) radiative attachment in which the $\bar{H}^+$ ion is formed due to spontaneous emission of a photon by a positron incident on $\bar{H}$; (ii) (induced) radiative attachment where the formation of $\bar{H}^+$ proceeds in the presence of a laser field via induced photoemission; and (iii) two-center dileptonic attachment which becomes possible in the presence of a neighboring (matter) atom B and in which an incident positron is attached to $\bar{H}$ via resonant transfer of energy to B driven by the two-center dileptonic interaction; atom B subsequently relaxes through spontaneous radiative decay. These three mechanisms are illustrated in Fig. 1.

To our knowledge, the mechanisms (ii) and (iii) have been considered neither for the formation of $\bar{H}^+$ nor for (the closely related process of) the formation of H^-. In contrast, the mechanism (i) was studied both for the $(e^- + H)$ and

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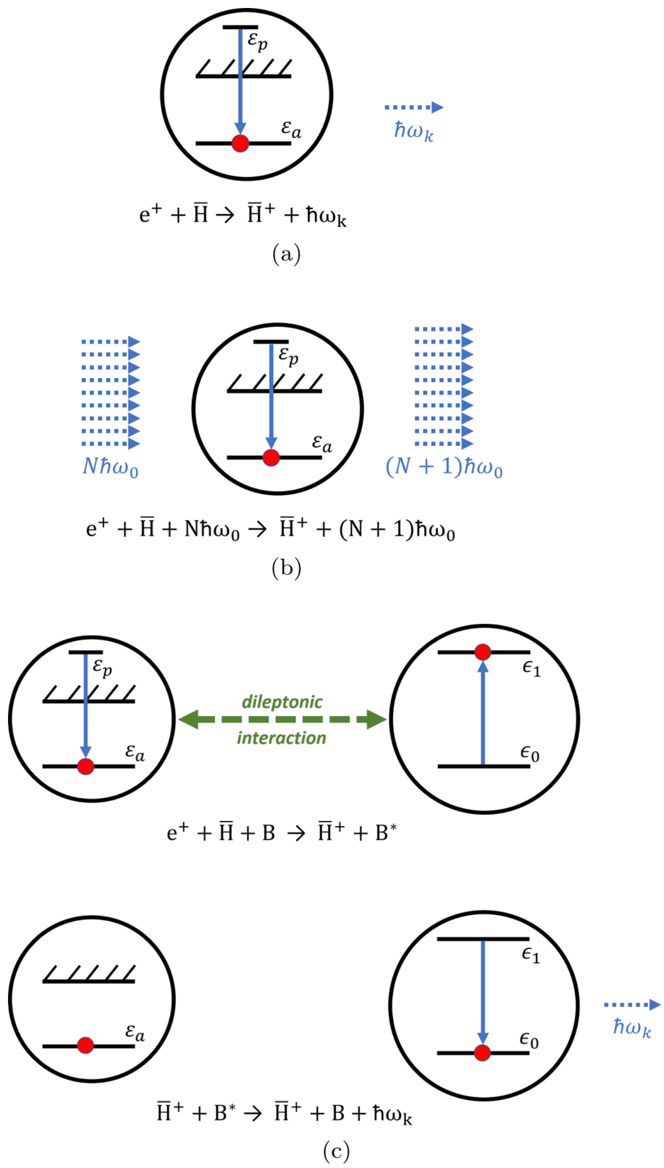

FIG. 1. Illustration of the radiative mechanisms for the formation of $\bar{H}^+$: (a) (single-center) spontaneous radiative attachment; (b) (single-center) laser-induced radiative attachment; (c) two-center dileptonic attachment.

($e^+ + \bar{H}$) systems (see, e.g., [18–22] and references therein). In this paper it is regarded as the reference mechanism and is considered for the sake of completeness (and convenience). In particular, since the spontaneous radiative attachment is characterized by relatively low formation rates, one of the main goals of this paper is to find out whether the other two mechanisms, (ii) and (iii), can be (much) more efficient.

Indeed, it will be shown that under certain conditions the laser-induced and two-center dileptonic attachment mechanisms can strongly dominate over the "simple" radiative attachment.

Besides, we shall also briefly discuss two types of (nonradiative) three-body attachment. The first involves two positrons incident on $\bar{H}$ with the interaction between them resulting in capture of one of the positrons whereas the other one carries away the energy release: it will be shown that this process, at energies of the incident positrons which are most favorable for radiative attachment, cannot compete with radiative attachment. In the other, instead of two positrons, a positron and an electron are incident on $\bar{H}$ and the formation of $\bar{H}^+$ proceeds via the positron-electron interaction: here we give some qualitative arguments suggesting that this type of three-body attachment can be much more efficient than the first one.

The paper is organized as follows. In the next section (Sec. II) we present the basic consideration of the mechanisms for radiative attachment. Section III contains numerical results and discussion of the radiative and three-body attachment mechanisms. Our main conclusions are summarized in Sec. IV.

Atomic units are used unless otherwise indicated.

II. THEORETICAL CONSIDERATION

A. Single-center spontaneous radiative attachment (1CRA)

Radiative recombination of an electron with a positive ion is a very well-known process, which has been studied for decades with energies of the incident electrons ranging from below 1 eV to relativistic values (see, e.g., [23–25] and references therein). Radiative attachment of an electron to a neutral atom is basically similar to radiative recombination and is governed by the same fundamental mechanism: the interaction of the electron-atom system with the radiation field.

From the point of view of quantum electrodynamics, the processes of radiative attachment of an electron to an atom and of a positron to an antiatom are in essence identical and their consideration may be based on the same treatment(s). Besides, it is known that good results for the detachment of an electron from H^- by photoabsorption (see, e.g., [21,22] and references therein) and by the impact of charged particles [26] as well as for radiative attachment of an electron to H [21,22] are obtained when these processes are treated as effectively single-electron processes in which the interaction between the "active" electron and the core of H^- is described by a short-range effective potential (although a more rigorous treatment [22] is expected to yield better results). It is obvious that the same can also be done for radiative attachment of a positron to $\bar{H}$ [21,22].

Therefore, referring to, e.g., [25], where radiative recombination in the single- (active-) electron approximation is considered, we may directly write the expression for the (total) rate per unit time for radiative attachment of a positron to an antihydrogen atom which reads as

$$\mathcal{R}^{1\mathrm{CRA}} = \frac{4\pi}{3V_p}\frac{\omega_A^3}{c^3}\frac{r_A^2}{p^2}. \tag{1}$$

Here, p is the momentum of the incident positron with respect to $\bar{H}$, $\omega_A = \varepsilon_p - \varepsilon_a$ is the transition frequency, where ε_p and ε_a ($\varepsilon_a \approx -0.75\,\mathrm{eV}$) are the energy of the incident positron and of the bound state of $\bar{H}^+$, respectively. Further, V_p is the normalization volume for the incident positron, c is the speed of light, and

$$r_A = \int_0^\infty dr\, r^3 g_0(r) g_{p1}(r) \tag{2}$$

is the radial matrix element for the transition of the incident positron into the bound state φ_a of $\bar{\text{H}}^+$, where g_{p1} and g_0 are the radial parts of the continuum and bound state, respectively.

