Symmetric informationally complete measurements identify the irreducible difference between classical and quantum systems

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We describe a general procedure for associating a minimal informationally complete quantum measurement with a purely probabilistic representation of the Born rule. Such representations provide a way to understand the Born rule as a consistency condition between probabilities assigned to the outcomes of one experiment in terms of the probabilities assigned to the outcomes of other experiments. In this setting, the difference between quantum and classical physics is the way their physical assumptions augment bare probability theory: Classical physics corresponds to a trivial augmentation—one just applies the law of total probability (LTP) between the scenarios—while quantum theory makes use of the Born rule expressed in one or another of the forms of our general procedure. To mark the irreducible difference between quantum and classical, one should seek the representations that minimize the disparity between the expressions. We prove that the representation of the Born rule obtained from a symmetric informationally complete measurement minimizes this distinction in at least two senses, the first to do with unitarily invariant distance measures between the rules and the second to do with available volume in a reference probability simplex (roughly speaking, a different kind of uncertainty principle). Both of these arise from a useful result in majorization theory. This work complements recent studies in quantum computation where the deviation of the Born rule from the LTP is measured in terms of negativity of Wigner functions.

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Quantum information theory represents a change of perspective. Rather than regarding quantum physics as a limitation on our abilities (the typical sentiment of older texts), we have learned that it can augment them. In frustrating some ambitions, it enables more subtle ones. Deviation from classicality is a *resource,* and the idea that this resource can be quantified as a modification of the classical probability calculus dates to the beginning of the field [1]. More recent inquiries have developed this notion precisely: The "negativity" in a Wigner-function representation of quantum states is now understood to be valuable in its own right [2–11]. However, what does this line of thinking say about quantum mechanics itself? Can one, following the lead of Carnot, take what might seem a statement of "mere" engineering and find a physical principle? In this paper, we prove some strong results in this regard in the context of finite-dimensional Hilbert spaces. In particular, we find the unique form of the quantum mechanical Born rule that makes it resemble the classical law of total probability (LTP) as closely as possible in at least two senses. Both come from a significant majorization result which may be of general interest for resource theory. This way of tackling the distinction between quantum and classical arises naturally in the quantum interpretive project of QBism [12,13], where the Born rule is seen as an empirically motivated constraint that one adds to probability theory when using it in the context of alternative (complementary) quantum experiments. We expect the techniques developed here to give an alternative way to explore the paradigm of negativity and to be of use for a range of practical problems.

The standard procedure in quantum theory for generating probabilities starts with an observer, or agent, assigning a quantum state ρ to a system. When the agent plans to measure the system, they represent the outcomes of their measurement with a positive-operator-valued measure (POVM) $\{D_j\}$. Assigning ρ implies that they assign the Born rule probabilities $Q(D_j) = \mathrm{tr}\rho D_j$ for the outcomes of their measurement. In this way, any quantum state ρ may be regarded as a compilation of probability distributions for all possible measurements. However, one does not have to consider all possible measurements to completely specify ρ. In fact, there exist measurements which are informationally complete (IC) in the sense that ρ is uniquely specified by the agent's expectations for the outcomes of that single measurement [14]. With respect to an IC measurement, any quantum state, pure or mixed, is equivalent to a single probability distribution. In this paper, we consider minimal informationally complete POVMs (MICs) for finite-dimensional quantum systems. These sets of operators form bases for the vector space of Hermitian operators and lead to probability distributions with the fewest number of entries necessary for reconstructing the quantum state. Minimal

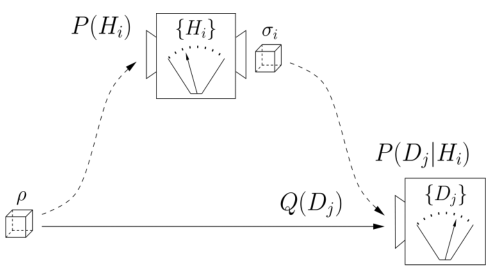

FIG. 1. Solid and dashed lines represent two hypothetical procedures an agent contemplates for a system assigned state ρ. The solid line represents making a direct measurement of a POVM $\{D_j\}$. The dotted line represents making the MIC $\{H_i\}$ first, preparing a postmeasurement state σ_i, and then finally making the $\{D_j\}$ measurement. For the solid path, the agent assigns one set of probabilities $Q(D_j)$. For the dotted path, the agent assigns two sets of probabilities $P(H_i)$ and $P(D_j|H_i)$. Unadorned by physical assumptions, probability theory does not suggest a relation between these paths. The Born rule in the form of Eq. (6) is such a relation.

informationally complete POVMs furnish a convenient way to bypass the language of quantum states, making quantum theory analogous to classical stochastic process theory, in which one puts probabilities in and gets probabilities out.

One can eliminate the need to use the operators ρ and D_j in the Born rule by reexpressing it as a relation between an agent's expectations for different experiments. Suppose our agent has a preferred reference process consisting of a measurement to which they ascribe the MIC $\{H_i\}$ and, upon obtaining outcome i, the preparation of a state σ_i, drawn from a linearly independent set of postmeasurement states $\{\sigma_i\}$. (See Fig. 1.) In their choice of this reference process, they require linearly independent postmeasurement states so that the inner products $\text{tr}D_j\sigma_i$ will uniquely characterize the operators D_j. Let $P(H_i)$ be their probabilities for the measurement $\{H_i\}$ and $P(D_j|H_i)$ be their conditional probabilities for a subsequent measurement of $\{D_j\}$. What consistency requirement among $Q(D_j)$, $P(H_i)$, and $P(D_j|H_i)$ does quantum physics entail?