If the momentum/energy of the incident positrons is not fixed, one should average the rate (1) over their momentum distribution. Assuming that this distribution is peaked at $\mathbf{p} = \mathbf{p}_c$ and that the energy width of this distribution is much smaller than the energy range ΔE_p on which the quantity r_A^2/p^2 noticeably varies (ΔE_p is the effective continuum width, for H^- and $\bar{\text{H}}^+$ one has $\Delta E_p \simeq 1\,\text{eV}$) we obtain that the averaged rate for radiative attachment $\langle \mathcal{R}^{1\text{CRA}} \rangle$ is given by

$$\langle \mathcal{R}^{1\text{CRA}} \rangle = \frac{4\pi}{3V_p} \frac{\omega_A^3}{c^3} \frac{r_A^2}{p_c^2}. \tag{3}$$

B. Single-center laser-induced radiative attachment (LIRA)

Suppose now that a positron is incident on an antihydrogen atom in the presence of a laser field [see Fig. 1(b)]. Now, the formation of the $\bar{\text{H}}^+$ ion can also proceed via induced photoemission. In order that the field only efficiently stimulates attachment with little destruction of the produced ions (and of course without destroying the atoms themselves), it should obey a couple of main conditions. First, it should be weak enough, $F_0 \ll F_a$, where F_0 is the strength of the laser field and F_a is the typical field, created by the ionic core, acting on the loosely bound positron in the $\bar{\text{H}}^+$ ion. Second, the frequency ω_0 of the laser field should be (nearly) resonant with respect to the positron transitions leading to the formation of $\bar{\text{H}}^+$: $\varepsilon_p - \varepsilon_a \approx \omega_0$, where ε_p and ε_a are the energy of the incident and bound positron, respectively.

We shall consider that the laser field is a classical electromagnetic wave of linear polarization which can be taken in the dipole approximation $\mathbf{F}(t) = \mathbf{F}_0 \sin(\omega_0 t)$. Then, the problem of laser-induced attachment (as well as of laser-induced resonant scattering) is described by the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = (\hat{H}_0 + \hat{W}(t))\psi, \tag{4}$$

where $\hat{H}_0$ is the Hamiltonian for the field-free ($\text{e}^+ + \bar{\text{H}}$) system and $\hat{W}$ is the interaction between the positron and the laser field. By expanding the wave function ψ into the "complete" set of states

$$\psi = a(t)\,\varphi_a + \int d^3\mathbf{p}\, b_{\mathbf{p}}(t)\, \varphi_{\mathbf{p}}, \tag{5}$$

where $\varphi_{\mathbf{p}}$ and φ_a refer to the continuum and bound state, respectively, of the field-free ($\text{e}^+ + \bar{\text{H}}$) system, and inserting (5) into (4) we obtain the set of equations for the unknown coefficients $a(t)$ and $b_{\mathbf{p}}(t)$:

$$i\frac{da}{dt} - \varepsilon_a\, a = \int d^3\mathbf{p}\, \langle \varphi_a|\hat{W}|\varphi_{\mathbf{p}}\rangle\, b_{\mathbf{p}}(t),$$
$$i\frac{db_{\mathbf{p}}}{dt} - \varepsilon_p\, b_{\mathbf{p}} = \langle \varphi_{\mathbf{p}}|\hat{W}|\varphi_a\rangle\, a \tag{6}$$

with the initial (boundary) conditions $a(t = t_i) = 0$, $b_{\mathbf{p}}(t = t_i) = \delta^3(\mathbf{p} - \mathbf{p}_0)$, where t_i is the moment of time when the field was switched on and $\mathbf{p}_0$ is the momentum of the incident positron.

We note that in (6) the laser-induced transitions between the continuum states were neglected, which is possible since the electromagnetic field is assumed to be sufficiently weak.

Employing, for definiteness, the so-called velocity gauge in which the electric field $\mathbf{F}$ is expressed solely via the vector potential $\mathbf{A}$, $\mathbf{F}(t) = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$, using the rotating-wave approximation (see, e.g., [27]) and assuming that the interaction with the field is switched on suddenly at $t = t_i = 0$, for the laser-induced formation rate $\mathcal{R}^{\text{LIRA}}$ per unit of time we obtain (see Appendix)

$$\mathcal{R}^{\text{LIRA}} = \frac{dP_a}{dt}, \tag{7}$$

where

$$P_a = |\,a\,|^2 = \frac{|W_{a,\mathbf{p}_0}|^2}{(\varepsilon_{p_0} - \varepsilon_a - \omega_0)^2 + \frac{\Gamma_a^2}{4}} \times \{1 + \exp(-\Gamma_a t) - 2\exp(-\Gamma_a t/2) \times \cos[(\varepsilon_{p_0} - \varepsilon_a - \omega_0)t]\}. \tag{8}$$

Here, $W_{a,\mathbf{p}} = -\frac{1}{2\omega_0}\langle \varphi_a|\mathbf{F}_0 \cdot \hat{\mathbf{p}}|\varphi_{\mathbf{p}}\rangle$ is the transition matrix element and $\Gamma_a = 2\pi \int d^2\Omega_{\mathbf{p}} |W_{a,\mathbf{p}}|^2_{p=|\mathbf{p}|=\sqrt{2(\varepsilon_a+\omega_0)}}$, where the integration runs over the angles of the emitted positron, is the width of the bound state φ_a of the ion which is caused by the interaction with the laser field. We note that the term in the first line of (8) has a familiar resonance structure with a maximum at the exact resonance $\varepsilon_{p_{res}} = \varepsilon_a + \omega_0$.

Let the incident positrons have a momentum/energy distribution. We assume, similarly as for 1CRA, that this distribution is peaked at $\mathbf{p}_0 = \mathbf{p}_c$ and is much narrower than the effective width of the continuum ($\Delta E_p \simeq 1\,\text{eV}$ for H^- and $\bar{\text{H}}^+$). Besides, we also suppose that this distribution is much broader than the width Γ_a (in a relatively weak laser field Γ_a amounts to just a tiny fraction of 1 eV so that our assumptions are very well compatible with each other). Then, after averaging (8) over the momentum distribution of the incident positrons we obtain

$$\langle P_a \rangle = \int d^3\mathbf{p}_0\, f(\mathbf{p}_0)\, P_a = \frac{F_0^2}{4\,\omega_0^2} \frac{2\pi}{\Gamma_a} [1 - \exp(-\Gamma_a t)] \times p_{res} \int d\Omega_{\mathbf{p}_0}\, f(\mathbf{p}_0) |\langle \varphi_a|\mathbf{e}_0 \cdot \hat{\mathbf{p}}|\varphi_{\mathbf{p}_0}\rangle|^2, \tag{9}$$

where $f(\mathbf{p})$ is the distribution function of the incident positrons and $\mathbf{e}_0 = \mathbf{F}_0/F_0$ is the unit polarization vector of the laser field. In (9) the integration over the angles is performed under a fixed value $p_0 = |\mathbf{p}_0| = p_{res} = \sqrt{2(\varepsilon_a + \omega_0)}$ of the positron momentum. The averaged formation rate then reads as

$$\langle \mathcal{R}^{\text{LIRA}} \rangle = \frac{d\langle P_a \rangle}{dt} = \pi \frac{F_0^2}{2\,\omega_0^2} \exp(-\Gamma_a t)\; p_{res} \times \int d\Omega_{\mathbf{p}_0} f(\mathbf{p}_0) |\langle \varphi_a|\mathbf{e}_0 \cdot \hat{\mathbf{p}}|\varphi_{\mathbf{p}_0}\rangle|^2, \tag{10}$$

where again $p_0 = p_{res}$. We note that it is advantageous for the rate to have $p_{res} = p_c$.

Let us assume for simplicity that all positrons are incident on $\bar{\mathrm{H}}$ along the field polarization and that their energies are uniformly distributed over an interval with a width $\Delta\varepsilon_p$ ($\Delta\varepsilon_p \ll \Delta E_p$), which includes the resonance region $\varepsilon_p \simeq \varepsilon_a + \omega_0$. Taking the laser field to be polarized along the z axis one has, under the above assumptions, that $f(\mathbf{p}_0) = \delta(1 - \cos\vartheta_{\mathbf{p}_0})/(2\pi\, p_{res}\, \Delta\varepsilon_p)$. Then, by making use of the relation $|\langle\varphi_a|\mathbf{e}_0\cdot\hat{\mathbf{p}}|\varphi_{\mathbf{p}_0}\rangle|^2 = (\varepsilon_{p_0} - \varepsilon_a)^2|\langle\varphi_a|\mathbf{e}_0\cdot\mathbf{r}|\varphi_{\mathbf{p}_0}\rangle|^2 = \omega_0^2|\langle\varphi_a|\mathbf{e}_0\cdot\mathbf{r}|\varphi_{\mathbf{p}_0}\rangle|^2$, expanding the state $\varphi_{\mathbf{p}_0}$ in partial waves (see, e.g., [28]) and taking into account that φ_a is an s state we obtain

$$\langle\mathcal{R}^{\mathrm{LIRA}}\rangle = \frac{\pi^2}{2V_p}\,\frac{F_0^2}{\Delta\varepsilon_p}\,\frac{r_A^2}{p_c^2}\,\exp(-\Gamma_a t). \tag{11}$$

We note that another choice of the collision geometry would merely modify the numerical prefactor in (11).

It follows from (10) and (11) that the formation rate substantially diminishes for those t where $\Gamma_a t \simeq 1$ and already essentially vanishes for t where $\Gamma_a t \gg 1$. This happens because in the presence of a laser field, in addition to induced attachment, photodetachment also occurs and with increasing the population of the bound state, the attachment and detachment events will eventually balance each other, leading to the zero net result for the formation rate. Therefore, for the laser-induced formation mechanism to be efficient during the whole laser pulse, its duration T ($0 \leqslant t \leqslant T$) should not be too long, $\Gamma_a T < 1$.

C. Two-center dileptonic attachment (2CDA)

Let us now consider attachment of a positron to $\bar{\mathrm{H}}$ which occurs in the presence of a neighbor atom B. At the moment, we disregard the possibility of annihilation and other processes involving antimatter embedded in matter (which will be discussed in Sec. III).

Suppose first that the distance R_0 ($R_0 \gg 1$ a.u.) between $\bar{\mathrm{H}}$ and B is fixed. If the energy, which is released in the process of $\mathrm{e}^+ + \bar{\mathrm{H}}$ attachment, is close to an excitation energy of a dipole-allowed transition in atom B, then the attachment can proceed by transferring, via the two-center positron-electron (dileptonic) interaction, the energy excess to atom B. The latter, as a result, undergoes a transition into an excited state. If afterwards the excited state of B radiatively decays to its initial (ground) state, then the two-center system becomes stable meaning that the $\bar{\mathrm{H}}^+$ ion has been formed. It is known that, due to its resonant nature, the two-center "recombination" channel may enhance the corresponding rate by orders of magnitude compared to the case when center B is absent [29,30].

Let now free positrons and a beam of antihydrogen atoms move in a (dilute) gas of atoms B. As was just mentioned, the two-center attachment process relies on the energy transfer resonant to a transition in B. However, the relative motion of $\bar{\mathrm{H}}$ and B leads to uncertainty in positron and electron transition energies (as they are viewed by the corresponding collision partner), effectively broadening them. Therefore, the efficiency of this two-center channel is restricted to low-velocity collisions, where the velocity v of $\bar{\mathrm{H}}$ with respect to B is much less than 1 a.u. (1 a.u. $= 2.18\times10^8$ cm/s) [31].

In a recent paper [31] the process of two-center dielectronic recombination of an electron with an atomic center A was considered when electrons and a beam of slow atoms A move in a gas of atomic centers B. The results of [31] can be straightforwardly adapted to the process of two-center dileptonic attachment.

Considering, following [31], only distant collisions between antihydrogens and matter atoms B and making use of the results of [31], for the contribution from these collisions to the total formation rate for positive antihydrogen ions per unit of time (per one $\mathrm{e}^+ + \bar{\mathrm{H}}$ pair) via the two-center dileptonic mechanism we obtain

$$\mathcal{R}^{\mathrm{2CDA}} = 3\pi^2\,\frac{n_B}{v\,b_{\min}^2}\,\frac{c^3\Gamma_r^B}{\omega_B^3}\,\frac{r_A^2}{p^2}\,\eta^2\big\{\sin^2\vartheta_{\boldsymbol{p}}K_1^2(\eta) + (1+\cos^2\vartheta_{\boldsymbol{p}})\,\eta\,K_0(\eta)\,K_1(\eta)\big\}. \tag{12}$$

Here, n_B is the density of atoms B, $b_{\min}$ ($b_{\min} \gg 1$ a.u.) is the minimum value of the impact parameter of the $\bar{\mathrm{H}} - B$ collisions, Γ_r^B is the radiative width of the excited state of atom B, and ω_B is the transition frequency between the ground and excited states of B. Further, p is the momentum of the incident positron, $\vartheta_{\boldsymbol{p}}$ is its incident angle (counted from the collision velocity $\mathbf{v}$), and $\eta = |\varepsilon_p - \varepsilon_a - \omega_B|\, b_{\min}/v$, where ε_p is the energy of the incident positron. Besides, $K_n(x)$ are the modified Bessel functions [32].

The functions $K_n(x)$ ($n = 0, 1, \ldots$) diverge at $x \to 0$ and decrease exponentially at $x > 1$ [32]. Therefore, in distant low-velocity collisions ($b_{\min} \gg 1$, $v \ll 1$) the most favorable conditions for the formation, according to (12), are realized when the energy of the incident positrons lies within the small interval centered at $\varepsilon_{p,r} = \varepsilon_a + \omega_B$ with the width $\delta\varepsilon_p \sim v/b_{\min}$. Since the quantity $v/b_{\min}$ is typically orders of magnitude larger than the natural width Γ_r^B we see that the collision strongly smears out the "static" resonance conditions $\varepsilon_a + \omega_B - \Gamma_r^B \lesssim \varepsilon_p \lesssim \varepsilon_a + \omega_B + \Gamma_r^B$ leading to a much broader range of "quasiresonance" energies of the incident positron.