Using the fact that $\{\sigma_i\}$ is a basis, we may write

$$\rho = \sum_j \alpha_j \sigma_j \tag{1}$$

for some set of real coefficients α_j. The probability of outcome H_i is then

$$P(H_i) = \sum_j \alpha_j \text{tr}H_i\sigma_j = \sum_j [\Phi^{-1}]_{ij}\alpha_j, \tag{2}$$

where we have defined the matrix Φ via its inverse

$$[\Phi^{-1}]_{ij} := \text{tr}H_i\sigma_j = h_i\text{tr}\rho_i\sigma_j \tag{3}$$

for $\rho_i := H_i/h_i$ and $h_i := \text{tr}H_i$. The invertibility of Φ is ensured by the linear independence of the MIC and postmeasurement sets. This implies that the coefficients of ρ in the σ_i basis may be written as an application of the Φ matrix on the vector of probabilities,

$$\rho = \sum_i \left[\sum_k [\Phi]_{ik}P(H_k)\right]\sigma_i. \tag{4}$$

The probability of D_j is given by another application of the Born rule, which becomes

$$Q(D_j) = \sum_{i=1}^{d^2}\left[\sum_{k=1}^{d^2}[\Phi]_{ik}P(H_k)\right]P(D_j|H_i), \tag{5}$$

where $P(D_j|H_i) = \text{tr}D_j\sigma_i$ is the probability for outcome D_j conditioned on obtaining H_i in the reference measurement. In more compact matrix notation, we can write

$$Q(D) = P(D|H)\Phi P(H), \tag{6}$$

where $P(D|H)$ is a matrix of conditional probabilities.

A symmetric IC POVM (SIC) [15–24] is a MIC for which all the H_i are rank 1 and

$$\text{tr}H_iH_j = \frac{1}{d^2}\frac{d\delta_{ij}+1}{d+1}. \tag{7}$$

Symmetric IC POVMs have yet to be proven to exist in all finite dimensions d, but they are widely believed to [23] have even been experimentally demonstrated in some low dimensions [25–28]. The *SIC projectors* associated with a SIC are the pure states $\rho_i = dH_i$. In dimension 2, a SIC can be represented as a regular tetrahedron inscribed in the Bloch sphere. (States defining a qubit SIC can be extracted from Ref. [29].) In higher dimensions, they are of course harder to visualize. When there is no chance of confusion, we will refer to the set of projectors as SICs as well. Prior work has given special attention to the reference procedure where the measurement and postmeasurement states are the same SIC [12,30,31]. In this case we denote Φ by Φ_{SIC} and Eq. (5) takes the particularly simple form

$$Q(D_j) = \sum_{i=1}^{d^2}\left[(d+1)P(H_i) - \frac{1}{d}\right]P(D_j|H_i). \tag{8}$$

Recall that the LTP expresses the simple consistency relation between the probabilities one assigns to the second of a sequence of measurements, the probabilities one assigns to the first, and the conditional probabilities for the second given the outcome of the first. Written in vector notation, this is

$$P(D) = P(D|H)P(H). \tag{9}$$

We write $P(D)$ as opposed to $Q(D)$ to indicate that it is the probability vector for the *second* of two measurements. On the other hand, $Q(D)$ is the vector of probabilities associated with a single measurement. Aside from the presence of Φ matrix, Eq. (6) is functionally equivalent to the LTP.

Although $P(H)$, $P(D|H)$, and $Q(D)$ are probabilities, $\Phi P(H)$ often is not. One may see by summing both sides of Eq. (5) over j that the vector is normalized, but in general it may contain negative numbers and values greater than 1. Such a vector is known as a quasiprobability, and matrices like Φ (real-valued matrices with columns summing to 1), which take probabilities to quasiprobabilities, are called column-quasistochastic matrices [32]. The subset of column-quasistochastic matrices with non-negative entries are the column-stochastic matrices. The inverse of a column-stochastic matrix is generally a column-quasistochastic matrix; in our case, inspection of Eq. (3) reveals that Φ^{-1} is column stochastic.

What would it mean if Φ could equal I? In this case we would have $Q(D) = P(D)$. Then, conceptually, it would not matter if the intermediate measurement were performed or not. Put another way, we could behave as though measurements simply revealed a preexisting property of the system, as in classical physics where measurements provide information about a system's coordinates in phase space.

Some amount of what makes quantum theory nonclassical resides in the fact that Φ cannot equal I. How close, then, can we make Φ to I by wisely choosing our MIC and postmeasurement states? It turns out that $\Phi_{\rm SIC}$ is closest to the identity with respect to the distance measure induced by any member of a large family of operator norms called unitarily invariant norms (see Sec. 3.5 in Ref. [33]). A unitarily invariant norm is one such that $\|A\| = \|UAV\|$ for all unitary matrices U and V. These norms include the Schatten p-norms (among which are the trace norm, the Frobenius norm, and the operator norm when $p = 1, 2$, and ∞, respectively) and the Ky Fan k-norms. This result codifies the intuition that Eq. (8) represents the "simplest modification one can imagine to the LTP" (see [34], p. 1971).

To prove this, we will make use of the theory of majorization [33,35]. Suppose x and y are vectors of N real numbers and that $x^\downarrow$ and $y^\downarrow$ are x and y sorted in nonincreasing order. Then we say that x weakly majorizes y from below, denoted by $x \succ_w y$, if

$$\sum_{i=1}^{k} x_i^\downarrow \geqslant \sum_{i=1}^{k} y_i^\downarrow \quad \text{for } k = 1, \dots, N. \tag{10}$$

If the last inequality is an equality, we say x majorizes y, denoted by $x \succ y$.