If the incident positrons do not have a fixed momentum $\boldsymbol{p}$, the rate (12) should be averaged over their momentum distribution function $f(\boldsymbol{p})$. This, in general, can be done only numerically.

However, a simple formula for the averaged rate, which enables one to establish a direct correspondence with the case of 2CDA at a fixed distance between $\bar{\mathrm{H}}$ and B, can be derived if we suppose the following: (i) the function $f(\boldsymbol{p})$ can be factorized as $f(\boldsymbol{p}) = f_\varepsilon^{(1)}(\varepsilon_p)\,f_\Omega^{(2)}(\Omega_{\boldsymbol{p}})$; (ii) the function $f_\varepsilon^{(1)}(\varepsilon_p)$ is distributed over an energy range which covers the interval of the "quasiresonance" energies, $\varepsilon_a + \omega_B - v/b_{\min} \lesssim \varepsilon_p \lesssim \varepsilon_a + \omega_B + v/b_{\min}$, and is much broader than this interval with $f_\varepsilon^{(1)}(\varepsilon_p)$ noticeably varying on a scale much larger than $\delta\varepsilon_p \sim v/b_{\min}$ [i.e., within the energy interval essential for 2CDA $f_\varepsilon^{(1)}(\varepsilon_p)$ is roughly a constant]. Then, taking into account that the continuum width $\Delta E_p \simeq 1\,\mathrm{eV}$ is much larger than $\delta\varepsilon_p \sim v/b_{\min}$, we can obtain a rough estimate for the averaged

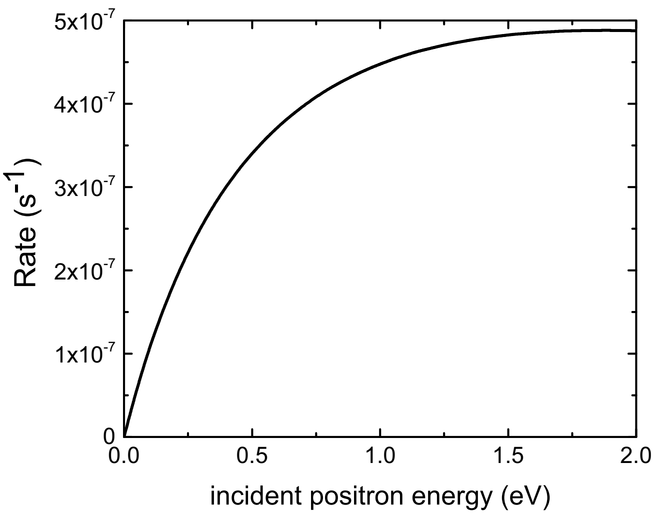

FIG. 2. The rate for (spontaneous) radiative attachment $\bar{\mathrm{H}} + e^+ \rightarrow \bar{\mathrm{H}}^+ + \hbar\omega_k$, given as a function of the energy of the incident positron.

rate given by (see [31])

$$\langle \mathcal{R}^{2\mathrm{CDA}} \rangle = \frac{9\pi^4}{16} \frac{n_B}{b_{\min}^3} \frac{\Gamma_r^B c^3}{\omega_B^3} \left(\frac{r_A^2}{p^2}\right)_{p=p_r} f_\varepsilon^{(1)}(\varepsilon_{p,r}) \times \int d\Omega_{\boldsymbol{p}} f_\Omega^{(2)}(\Omega_{\boldsymbol{p}}) \left(1 + \frac{1}{2}\sin^2\vartheta_{\boldsymbol{p}}\right), \quad (13)$$

where $p_r = \sqrt{2\varepsilon_{p,r}} = \sqrt{2(\varepsilon_a + \omega_B)}$. Assuming for simplicity that all positrons are incident under the angle $\vartheta_{\boldsymbol{p}} = \pi/2$ and are homogeneously distributed over the energy interval $\Delta\varepsilon_p$, one can get (see [31])

$$\langle \mathcal{R}^{2\mathrm{CDA}} \rangle = \frac{3^3\pi^4}{2^5} \frac{n_B}{b_{\min}^3} \frac{c^3}{\omega_B^3} \frac{\Gamma_r^B}{\Delta\varepsilon_p} \left(\frac{r_A^2}{p^2}\right)_{p=p_r}. \quad (14)$$

As it follows from (12), (13), and (14) the results for 2CDA depend on the parameter $b_{\min}$. Based on the discussion of collisional two-center dielectronic recombination, given in [31], we set here $b_{\min} = 5$ a.u. (that enables one to satisfy all assumptions which our approach to the two-center process relies on).

Since only contributions to 2CDA from relatively large impact parameters $b \geqslant b_{\min}$ are accounted for, the results (12) and (14) represent a lower boundary for the rate of 2CDA.

III. RESULTS AND DISCUSSION

A. Single-center spontaneous radiative attachment (1CRA)

(Single-center) spontaneous radiative attachment of an electron to atomic hydrogen has been considered and is well known (see, e.g., [18–22] and references therein). The corresponding results can be straightforwardly applied also for the attachment of a positron to antihydrogen [21,22].

Therefore, in our discussion of the attachment mechanisms we take, as a reference, the spontaneous radiative attachment: $e^+ + \bar{\mathrm{H}} \rightarrow \bar{\mathrm{H}}^+ + \hbar\omega_k$. The rate for the formation of $\bar{\mathrm{H}}^+$ via this mechanism is shown in Fig. 2 as a function of the energy ε_p of the incident positron assuming a density of positrons of 10^8 cm^{-3}. The rate was calculated by approximating the incident positron by a plane wave and the bound state of $\bar{\mathrm{H}}^+$ by the following wave function:

$$\varphi_a(\mathbf{r}) = \sqrt{\frac{\alpha\,\beta\,(\alpha+\beta)}{2\,\pi\,(\beta-\alpha)^2}}\; \frac{\exp(-\alpha\,r) - \exp(-\beta\,r)}{r}, \quad (15)$$

where $\alpha = 0.235$ a.u. ($\alpha^2/2 = 0.0275$ a.u. ≈ 0.748 eV is the binding energy) and $\beta = 0.913$ a.u. The wave function (15) was obtained by using a nonlocal separable potential of Yamaguchi [33] to describe a short-range effective interaction of the "active" positron with the core of the $\bar{\mathrm{H}}^+$. Our calculation yields the attachment cross section which has about the same shape as predicted by a more accurate approach [22] but is $\approx 30\%$ larger. Such an accuracy is quite sufficient for this study, where we are focused on a comparison of laser-induced and two-center attachment channels to the reference one, represented by the spontaneous radiative attachment. Indeed, the rates for all the radiative formation mechanisms involve the same dipole transition matrix element between the continuum and bound states of $\bar{\mathrm{H}}^+$ and, therefore, the ratios do not depend on it.

It is seen in Fig. 2 that in the interval of positron energies $0\,\mathrm{eV} \leqslant \varepsilon_p \leqslant 2\,\mathrm{eV}$ with increasing ε_p the rate initially rapidly grows and then saturates becoming almost a constant.

B. Single-center laser-induced radiative attachment (LIRA)

The relative effectiveness of laser-induced radiative attachment (with respect to 1CRA) can be characterized by the ratio $\bar{\mu}_{\mathrm{LIRA/1CRA}}$ of their (averaged) formation rates. Using Eqs. (3) and (11) for this ratio we obtain

$$\bar{\mu}_{\mathrm{LIRA/1CRA}} = \frac{\langle \mathcal{R}^{\mathrm{LIRA}} \rangle}{\langle \mathcal{R}^{1\mathrm{CRA}} \rangle} \approx \frac{3\,\pi}{8} \frac{F_0^2\,c^3}{\Delta\varepsilon_p\,\omega_0^3} \exp(-\Gamma_a t). \quad (16)$$

Choosing $\omega_0 \approx 1.5$ eV, $\Delta\varepsilon_p = 0.1$ eV, and assuming that the duration T of the laser pulse is not too long, $\exp(-\Gamma_a T) \approx 1$, we obtain that $\bar{\mu}_{\mathrm{LIRA/1CRA}} \approx 1$ at $I_0 \approx 1.2 \times 10^4$ W/cm^2 where $I_0 = cF_0^2/(8\,\pi)$ is the (averaged over the period) intensity of the laser field. The above value is rather low and by increasing the intensity by two to three orders of magnitude one can increase the ratio $\bar{\mu}_{\mathrm{LIRA/1CRA}}$ by the same amount.

Unlike spontaneous radiative attachment, laser-induced attachment is a resonant process which effectively proceeds only when the energy of the incident positron ε_p lies within a very narrow interval (with the effective width of a few Γ_a's) in the vicinity of the resonance energy $\varepsilon_{p_{\mathrm{res}}} = \varepsilon_a + \omega_0$. At an intensity $I_0 \approx 1.2 \times 10^4$ W/cm^2 the width Γ_a is very small: $\Gamma_a \approx 1.5 \times 10^{-10}$ eV. This means that for the resonant energies of the incident positron, $\varepsilon_a + \omega_0 - \Gamma_a \lesssim \varepsilon_p \lesssim \varepsilon_a + \omega_0 + \Gamma_a$, the enhancement of the $\bar{\mathrm{H}}^+$ formation due to the interaction with the laser field of intensity $I_0 \approx 1.2 \times 10^4$ W/cm^2 reaches $10^8 - 10^9$.

One should note that it was proposed ([34], see also [12]) to enhance the formation of antihydrogen atoms $\bar{\mathrm{H}}$ via laser-induced recombination of positrons with antiprotons. We, however, are not aware of any experimental evidence for this mechanism in collisions between positrons and antiprotons. In particular, this mechanism was not confirmed in an experiment [35] where no effect of the laser field on the formation rate was observed. This was attributed to the dominance of three-body recombination under the conditions of the experiment in which positrons with very low energies were incident on antiprotons and the formation of antihydrogen was most likely to proceed via positron capture into highly excited Rydberg states of $\bar{\mathrm{H}}$.

In case of formation of $\bar{\mathrm{H}}^+$, such states are absent. Besides, since an antihydrogen atom is a neutral object, the mutual repulsions between the incident positrons as well as between the newly bound and outgoing positrons will no longer be counterbalanced by their attraction to the antiproton (screened now by the bound atomic positron) that strongly weakens chances for the three-body attachment. Therefore, as it will be shown below, at positron energies $\varepsilon_p \simeq 1$ eV, which are most favorable for all the radiative mechanisms, three-body attachment is much weaker than 1CRA. Thus, under such conditions the LIRA mechanism is not expected to be masked by the nonradiative attachment channel.

C. Two-center dileptonic attachment (2CDA)

Similarly, the relative effectiveness of two-center dileptonic attachment can be described by the ratio $\bar{\mu}_{\mathrm{2CDA/1CRA}}$ of the corresponding (averaged) formation rates. Using Eqs. (3) and (14) we obtain

$$\bar{\mu}_{\mathrm{2CDA/1CRA}} = \frac{\langle \mathcal{R}^{\mathrm{2CDA}} \rangle}{\langle \mathcal{R}^{\mathrm{1CRA}} \rangle} \approx \frac{3^4 \pi^3}{2^7} \frac{n_B}{b_{\mathrm{min}}^3} \frac{c^6}{\omega_A^3 \, \omega_B^3} \frac{\Gamma_r^B}{\Delta\varepsilon_p}. \quad (17)$$

Let us, as an example of 2CDA, consider that positrons and a beam of slow $\bar{\mathrm{H}}$ move in a gas of Cs atoms. In this case the process of 2CDA of interest is $(\bar{\mathrm{H}} + \mathrm{e}^+) + \mathrm{Cs}(6\ ^2S_{1/2}) \rightarrow \bar{\mathrm{H}}^+ + \mathrm{Cs}(6\ ^2P_{3/2}) \rightarrow \bar{\mathrm{H}}^+ + \mathrm{Cs}(6\ ^2S_{1/2}) + \hbar\omega_k$, in which positron capture by antihydrogen proceeds via the transfer of the energy release to Cs exciting the $6\ ^2S_{1/2} \rightarrow 6\ ^2P_{3/2}$ dipole transition with its consequent deexcitation via spontaneous radiative decay. Taking into account that $\omega_B = 1.455$ eV and $\Gamma_r^B = 2.14 \times 10^{-8}$ eV, and choosing $b_{\mathrm{min}} = 5$ a.u. and $\Delta\varepsilon_p = 0.1$ eV we obtain that $\bar{\mu}_{\mathrm{2CDA/1CRA}} = 1$ at $n_B \simeq 10^{12}$ cm^{-3}. Thus, already for a rather dilute gas the 2CDA mechanism can outperform the simple one-center radiative attachment.

More information about the relationship between these two mechanisms is obtained by considering the energy-dependent ratio between them. Using Eqs. (1) and (12) for this ratio we obtain

$$\begin{aligned} \mu_{\mathrm{2CDA/1CRA}} &= \frac{\mathcal{R}^{\mathrm{2CDA}}}{\mathcal{R}^{\mathrm{1CRA}}} \\ &= \frac{9\pi}{4} \frac{n_B}{v\, b_{\mathrm{min}}^2} \frac{c^6 \Gamma_r^B}{\omega_A^3\, \omega_B^3} \tilde{\eta}^2 \big\{ \sin^2 \vartheta_p K_1^2(\tilde{\eta}) \\ &\quad + (1 + \cos^2 \vartheta_p)\, \tilde{\eta}\, K_0(\tilde{\eta})\, K_1(\tilde{\eta}) \big\}. \quad (18) \end{aligned}$$

In Fig. 3 the ratio $\mu_{\mathrm{2CDA/1CRA}}$ is plotted as a function of the energy of the incident positron for collision velocities ranging between 0.01 and 0.1 a.u. corresponding to the interval of impact energies from 2.5 to 250 eV/u. It is seen in the figure that at the lowest velocities this ratio reaches a maximum close to the position of the resonance $\varepsilon_{p,r} = \varepsilon_0 + \omega_B \approx 0.7$ eV, and is roughly symmetric with respect to this point. The maximum is rather broad: its width is caused by the relative motion of $\bar{\mathrm{H}}$ and B and even for the lowest velocity considered in Fig. 3 it exceeds the corresponding radiative width ($\Gamma_r^B \sim 2 \times 10^{-8}$ eV) of the excited state of Cs by orders of magnitude.