Another variant of majorization, called log-majorization or multiplicative majorization, is also studied [35]. We say that x weakly log-majorizes y from below, denoted by $x \succ_{w\log} y$, if

$$\prod_{i=1}^{k} x_i^\downarrow \geqslant \prod_{i=1}^{k} y_i^\downarrow \quad \text{for } k = 1, \dots, N. \tag{11}$$

If the last inequality is an equality, we say x log-majorizes y, denoted by $x \succ_{\log} y$. Taking the logarithm of both sides of Eq. (11) demonstrates that log-majorization is majorization between the vectors after an elementwise application of the log-map. Log-majorization is strictly stronger than regular majorization; $x \succ_{w\log} y \Rightarrow x \succ_w y$, but the reverse implication is not true. Majorization is a partial order on vectors of real numbers sorted in nonincreasing order.

Throughout this paper we will make use of the standard inequalities between the arithmetic, geometric, and harmonic means for vectors of n positive numbers x_i,

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geqslant \left(\prod_{i=1}^{n} x_i\right)^{1/n} \geqslant \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1}, \tag{12}$$

with equality in all cases if and only if $x_i = c$ for all i. We now turn to two lemmas.

Lemma 1. Let $\Phi_{\rm p}$ denote the column-quasistochastic matrix associated with a MIC and a proportional postmeasurement set. Then $\det \Phi_{\rm p} \geqslant \det \Phi_{\rm SIC}$ with equality if and only if the MIC is a SIC.

Proof. We may write $\Phi_{\rm p}^{-1} = GA^{-1}$, where $G_{ij} := {\rm tr} H_i H_j$ is the Gram matrix of the MIC elements and $A_{ij} := h_i \delta_{ij}$. Note that $\Phi_{\rm p}^{-1}$ has real positive eigenvalues because it has the same spectrum as the positive definite matrix $A^{-1/2} G A^{-1/2}$. Also note that

$$\sum_i \frac{1}{\lambda_i(\Phi_{\rm p})} = {\rm tr}\Phi_{\rm p}^{-1} = \sum_i h_i {\rm tr}\rho_i\sigma_i \leqslant \sum_i h_i = d. \tag{13}$$

One of the eigenvalues of $\Phi_{\rm p}$, which we denote by $\lambda_{d^2}(\Phi_{\rm p})$, must equal 1 because an equal-entry row vector is always a left eigenvector with eigenvalue 1 of a matrix with columns summing to unity. Therefore, we may write

$$\sum_{i<d^2} \frac{1}{\lambda_i(\Phi_{\rm p})} \leqslant d - 1. \tag{14}$$

The reciprocal of this expression is proportional to the harmonic mean of the first $d^2 - 1$ eigenvalues of $\Phi_{\rm p}$. Thus, because the geometric mean is always greater than or equal to the harmonic mean,

$$\left(\prod_{i=1}^{d^2-1} \lambda_i(\Phi_{\rm p})\right)^{1/(d^2-1)} \geqslant \left(\frac{1}{d^2-1}\sum_{i=1}^{d^2-1}\frac{1}{\lambda_i(\Phi_{\rm p})}\right)^{-1} \geqslant d+1, \tag{15}$$

which, noting that $\lambda_{d^2}(\Phi_{\rm p}) = 1$, implies

$$\det \Phi_{\rm p} \geqslant (d+1)^{d^2-1} = \det \Phi_{\rm SIC}. \tag{16}$$

Equality is achieved in this if and only if all the $\lambda_i(\Phi_{\rm p})$ are equal, so Eq. (16) is saturated if and only if $\lambda(\Phi_{\rm p}) = \lambda(\Phi_{\rm SIC})$. We next show this implies that in fact the MIC is a SIC.

For any $\Phi_{\rm p}^{-1}$, we may write $\Phi_{\rm p}^{-1} = P^{-1} D P$, where the rows of P are the left eigenvectors of $\Phi_{\rm p}^{-1}$ and D is the diagonal matrix of eigenvalues of $\Phi_{\rm p}^{-1}$. Since $\Phi_{\rm p}^{-1}$ is column stochastic, the row vector $(1/d, \dots, 1/d)$ is the (scaled) left eigenvector of $\Phi_{\rm p}^{-1}$ with eigenvalue 1 and so it is the first row of P when the eigenvalues are in descending order. Left eigenvectors of a matrix are right eigenvectors of the transpose of the matrix, so we have

$$\begin{aligned} \left(\Phi_{\rm p}^{-1}\right)^T |v\rangle &= A^{-1}G|v\rangle = A^{-1}GA^{-1}A|v\rangle \\ &= A^{-1}\Phi_{\rm p}^{-1}A|v\rangle = \lambda|v\rangle, \\ &\Rightarrow \Phi_{\rm p}^{-1}A|v\rangle = \lambda A|v\rangle, \end{aligned} \tag{17}$$

where $\langle v|$ is an arbitrary left eigenvector of $\Phi_{\rm p}^{-1}$. Combined with our choice of scale for the first row of P, we conclude that the first column of P^{-1} is $(h_1, h_2, \dots, h_{d^2})^T$.

Now suppose $\Phi_{\rm p}$ is such that $\lambda(\Phi_{\rm p}) = \lambda(\Phi_{\rm SIC})$. Then $G = P^{-1}DPA$, where $[D]_{ij} = \frac{1}{d+1}(\delta_{ij} + d\delta_{i1}\delta_{j1})$ and

$$\begin{aligned} [G]_{ij} &= \sum_{klm} [P^{-1}]_{ik}[D]_{kl}[P]_{lm}[A]_{mj} \\ &= \sum_{klm} [P^{-1}]_{ik}\left[\frac{1}{d+1}(\delta_{kl} + d\delta_{k1}\delta_{l1})\right][P]_{lm}\delta_{mj}h_m \\ &= \frac{1}{d+1}\sum_{kl} [P^{-1}]_{ik}(\delta_{kl} + d\delta_{k1}\delta_{l1})[P]_{lj}h_j \end{aligned}$$

$$= \frac{1}{d+1}(h_j\delta_{ij} + dh_j[P^{-1}]_{i1}[P]_{1j})$$
$$= \frac{1}{d+1}(h_j\delta_{ij} + h_ih_j). \tag{18}$$