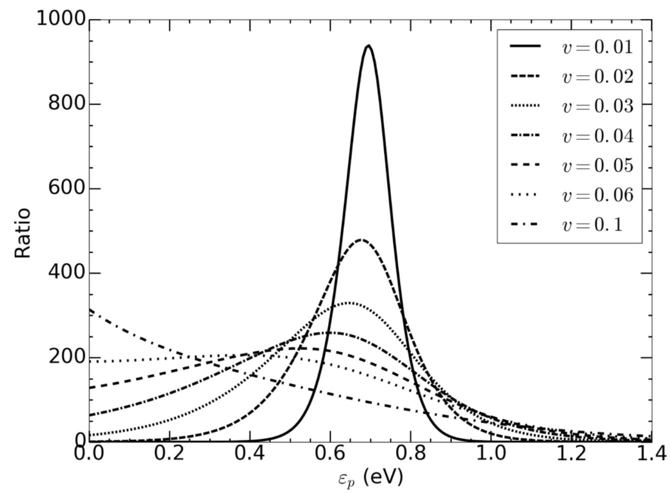

FIG. 3. The ratio (18) as a function of the energy ε_p of the incident positron at $\vartheta_p = \pi/2$ for collision velocities ranging from $v = 0.01$ to 0.1 a.u. The collision system is $(\bar{\mathrm{H}} + \mathrm{e}^+)$-Cs$(6\ ^2S_{1/2})$, $b_{\mathrm{min}} = 5$ a.u., and $n_B = 10^{15}$ cm^{-3}.

When the collision velocity grows, the resonance structure in Fig. 3 broadens and its maximum is shifted to lower positron energies. With a further increase in the collision velocity, the resonance structure eventually disappears (and this occurs at velocities which are still much smaller than 1 a.u.).

The above consideration suggests that the 2CDA mechanism can be much more efficient than 1CRA (we remind that our results do not take into account the contribution to 2CDA from collisions with smaller impact parameters and thus represent a lower boundary for the 2CDA production rate). However, it involves the interaction between matter and antimatter and the question naturally arises whether other processes, which will be present in such an environment (for instance, annihilation), would not effectively eliminate this mechanism.

Let us first consider what can happen to a positron which moves in a gas of neutral atoms. By far the dominant process in this case is elastic positron scattering (see, e.g., [36] and references therein) and at the positron energies of interest ($\varepsilon_p \lesssim 1$ eV) its cross section reaches several tens of 10^{-16} cm^2. However, since elastic scattering influences neither the positron numbers nor their energy, it does not have a substantial impact on the efficiency of the 2CDA mechanism.

Cross sections for atomic excitation by positron impact can be rather substantial. However, at the positron energies of interest, atomic excitation (from the ground state) is not allowed by the energy conservation.

In a collision between a positron and an atom, a positronium Ps can be formed. In particular, in collisions with Cs atoms the cross section for the formation of Ps (at positron energies of interest) is $\simeq 2 \times 10^{-16}$ cm^2 [37]. The mean-free path of a positron in a gas of Cs atoms with respect to this process would then be $\simeq 5 \times 10^3$ cm and $\simeq 5$ cm at $n_B \simeq 10^{12}$ cm^{-3} and $\simeq 10^{15}$ cm^{-3}, respectively. Thus, even though this channel reduces the number of positrons available for 2CDA, it is not expected to have a crucial impact on the efficiency of the 2CDA mechanism (unless the density of atoms B reaches quite high values). Besides, one should note that the formation of Ps does not close the pathway for the formation of $\bar{\mathrm{H}}^+$. Indeed, the latter can still be formed via the capture reaction $\bar{\mathrm{H}} + \mathrm{Ps} \rightarrow \bar{\mathrm{H}}^+ + \mathrm{e}^-$.

One more reaction, in which a positron can participate and which can preclude it to take part in the interaction with $\bar{\mathrm{H}}$, is related to the fact that a positron can form a bound state with some neutral atoms. We note, however, that positrons are not likely to form a bound state with Cs or Rb atoms [38].

The last channel to consider here is positron annihilation. Unlike the previous channels, annihilation of positrons would fully terminate the formation of $\bar{\mathrm{H}}^+$. However, this process has quite a low probability. Indeed, assuming that a positron annihilates mainly with the outer-shell atomic electrons via two-photon emission and that this process is essentially annihilation of a free positron-electron pair, we obtain that at the positron energies of interest the annihilation cross section is $\simeq 1.7 \times 10^{-22}$ cm^2 [39], i.e., is very small. In particular, the corresponding mean-free path in a gas of Cs atoms would exceed 1 km (at a density $n_B = 10^{15}$ cm^{-3}). Thus, positron annihilation is not expected to have any noticeable impact on the efficiency of the 2CDA mechanism.

Aside from free positrons, one should also consider what can occur with $\bar{\mathrm{H}}$ and $\bar{\mathrm{H}}^+$ which move in a gas of matter atoms. However, we could not locate in the literature any data for processes involving $\bar{\mathrm{H}}^+$ penetrating matter. Concerning $\bar{\mathrm{H}}$, at the impact energies of interest ($\simeq$10–150 eV/u) there exist, to our knowledge, only (theoretical) results for collisions of this antiatom with the simplest atoms and molecules (H, He, H_2, and H_2^+) (see recent review [41]). Based on them one could expect that at impact energies above 20–30 eV/u annihilation of antiprotons in such collisions will not be the main channel determining the losses of $\bar{\mathrm{H}}$ and $\bar{\mathrm{H}}^+$.

It also seems to be reasonable to assume that, due to much lower binding energy and much bigger size, the loss of $\bar{\mathrm{H}}^+$ in collisions will be substantially larger than that of $\bar{\mathrm{H}}$. Therefore, in order to have at least some (rather crude) estimate, we suppose that any collision with a matter atom, which occurs within the impact parameter range $0 \leqslant b \leqslant r_{\bar{\mathrm{H}}^+}$, where $r_{\bar{\mathrm{H}}^+} \simeq 4.26$ a.u. is the size of $\bar{\mathrm{H}}^+$, will lead to the destruction of this ion (due to one or other reason). This results in the total loss cross section of $\pi r_{\bar{\mathrm{H}}^+}^2 \simeq 1.6 \times 10^{-15}$ cm^2 and the corresponding mean-free path of $\simeq$500 cm and $\simeq$0.5 cm for $n_B \simeq 10^{12}$ cm^{-3} and $\simeq 10^{15}$ cm^{-3}, respectively.