In the last step we used that $[P]_{1j} = 1/d$ and $[P^{-1}]_{i1} = h_i$. If this Gram matrix comes from a MIC, one may use

$$[G]_{ii} = h_i^2 \mathrm{tr}\rho_i^2 = \frac{1}{d+1}\left(h_i + h_i^2\right) \tag{19}$$

and the fact that $\mathrm{tr}\rho_i \leqslant 1$ to show that $h_i \geqslant 1/d$. As the average h_i value must be $1/d$, this implies that $h_i = 1/d$ for all i and furthermore that each ρ_i is rank 1. Substituting this into Eq. (18) gives

$$[G]_{ij} = \frac{d\delta_{ij} + 1}{d^2(d+1)}, \tag{20}$$

that is, the MIC is a SIC and $\Phi_{\rm p} = \Phi_{\rm SIC}$. ■

Let $s(A)$ denote the vector of singular values of the matrix A in nonincreasing order. The proof of the following lemma may be found in Appendix A.

Lemma 2. Let Φ be the column-quasistochastic matrix associated with an arbitrary reference process. Then

$$s(\Phi) \succ_{w\,\log} s(\Phi_{\rm SIC}) \tag{21}$$

with equality if and only if the MIC and postmeasurement states are SICs.

We are now poised to prove the following.

Theorem 1. Let Φ be the column-quasistochastic matrix associated with an arbitrary reference process. Then for any unitarily invariant norm $\|\cdot\|$,

$$\|I - \Phi\| \geqslant \|I - \Phi_{\rm SIC}\|, \tag{22}$$

with equality if and only if the MIC and postmeasurement states are SICs.

Proof. By Corollary 3.5.9 in Ref. [33], every unitarily invariant norm is monotone with respect to the partial order on matrices induced by weak majorization of the vector of singular values. Further, $I - \Phi$ is singular with exactly one eigenvalue equal to zero, so one of its singular values is zero as well. Then

$$s(I-\Phi) \succ \left\{\frac{\sum_i s_i(I-\Phi)}{d^2-1}, \dots, \frac{\sum_i s_i(I-\Phi)}{d^2-1}\right\}$$
$$\succ_w \{d, \dots, d\} = s(I - \Phi_{\rm SIC}) \tag{23}$$

if

$$\sum_i s_i(I-\Phi) \geqslant d(d^2-1). \tag{24}$$

We have

$$\sum_i s_i(I-\Phi) \geqslant \sum_i |\lambda_i(I-\Phi)| = \sum_i |\lambda_i(\Phi) - 1|$$
$$\geqslant \sum_i [|\lambda_i(\Phi)| - 1] \geqslant \sum_i \lambda_i(\Phi_{\rm SIC}) - d^2$$
$$= d(d^2-1), \tag{25}$$

where the first inequality follows Eq. (3.3.13a) in Ref. [33], the second follows from the triangle inequality, and the last follows from Lemma 2. ■

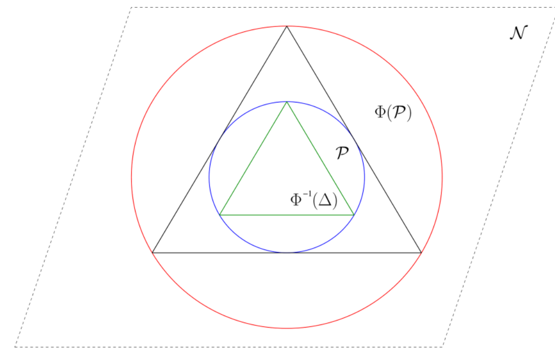

FIG. 2. Here $\mathcal{N}$ is the normalized hyperplane of d^2-element quasiprobability vectors and the black outer triangle represents the (d^2-1)-simplex Δ of probabilities. For a given MIC, the green inner triangle is the simplex $\Phi^{-1}(\Delta)$, the blue circle is the image of $\mathcal{Q}_d$ under the Born rule, denoted by $\mathcal{P}$, and the red circle is $\Phi(\mathcal{P})$. Here $\mathcal{P}$ and $\Phi(\mathcal{P})$ are portrayed with circles to capture convexity and inclusion relationships only; they need not bear any resemblance to spheres.

It is known that no quasiprobability representation of quantum theory can be entirely non-negative [36]. What does this mean in our formalism?

Let $\mathcal{N}$ be the normalized hyperplane of d^2-element quasiprobability vectors. Within this is the (d^2-1)-simplex of probability vectors Δ. For any MIC, d-dimensional quantum state space $\mathcal{Q}_d$ is mapped by the Born rule to a convex subset of Δ, denoted by $\mathcal{P}$. Note that $\Phi^{-1}(\Delta)$ is equal to the convex hull of the d^2 probability vectors $\mathrm{tr}H_j\sigma_i$, that is, the probabilities for the MIC for each postmeasurement state. Consequently, $\Phi^{-1}(\Delta) \subset \mathcal{P}$, which implies $\Delta \subset \Phi(\mathcal{P})$. These inclusions must be strict, i.e., $\Phi \neq I$: When the MIC and postmeasurement states are rank 1, the vertices of the simplex will be among the pure-state probability vectors, but $\mathcal{P}$ contains more pure states than there are vertices of $\Phi^{-1}(\Delta)$. Since the image of some probability vectors consistent with quantum theory must leave the probability simplex under the application of Φ, we have demonstrated that the appearance of negativity is unavoidable in our framework and is in fact characterized by the fact that Φ cannot equal the identity. Figure 2 illustrates the situation.

The weak-log-majorization result of Lemma 2 has at least one more important implication for quantifying the quantum deviation from classicality. Instead of looking at the functional form of Eq. (6) and considering how much of a deviation from the LTP it represents, one may approach the problem from a geometric perspective.

Classically, one can always imagine assigning probability 1 to an outcome of a putative maximally informative measurement, for instance, when one knows the system's exact phase-space point. However, in an interpretation of quantum theory without hidden variables, whatever one might mean by "maximally informative," one cannot mean that the reference measurement's full probability simplex is available. Indeed, quantum mechanics does not allow probability 1 for the outcome of any MIC [37]. Thus deviation from classicality can also be captured by the fact that the region of probabilities compatible with quantum states is strictly smaller than the full (d^2-1)-simplex. In this setting, the irreducible deviation from classicality is defined by the largest possible region for a reference measurement's probability simplex. The following theorem establishes that a SIC uniquely maximizes the Euclidean volume of this region, thereby answering a question raised in Ref. [34] (see pp. 475 and 571 therein).

Theorem 2. For any MIC in dimension d, let $\mathcal{P}$ denote the image of $\mathcal{Q}_d$ under the Born rule and let $\mathrm{vol}_{\mathrm{E}}(\mathcal{P})$ denote its Euclidean volume. Then

$$\mathrm{vol}_{\mathrm{E}}(\mathcal{P}) \leqslant \mathrm{vol}_{\mathrm{E}}(\mathcal{P}_{\mathrm{SIC}}) \tag{26}$$

with equality if and only if the MIC is a SIC. Furthermore,

$$\mathrm{vol}_{\mathrm{E}}(\mathcal{P}_{\mathrm{SIC}}) = \sqrt{\frac{(2\pi)^{d(d-1)}}{d^{d^2-2}(d+1)^{d^2-1}}\frac{\Gamma(1)\cdots\Gamma(d)}{\Gamma(d^2)}}. \tag{27}$$

The proof of Theorem 2 involves methods of differential geometry which would be distracting here. We direct the interested reader to Appendix B for details.

The (d^2-1)-simplex Δ has Euclidean volume [38]

$$\mathrm{vol}_{\mathrm{E}}(\Delta) = \frac{d}{\Gamma(d^2)}, \tag{28}$$

so we can calculate the ratio of the Euclidean volumes of $\mathcal{P}_{\mathrm{SIC}}$ and the simplex it lies within,

$$\frac{\mathrm{vol}_E(\mathcal{P}_{\mathrm{SIC}})}{\mathrm{vol}_{\mathrm{E}}(\Delta)} = \sqrt{\frac{(2\pi)^{d(d-1)}}{d^{d^2}(d+1)^{d^2-1}}\Gamma(1)\cdots\Gamma(d)}. \tag{29}$$

When $d=2$, quantum state space is the Bloch ball and $\mathcal{P}_{\mathrm{SIC}}$ is the largest ball which can be inscribed in the regular tetrahedron Δ_3,

$$\frac{\mathrm{vol}_{\mathrm{E}}(\mathcal{P}_{\mathrm{SIC}})}{\mathrm{vol}_{\mathrm{E}}(\Delta_3)} = \frac{\pi}{6\sqrt{3}} \approx 0.3023. \tag{30}$$

When $d=3$,

$$\frac{\mathrm{vol}_{\mathrm{E}}(\mathcal{P}_{\mathrm{SIC}})}{\mathrm{vol}_{\mathrm{E}}(\Delta_8)} = \frac{\pi^3}{1296\sqrt{3}} \approx 0.0138. \tag{31}$$

In general, the ratio is very rapidly decreasing, signifying a greater and greater deviation from classicality with each Hilbert space dimension.

Theorems 1 and 2 show that the SICs provide a way of casting the Born rule in wholly probabilistic terms, which by two different standards make the difference between classical and quantum as small as possible. Of all the representations deriving from our general procedure, the representation given by Eq. (8) is the essential one for specifying how quantum is quantum.

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APPENDIX A: PROOF OF LEMMA 2

For a MIC $\{E_i\}$ and a postmeasurement set $\{\sigma_j\}$,

$$[\Phi^{-1}]_{ij} = \mathrm{tr} E_i \sigma_j. \tag{A1}$$

The elements of the MIC may be expanded in the SIC basis

$$E_i = \sum_k [\alpha]_{ik} H_k, \tag{A2}$$

so we may write

$$[\Phi^{-1}]_{ij} = \sum_k [\alpha]_{ik} \mathrm{tr} H_k \sigma_j = \sum_k [\alpha]_{ik} p(k|j), \tag{A3}$$

where $p(k|j)$ is the probabilistic representation of the state σ_j with respect to the SIC $\{H_k\}$. The α matrix must be invertible because it is a transformation between two bases, so the probability vectors can be written

$$p(i|j) = \sum_k [\alpha^{-1}]_{ik} [\Phi^{-1}]_{kj}. \tag{A4}$$

We know that SIC probability vectors satisfy [12]

$$\sum_i p(i|j)^2 \leqslant \frac{2}{d(d+1)} \,\forall j, \tag{A5}$$

so we have

$$\sum_i \left(\sum_k [\alpha^{-1}]_{ik}[\Phi^{-1}]_{kj}\right)^2 \leqslant \frac{2}{d(d+1)} \,\forall j. \tag{A6}$$

Summing over j, we then have

$$\sum_{ij} \left(\sum_k [\alpha^{-1}]_{ik}[\Phi^{-1}]_{kj}\right)^2 \leqslant \frac{2d}{d+1}. \tag{A7}$$

This expression is the sum of the absolute square entries of a matrix, which is equivalent to the square of the Frobenius norm of the matrix

$$\|\alpha^{-1}\Phi^{-1}\|_2^2 = \sum_i s^2(\alpha^{-1}\Phi^{-1}) \leqslant \frac{2d}{d+1}. \tag{A8}$$

From Eq. (3.1.11) in Ref. [33], for any square matrix A,

$$\sum_i |\lambda_i(A)|^2 \leqslant \sum_i \varsigma^2(A), \tag{A9}$$

so we have a general bound on the absolute squared spectrum

$$\sum_i |\lambda_i(\alpha^{-1}\Phi^{-1})|^2 \leqslant \frac{2d}{d+1}. \tag{A10}$$