D. Three-body attachment (3BA)

As was already mentioned in the Introduction, an incident positron can also form a bound state with $\bar{\mathrm{H}}$ by interacting with another free positron, which carries away the energy released in the attachment process (see Fig. 4).

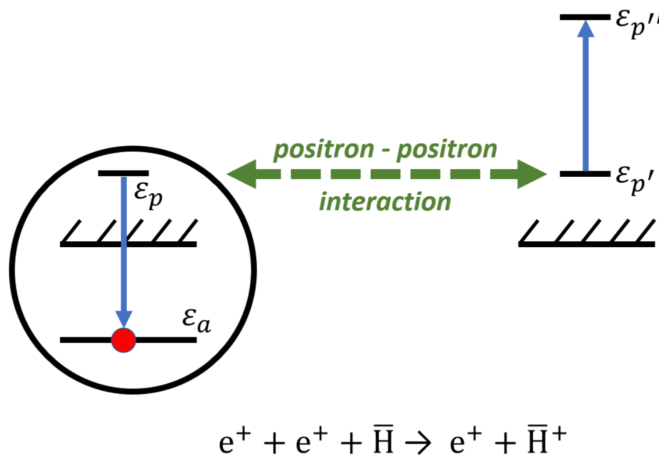

FIG. 4. Three-body attachment.

However, compared to the corresponding reaction resulting in the production of an antihydrogen atom, $\mathrm{e}^+ + \mathrm{e}^+ + \bar{p} \rightarrow \bar{\mathrm{H}} + \mathrm{e}^+$, the efficiency of this channel for formation of $\bar{\mathrm{H}}^+$ is strongly diminished by the mutual repulsion of the incident positrons, which is now not balanced by their attraction to the antiproton largely screened by the bound (anti)atomic positron, and also by the repulsion between the $\bar{\mathrm{H}}^+$ and the positron in the final state.

Further, the rate for the three-body attachment is proportional to the positron density squared. Therefore, one more point, which would limit in practice the efficiency of this channel, is that the highest reachable densities $\simeq(1-2) \times 10^8$ cm^{-3} are still relatively very low.

For instance, let us make an estimate by assuming that incident positrons have energy $\simeq$0.7 eV and fully ignoring the repulsions between the incident positrons in the initial state and between the positive anti-ion and the outgoing positron in the final state. According to such a crude approach, which strongly overestimates the production rate, this three-body attachment mechanism would outperform (one-center) radiative attachment starting with positron densities of $\simeq 2 \times 10^{12}$ cm^{-3}. For positrons, such densities are quite high and, at present, to our knowledge, cannot be realized in laboratories. At the highest reachable densities $\simeq(1-2) \times 10^8$ cm^{-3}, our (strongly overestimated) rate for this channel is still more than four orders of magnitude smaller than the rate for 1CRA.

E. Dileptonic three-body attachment

Taking into account that the reaction, briefly discussed in the previous subsection, does not seem to be efficient and that the cross section for positron-electron annihilation is quite low one could envisage dileptonic three-body attachment where the reaction (i) $\mathrm{e}^+ + \mathrm{e}^+ + \bar{\mathrm{H}} \rightarrow \bar{\mathrm{H}}^+ + \mathrm{e}^+$ is replaced by (ii) $\mathrm{e}^+ + \mathrm{e}^- + \bar{\mathrm{H}} \rightarrow \bar{\mathrm{H}}^+ + \mathrm{e}^-$ in which the "assisting" particle is a free electron instead of a free positron (see Fig. 5). We note that the latter would in some sense be similar to two-center dileptonic attachment in which the "assisting" particle is represented by a bound atomic electron.

Unlike two incident positrons, the incident positron and electron attract each other. Besides, in the final state instead of the repulsion between the $\bar{\mathrm{H}}^+ + \mathrm{e}^+$ one now has the attraction between $\bar{\mathrm{H}}^+ + \mathrm{e}^-$. Already, these two points are

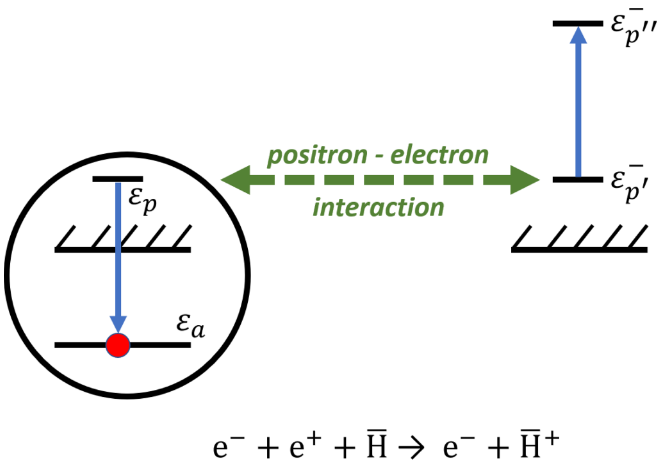

FIG. 5. Dileptonic three-body attachment.

expected to strongly increase the probability for three-body attachment [42]. Moreover, one can produce electron beams with much higher densities than the corresponding positron beams. Therefore, the reaction (ii) could have rates which are many orders of magnitude larger than those reachable for the reaction (i).

IV. CONCLUSIONS

We have considered the formation of positive ions of antihydrogen via radiative attachment of incident positrons to antihydrogen atoms. Three formation mechanisms were discussed in some detail with all of them involving photoemission as a common underlying feature.

The first of them is "simple" spontaneous radiative attachment of an incident positron to antihydrogen. This mechanism is driven by the interaction of the ($e^+ + \bar{H}$) system with the radiation field leading to spontaneous emission of a photon which carries away the energy released in the attachment process.

The second mechanism is laser-induced radiative attachment. It occurs in the presence of a relatively weak laser field which is resonant to positron transitions. The driving force of this mechanism is the interaction of the ($e^+ + \bar{H}$) system with the laser field and the process proceeds via induced photoemission.

The last radiative formation mechanism considered is two-center dileptonic attachment. It takes place when beams of positrons and antihydrogens cross in a gas of (matter) atoms B. Here, the attachment proceeds via (nearly) resonant transfer of energy from the ($e^+ + \bar{H}$) subsystem to atom B caused by the two-center dileptonic interaction. As a result, the formation of $\bar{H}^+$ is accompanied by excitation of B with its subsequent relaxation through spontaneous radiative decay. Thus, similar to the laser-induced attachment, the two-center dileptonic attachment is a resonant process (although its resonance nature is strongly affected by the relative motion of $\bar{H}$ and B). However, unlike the second (and the first) mechanism, the two-center one involves two interactions: the (Coulomb) interaction between ($e^+ + \bar{H}$) and atom B and the interaction of B with the radiation field.