Equation (A4) shows that $\alpha^{-1}\Phi^{-1}$ is column stochastic and thus that one of its eigenvalues is 1, so we may write

$$\sum_{i>1} |\lambda_i(\alpha^{-1}\Phi^{-1})|^2 \leqslant \frac{2d}{d+1} - 1 = \frac{d-1}{d+1}. \tag{A11}$$

Now, using the arithmetic-geometric mean inequality,

$$\begin{aligned}\frac{d-1}{d+1} &\geqslant \sum_{i>1} |\lambda_i(\alpha^{-1}\Phi^{-1})|^2 \\ &\geqslant (d^2-1)\left(\prod_{i>1} |\lambda_i(\alpha^{-1}\Phi^{-1})|^2\right)^{1/(d^2-1)} \\ &= (d^2-1)|\det \alpha^{-1}\Phi^{-1}|^{2/(d^2-1)},\end{aligned} \tag{A12}$$

which implies

$$|\det \alpha^{-1}\Phi^{-1}| \leqslant \left(\frac{d-1}{(d+1)(d^2-1)}\right)^{(d^2-1)/2} = \left(\frac{1}{d+1}\right)^{d^2-1} = \det \Phi_{\mathrm{SIC}}^{-1}. \tag{A13}$$

From Eq. (A2) we can write

$$\mathrm{tr} E_i E_j = \sum_{kl} \alpha_{ik}\alpha_{jl} \mathrm{tr} H_k H_l \iff G = \alpha G_{\mathrm{SIC}} \alpha^T \iff \det G = (\det \alpha)^2 \det G_{\mathrm{SIC}}, \tag{A14}$$

where G is the MIC Gram matrix and G_{SIC} is the SIC Gram matrix. Recall the definition of the A matrix from the proof of Lemma 1. The arithmetic-geometric mean inequality shows $\det A \leqslant (1/d)^{d^2}$ with equality if and only if $h_i = 1/d$. Then, since $G = \Phi_{\mathrm{p}}^{-1} A$, Lemma 1 shows

$$\det G = \left(\det \Phi_{\mathrm{p}}^{-1}\right)(\det A) \leqslant \left(\det \Phi_{\mathrm{SIC}}^{-1}\right)(1/d)^{d^2} = \det G_{\mathrm{SIC}} \tag{A15}$$

with equality if and only if the MIC is a SIC. This implies $(\det \alpha)^2 \leqslant 1$, and so $|\det \alpha| \leqslant 1$. Since $|\det \alpha^{-1}\Phi^{-1}| = |\det \alpha^{-1}||\det \Phi^{-1}|$, we conclude that

$$|\det \Phi^{-1}| \leqslant \det \Phi_{\mathrm{SIC}}^{-1}. \tag{A16}$$

Equivalently, $\det \Phi_{\mathrm{SIC}} \leqslant |\det \Phi|$. Theorem 3.3.2 in Ref. [33] shows $s(A) \succ_{\log} |\lambda(A)|$ for an arbitrary matrix A. To show the desired weak-log-majorization result, we wish to prove $|\lambda(\Phi)| \succ_{w\log} \lambda(\Phi_{\mathrm{SIC}})$. For this we show weak majorization of the logarithm of the entries:

$$\log|\lambda(\Phi)| \succ \left(\frac{\sum_{i=1}^{d^2}\log|\lambda_i(\Phi)|}{d^2-1}, \dots, \frac{\sum_{i=1}^{d^2}\log|\lambda_i(\Phi)|}{d^2-1}, 0\right) = \left(\frac{\log|\det\Phi|}{d^2-1}, \dots, \frac{\log|\det\Phi|}{d^2-1}, 0\right)$$
$$\succ_w \left(\frac{\log\det\Phi_{\mathrm{SIC}}}{d^2-1}, \dots, \frac{\log\det\Phi_{\mathrm{SIC}}}{d^2-1}, 0\right) = (\log(d+1), \dots, \log(d+1), 0) = \lambda(\log\Phi_{\mathrm{SIC}}). \tag{A17}$$

Thus,

$$s(\Phi) \succ_{\log} |\lambda(\Phi)| \succ_{w\log} \lambda(\Phi_{\mathrm{SIC}}) = s(\Phi_{\mathrm{SIC}}). \tag{A18}$$

If $\{H_i\}$ and $\{\sigma_j\}$ are SICs, $\Phi_{ij}^{-1} = \frac{1}{d}\mathrm{tr}\Pi_i\Pi'_j$, where $\{\Pi_i\}$ and $\{\Pi'_j\}$ are SIC projectors in dimension d. Then

$$[\Phi^{-1}\Phi^{-1\dagger}]_{ij} = \frac{1}{d^2}\sum_k (\mathrm{tr}\Pi_i\Pi'_k)(\mathrm{tr}\Pi_j\Pi'_k) = \frac{1}{d^2}\mathrm{tr}\left[(\Pi_i\otimes\Pi_j)\left(\sum_k \Pi'_k\otimes\Pi'_k\right)\right]$$
$$= \frac{1}{d^2}\mathrm{tr}\left[(\Pi_i\otimes\Pi_j)\left(\frac{2d}{d+1}P_{\mathrm{sym}}\right)\right] = \frac{1}{d(d+1)}\mathrm{tr}\left[(\Pi_i\otimes\Pi_j)\left(I\otimes I + \sum_{kl}^d |k\rangle\langle l|\otimes|l\rangle\langle k|\right)\right]$$
$$= \frac{1+\mathrm{tr}\Pi_i\Pi_j}{d(d+1)} = \frac{d\delta_{ij}+d+2}{d(d+1)^2} = \left[\Phi_{\mathrm{SIC}}^{-2}\right]_{ij}, \tag{A19}$$

where P_{sym} is the projector onto the symmetric subspace of $\mathcal{H}_d^{\otimes 2}$ and in the third step we employed the fact that the SICs form a minimal 2-design [16]. This shows that the modulus of Φ is equal to Φ_{SIC} and thus the singular values of Φ and Φ_{SIC} coincide.

On the other hand, suppose $s(\Phi) = s(\Phi_{\mathrm{SIC}})$. The product of all the singular values is the absolute value of the determinant [33], so $|\det\Phi^{-1}| = \det\Phi_{\mathrm{SIC}}^{-1} \Rightarrow |\det\alpha| = 1 \Rightarrow \det G = \det G_{\mathrm{SIC}} \iff \{E_i\}$ is a SIC. Carrying through the consequences of the MIC being a SIC allows us to see from Eq. (A6) that σ_j is rank 1 because the upper bound is saturated for SIC probability vectors. We may expand the $\{\sigma_j\}$ in the SIC projector basis,

$$\sigma_j = \sum_k [\beta]_{jk}\Pi_k. \tag{A20}$$

Acting on both sides by a SIC element and computing the trace of both sides, we see

$$[\Phi^{-1}]_{ij} = \mathrm{tr}E_i\sigma_j = \sum_k [\beta]_{jk}\mathrm{tr}E_i\Pi_k = \left[\Phi_{\mathrm{SIC}}^{-1}\beta^T\right]_{ij}, \tag{A21}$$

so $|\det\Phi^{-1}| = |\det\Phi_{\mathrm{SIC}}^{-1}||\det\beta^T| = \det\Phi_{\mathrm{SIC}}^{-1}$ implies $|\det\beta| = 1$. Denoting the Gram matrix of states by g, we have, in the same way as before,

$$\det g = (\det\beta)^2\det g_{\mathrm{SIC}} = \det g_{\mathrm{SIC}}. \tag{A22}$$

We now prove that $\det g = \det g_{\mathrm{SIC}}$ implies that the basis of projectors forms a SIC. The following lemma is due to Zhu [39]. We only use part of Zhu's conclusion, but the lemma is of enough interest to present in full.

Lemma 3. Let λ be the spectrum of the Gram matrix g of a normalized basis of positive-semidefinite operators Π_j sorted in nonincreasing order. Then $\lambda \succ \lambda_{\rm SIC}$ with equality if and only if Π_j forms a SIC.

Proof. By assumption $\mathrm{tr}\Pi_j^2 = 1$ for all j. Since the eigenvalues of Π_j are non-negative,

$$1 = \mathrm{tr}\Pi_j^2 = \sum_i \lambda_i^2(\Pi_j) \leqslant \sum_i \lambda_i(\Pi_j) = \mathrm{tr}\Pi_j. \tag{A23}$$

Define the frame superoperator

$$\mathcal{F} = \sum_j \|\Pi_j\rangle\!\rangle\langle\!\langle\Pi_j\|, \tag{A24}$$

where $\|A\rangle\!\rangle := \sum_{ij}[A]_{ij}|i\rangle|j\rangle$. In addition, $\mathcal{F}$ has the same spectrum as the Gram matrix $[g]_{ij} = \langle\!\langle\Pi_i\|\Pi_j\rangle\!\rangle = \mathrm{tr}\Pi_i\Pi_j$. To see this, form a projector out of the state $\sum_i \|\Pi_i\rangle\!\rangle|i\rangle$, where $|i\rangle$ is an orthonormal basis in $\mathcal{H}_{d^2}$, and perform partial traces over each subsystem. The results are $g^{\rm T}$ and $\mathcal{F}$, and so, by the Schmidt theorem, the spectra of $\mathcal{F}$ and g are equal: $\lambda(g) = \lambda(\mathcal{F}) = \lambda$.

The expectation value of any operator with respect to an arbitrary normalized state is less than or equal to its maximal eigenvalue. Thus, a lower bound on the maximal eigenvalue λ_1 of $\mathcal{F}$ is given by

$$\lambda_1 \geqslant \frac{1}{d}\langle\!\langle I\|\mathcal{F}\|I\rangle\!\rangle = \frac{1}{d}\sum_j(\mathrm{tr}\Pi_j)^2 \geqslant d. \tag{A25}$$

As our basis is normalized, $\mathrm{tr}g = d^2$, so $\sum_i \lambda_i = d^2$. With this constraint and our bound on the maximal eigenvalue, we have

$$\lambda \succ \left(\lambda_1, \frac{d^2-\lambda_1}{d^2-1}, \dots, \frac{d^2-\lambda_1}{d^2-1}\right) \succ \left(d, \frac{d}{d+1}, \dots, \frac{d}{d+1}\right) = \lambda_{\rm SIC}. \tag{A26}$$

The second majorization becomes an equality when $\lambda_1 = d$. From Eq. (A25) we can see that all Π_j must be rank 1 for this condition to be satisfied. Furthermore, we see that in this case $\frac{1}{\sqrt{d}}\|I\rangle\!\rangle$ is an eigenvector of $\mathcal{F}$ which achieves the maximal eigenvalue d. When both majorizations are equalities the spectrum $\lambda_{\rm SIC}$ tells us that $\mathcal{F}$ takes the form of a weighted sum of a projector and the identity superoperator $\mathbf{I}$, specifically

$$\mathcal{F} = \frac{d}{d+1}(\mathbf{I} + \|I\rangle\!\rangle\langle\!\langle I\|). \tag{A27}$$