We have shown that under certain conditions the second and third radiative mechanisms can strongly outperform the simple radiative attachment. Besides, according to our estimates, in the range of energies of the incident positrons $\varepsilon_p \simeq 1$ eV, which is most favorable for the radiative mechanisms, the (nonradiative) three-body attachment, where one of two positrons incident on $\bar{H}$ is attached whereas the other one carries away the energy excess, cannot compete with 1CRA even when the positron density is taken to be the highest possible for realization in laboratories. This, however, can drastically change if three-body attachment proceeds in an environment where, in addition to the antihydrogen and positron beams, one adds a beam of electrons with a density much higher than that available for positrons.

ACKNOWLEDGMENTS

We acknowledge the support from the CAS President's International Fellowship Initiative, the National Key Research and Development Program of China (Grant No. 2017YFA0402300) and the German Research Foundation (DFG) under Grant No. 349581371 (the projects MU 3149/4-1 and VO 1278/4-1). A.B.V. is grateful for the hospitality of the Institute of Modern Physics.

APPENDIX

Here, we discuss how formula (8) can be derived. In the dipole approximation, the magnetic field of the electromagnetic wave vanishes, the electric field is space independent, and the interaction of the positron with the field (in the velocity gauge) is given by $\hat{W} = -\frac{1}{c}\mathbf{A}(t)\cdot\hat{\mathbf{p}} + \frac{1}{2c^2}\mathbf{A}^2(t)$. Then, since the states φ_a and $\varphi_{\mathbf{p}}$ are orthogonal, one has $\langle\varphi_a|\hat{W}|\varphi_{\mathbf{p}}\rangle = -\frac{1}{c}\langle\varphi_a|\mathbf{A}(t)\cdot\hat{\mathbf{p}}|\varphi_{\mathbf{p}}\rangle$. Assuming, for definiteness, that the time dependence of the electric field $\mathbf{F}(t) = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$ is given by $\mathbf{F}(t) = \mathbf{F}_0 \sin\omega_0 t$ ($\mathbf{A}(t) = \frac{c\mathbf{F}_0}{\omega_0}\cos\omega_0 t$) and applying the rotating-wave approximation [27] to Eqs. (6) we obtain

$$i\frac{da}{dt} - \varepsilon_a\, a = \int d^3\mathbf{p}\, W_{a,\mathbf{p}}\, \exp(i\omega_0 t)\, b_{\mathbf{p}}(t),$$
$$i\frac{db_{\mathbf{p}}}{dt} - \varepsilon_p\, b_{\mathbf{p}} = W^*_{a,\mathbf{p}}\, \exp(-i\omega_0 t)\, a(t), \tag{A1}$$

where $W_{a,\mathbf{p}} = -\frac{1}{2\omega_0}\langle\varphi_a|\mathbf{F}_0\cdot\hat{\mathbf{p}}|\varphi_{\mathbf{p}}\rangle$ and $W^*_{a,\mathbf{p}}$ is the complex conjugate of $W_{a,\mathbf{p}}$. The system of equations (A1) with the assumption that the interaction with the field is switched on suddenly at $t = t_i = 0$ is conveniently solved by using the Laplace transformation

$$L(\lambda) = \int_0^{+\infty} dt\, f(t)\exp(-\lambda t),$$
$$f(t) = \frac{1}{2\pi i}\int_{\lambda_0 - i\infty}^{\lambda_0 + i\infty} d\lambda\, L(\lambda)\exp(\lambda t), \tag{A2}$$

where λ_0 is a real constant exceeding the real part of any of the singular points of $L(\lambda)$ (see, e.g., [43]).

Multiplying each of the equations in (A1) by $\exp(-\lambda t)$, integrating them over the time from 0 to $+\infty$ and making use of the initial conditions for $a(t)$ and $b_{\mathbf{p}}(t)$ [$a(t = 0) = 0$ and

$b_{\mathbf{p}}(t=0) = \delta^3(\mathbf{p}-\mathbf{p}_0)$] we obtain

$$(i\lambda - \varepsilon_a) L_a(\lambda) = \int d^3\mathbf{p}\, W_{a,\mathbf{p}}\, L_{\mathbf{p}}(\lambda - i\omega_0),$$
$$(i\lambda - \varepsilon_p) L_{\mathbf{p}}(\lambda) = i\,\delta^3(\mathbf{p}-\mathbf{p}_0) + W^*_{a,\mathbf{p}}\, L_a(\lambda + i\omega_0). \quad (A3)$$

Solving (A3) yields

$$L_a(\lambda) = \frac{iW_{a,\mathbf{p}_0}}{(i\lambda + \omega_0 - \varepsilon_{p_0})} \frac{1}{\left(i\lambda - \varepsilon_a - \int d^3\mathbf{p}\, \frac{|W_{a,\mathbf{p}}|^2}{i\lambda + \omega_0 - \varepsilon_p}\right)}. \quad (A4)$$

Using (A4) and the inverse Laplace transformation [given by the second line in (A2)] we get

$$a(t) = \frac{1}{2\pi i} \int_{\lambda_0 - i\infty}^{\lambda_0 + i\infty} d\lambda\, L_a(\lambda) \exp(\lambda t)$$
$$= \frac{iW_{a,\mathbf{p}_0}}{2\pi} \int_{-\infty + i0}^{+\infty + i0} dz\, \frac{\exp(-izt)}{z + \omega_0 - \varepsilon_{p_0}}$$
$$\times \frac{1}{z - \varepsilon_a - \int d^3\mathbf{p}\, \frac{|W_{a,\mathbf{p}}|^2}{z + \omega_0 - \varepsilon_p}}, \quad (A5)$$

where $z = -i\lambda$. The integral in (A5) is easily calculated by using the so-called pole approximation [44] in which $\int d^3\mathbf{p}\, \frac{|W_{a,\mathbf{p}}|^2}{x + \omega_0 - \varepsilon_p + i0} \approx \Delta_a - i\,\Gamma_a/2$, where Δ_a and Γ_a are the shift and the width of the resonance, respectively. They are assumed to be sufficiently weakly dependent on x such that $\Delta_a = \mathrm{P.V.} \int d^3\mathbf{p}\, \frac{|W_{a,\mathbf{p}}|^2}{\varepsilon_a + \omega_0 - \varepsilon_p}$ and $\Gamma_a = 2\pi \int d^2\Omega_{\mathbf{p}} |W_{a,\mathbf{p}}|^2_{p=|\mathbf{p}|=\sqrt{2(\varepsilon_a+\omega_0)}}$. Then, performing the integration in (A5) we obtain

$$a(t) = \frac{W_{a,\mathbf{p}_0}}{\varepsilon_{p_0} - \varepsilon_a - \omega_0 + i\Gamma_a/2} \{\exp[-i(\varepsilon_{p_0} - \omega_0)t]$$
$$- \exp(-i\varepsilon_a t - \Gamma_a t/2)\}, \quad (A6)$$

where the small shift Δ_a is assumed to be already included into the energy ε_a. Taking the absolute square of (A6), we arrive at expression (8).

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[44] The pole approximation has an excellent accuracy provided $\varepsilon_a + \omega_0 \gg \max\{\Delta_a, \Gamma_a\}$. It, in particular, is very often used when calculating the time development of various decay processes (for instance, the spontaneous radiative decay of excited atomic states).

Correction: Minor errors in Eqs. (12) and (18) have been fixed.