By Corollary 1 in Ref. [40], this implies the Π_j form a SIC.■

As in Lemma 3, denote by λ the spectrum of g sorted in nonincreasing order. Here $\mathrm{tr}g = d^2$, so

$$\sum_{i>1}\lambda_i = d^2 - \lambda_1. \tag{A28}$$

Then because the arithmetic mean is greater than or equal to the geometric mean with equality if and only if the elements are all equal, we have

$$\frac{1}{d^2-1}\sum_{i>1}\lambda_i = \frac{d^2-\lambda_1}{d^2-1} \geqslant \left(\prod_{i>1}\lambda_i\right)^{1/(d^2-1)}, \tag{A29}$$

which implies

$$\det g \leqslant \lambda_1\left(\frac{d^2-\lambda_1}{d^2-1}\right)^{d^2-1}, \tag{A30}$$

with equality if and only if $\lambda_2 = \cdots = \lambda_d^2 = \frac{d^2-\lambda_1}{d^2-1}$. When $\lambda_1 = d$, we then have

$$\det g = \frac{d^{d^2}}{(d+1)^{d^2-1}} = \det g_{\rm SIC}, \tag{A31}$$

with equality if and only if $\lambda = \lambda_{\rm SIC}$. By Lemma 3 we have equality if and only if the postmeasurement states form a SIC.

APPENDIX B: PROOF OF THEOREM 2

Equation (4) expanded instead in the ρ_i basis allows us to relate the differential elements of operator space and probability space for any MIC basis:

$$d\sigma = \sum_{i,j}[\Phi]_{ij}\rho_i dp^j. \tag{B1}$$

The Hilbert-Schmidt line element is then

$$ds_{\rm HS}^2 = \mathrm{tr}(d\sigma)^2 = \sum_{ijkl}[\Phi]_{ij}[\Phi]_{kl}(\mathrm{tr}\rho_i\rho_k)dp^j dp^l. \tag{B2}$$

As in the proof of Lemma 1, we write $\Phi = AG^{-1}$, where $[G]_{ij} = \mathrm{tr}H_iH_j$ is the Gram matrix for the MIC and $[A]_{ij} = h_i\delta_{ij}$. Note further that $\mathrm{tr}\rho_i\rho_j = [A^{-1}GA^{-1}]_{ij}$. Then Eq. (B2) simplifies to

$$ds_{\rm HS}^2 = \sum_{ij}[G^{-1}]_{ij}dp^i dp^j. \tag{B3}$$

The Hilbert-Schmidt volume element on the space of Hermitian operators in $\mathcal{L}(\mathcal{H}_d)$ may now be related to the Euclidean volume element in $\mathbb{R}^{d^2}$,

$$d\Omega_{\rm HS} = \sqrt{|\det G^{-1}|}dV_{\rm E}, \tag{B4}$$

or, equivalently,

$$dV_{\rm E} = \sqrt{|\det G|}d\Omega_{\rm HS}. \tag{B5}$$

The larger the $\det G$, the larger the corresponding Euclidean volume. Recall Eq. (A15), which says

$$\det G \leqslant \det G_{\rm SIC}, \tag{B6}$$

with equality if and only if the MIC is a SIC. Thus, for any region in operator space, the Euclidean volume is maximal with respect to the SIC basis. In particular, the SIC basis gives the largest volume among positive-semidefinite operators A satisfying $1-\epsilon \leqslant \mathrm{tr}A \leqslant 1+\epsilon$ for any $\epsilon > 0$. As $\epsilon \to 0$, we obtain quantum state space $\mathcal{Q}_d$ and the corresponding region in $\mathbb{R}^{d^2}$ will have the largest hyperarea within Δ when computed with the SIC basis.

To calculate this hyperarea, we need to find the metric on Δ induced by the Hilbert-Schmidt metric in the SIC basis. We may parametrize Δ by

$$X = \left(p^1, \dots, p^{d^2-1}, 1 - \sum_{i=1}^{d^2-1}p^i\right), \tag{B7}$$

which has partial derivatives $\partial_i X^\mu = \delta_i^\mu - \delta_{d^2}^\mu$ where the latin index runs from 1 to d^2-1 and the greek index runs from 1 to d^2. For any MIC, the induced metric g is given by

$$[g]_{ij} = \sum_{\mu,\nu=1}^{d^2} \partial_i X^\mu \partial_j X^\nu [G^{-1}]_{\mu\nu}. \tag{B8}$$

It is easily seen that $G_{\rm SIC}^{-1} = d(d+1)I - J$, where J is the Hadamard identity. One may then calculate $g_{\rm SIC} = d(d+1)(I+J)$ and $\det g_{\rm SIC} = d^2(d^2+d)^{d^2-1}$. The induced volume element on Δ is then

$$d\omega_{\rm HS} = d\sqrt{(d^2+d)^{d^2-1}}\,dp^1\cdots dp^{d^2-1}. \tag{B9}$$

In a similar way, it may be checked that the Euclidean metric in $\mathbb{R}^{d^2}$ induces a volume element $d\mathcal{A}_{\rm E}$ on Δ satisfying

$$\frac{1}{d}d\mathcal{A}_{\rm E} = dp^1\cdots dp^{d^2-1}, \tag{B10}$$

and so

$$d\omega_{\rm HS} = \sqrt{(d^2+d)^{d^2-1}}\,d\mathcal{A}_{\rm E}. \tag{B11}$$

We may now integrate over quantum state space to obtain

$$\mathrm{vol}_{\rm HS}(\mathcal{Q}_d) = \sqrt{(d^2+d)^{d^2-1}}\,\mathrm{vol}_{\rm E}(\mathcal{P}_{\rm SIC}). \tag{B12}$$

Życzkowski and Sommers [41] calculate the Hilbert-Schmidt volume of finite-dimensional quantum state space to be

$$\mathrm{vol}_{\rm HS}(\mathcal{Q}_d) = \sqrt{d}(2\pi)^{d(d-1)/2}\frac{\Gamma(1)\cdots\Gamma(d)}{\Gamma(d^2)}, \tag{B13}$$

from which Eq. (27) follows.

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