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Rapid Communications

All tight correlation Bell inequalities have quantum violations

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It is by now well established that there exist nonlocal games for which the best entanglement-assisted performance is not better than the best classical performance. Here we show in contrast that any two-player XOR game, for which the corresponding Bell inequality is tight, has a quantum advantage. In geometric terms, this means that any correlation Bell inequality for which the classical and quantum maximum values coincide, does not define a facet, i.e., a face of maximum dimension, of the local (Bell) polytope. Indeed, using semidefinite programming duality, we prove upper bounds on the dimension of these faces, bounding it far away from the maximum. In the special case of nonlocal computation games, it had been shown before that they are not facet defining; our result generalizes and improves this. As a by-product of our analysis, we find a similar upper bound on the dimension of the faces of the convex body of quantum correlation matrices, showing that (except for the trivial ones expressing the non-negativity of probability) it does not have facets.

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Introduction. In 1964, Bell [1] proved that some predictions of quantum theory regarding the correlations between distant events cannot be explained by any classical, i.e., local realistic theory. He derived a simple observable criterion that any classical theory must obey, and showed that particular measurements performed by two parties on a maximally entangled state could violate it. What we now call a Bell inequality was introduced in Ref. [2], as an upper bound on a single linear function of observable probabilities, i.e., an operational expectation value. This quantity has been experimentally measured [3,4] and shown to exceed the classical upper bound, and thereby elevated Bell's theorem to one of the deepest results in science, with a momentous impact on the way we understand the physical world. Quantum entanglement is responsible for these observed correlations and it is also the key ingredient in most of the quantum informational advantage in computation, communication, and sensing applications. Nonlocality on its own has also been identified as a valuable resource in applications such as secure key distribution [5], certified randomness [6], reduced communication complexity [7], self-testing [8,9], and computation [10–12].

In order to advance in the fundamental understanding of the perplexing features of nonlocal correlations and their technological spin-offs, in recent decades important efforts have been devoted to their characterization and exploitation [13]. Tsirelson [14] computed the maximal violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [2] attainable by quantum mechanics; later, Popescu and Rohrlich [15] (see also [16]) showed that although quantum correlations belong to the set of no-signaling correlations, which are those not allowing for instantaneous communication, they do not attain the full strength allowed in principle by the no-signaling condition. These results reveal astonishing features of the convex sets of classical ($\mathcal{C}$), quantum ($\mathcal{Q}$), and no-signaling ($\mathcal{NS}$) correlations, in particular the strict inclusion $\mathcal{C} \subset \mathcal{Q} \subset \mathcal{NS}$. However, a lot remains to be understood, both at the conceptual level and the mathematical level. For instance, the fact that $\mathcal{Q} \subset \mathcal{NS}$ spurred the search for underlying operational principles that would single out quantum correlations among general no-signaling ones [17–19]. An approach that may assist in identifying such operationally defined principles and that may unveil new applications of nonlocality is based on cooperative games [20], where two (or more) remote parties cooperate to win a probabilistic game against a referee. Indeed, an increased winning probability when the two parties use quantum resources instead of classical ones, is equivalent to the violation of a Bell inequality. Gill [21,22] asked the fruitful question whether all *tight* Bell inequalities are violated by quantum mechanics. Here, tightness means that the inequality cannot be expressed as a positive linear combination of other Bell inequalities, or in geometric terms, that the Bell inequality defines a facet of the polytope of classical correlations (see below). Linden *et al.* [23] (motivated by [24]) found the first class of two-player games, called *nonlocal computation (NLC)*, that have no quantum advantage; the tightness of their Bell inequalities was posed as an open question in Ref. [23]. Almeida *et al.* [25] presented another case, the multiparty *guess your neighbor's input (GYNI)* game, shown to define a tight Bell inequality without quantum violation. Around the same time, (after

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numerical confirmation of several special cases [25, Appx. D]), it was understood that NLC games never define facets of the Bell polytope [26], though this result was never written up; a proof was eventually published by Ramanathan *et al.* [27].

In this Rapid Communication we prove that XOR games without a quantum advantage never define a facet of the Bell polytope (thus extending the result for NLC). Moreover, since XOR games fully characterize the correlation polytope, we answer Gill's question in the affirmative for the correlation polytope: all nontrivial tight correlation Bell inequalities (in particular those violated by no-signaling) have quantum violations. The remainder of this Rapid Communication is structured as follows: (i) we first introduce the general formalism to describe the set of no-signaling, local classical and quantum correlations; (ii) we briefly present the XOR games and give general expressions for the winning probabilities under different locality scenarios; (iii) we present our main theorem and the main ideas of its proof; and (iv) we extend our result to the quantum set of correlations, and conclude with a discussion and outlook. In the Supplemental Material (SM) [28] we give the full proofs of our result, and present a simplified analysis in the particular case of NLC, reconstructing the argument alluded to in Ref. [26], and improving [27] by giving a bound on the dimension of the face.

No-signaling behaviors. Consider a bipartite system where two parties, Alice and Bob, can perform measurements $x \in [m_A]$ and $y \in [m_B]$, respectively, obtaining the respective outcomes a and b, which are binary. The event of obtaining a and b when the local measurements x and y are performed, is given according to a conditional probability $p(a, b|x, y)$. In addition p must satisfy the *no-signaling property*, i.e., $\sum_b p(a, b|x, y) = \sum_b p(a, b|x, y')\ \forall a, x, y, y'$ and analogously summing Alice's outcomes, which in physical terms excludes that any party signals to another party by their choice of input.

The set of all probabilities satisfying the above no-signaling property, $\mathcal{NS}$ set, constitutes a polytope of dimension $D = m_A m_B + m_A + m_B$ [29]. In an attempt to explain the phenomena locally, one may consider the existence of classical local (hidden) variables $\lambda \in \Lambda$, distributed according to a probability law $\rho(\lambda)$, such that the probability of an observed event can be written as $p(a, b|x, y) = \int_\Lambda d\lambda \rho(\lambda) p(a|x, \lambda) p(b|y, \lambda)$. The set of such probabilities forms the so-called Bell or local polytope $\mathcal{C}$ and has the same dimension as the no-signaling polytope [29]. A polytope $\mathcal{P}$ can equivalently be defined as the convex hull of a finite set of points, $\mathcal{P} = \text{conv}\{v_j : j = 1, \dots, k\}$, or as a bounded intersection of finitely many closed half-spaces, $\mathcal{P} = \{\vec{v} : \forall i = 1, \dots, \ell\ \ \vec{u}_i \cdot \vec{v} \leqslant w_i\}$ [30]. A linear inequality for $\mathcal{P}$ is a $\vec{u} \cdot \vec{v} \leqslant w$ that holds for all $\vec{v} \in \mathcal{P}$; in geometry, $H = \{\vec{v} : \vec{u} \cdot \vec{v} = w\}$ is also called a *supporting hyperplane* of $\mathcal{P}$. For a given supporting hyperplane, the set of point $\vec{v} \in \mathcal{P}$ achieving the equality is called a *face* of the polytope $\mathcal{P}$, $\mathcal{F} = \mathcal{P} \cap H$. In the case of $\mathcal{C}$, its corresponding inequalities are precisely the Bell inequalities. The faces of maximum dimension $D - 1$ are called facets (see Fig. 1); when dealing with $\mathcal{C}$, the corresponding inequalities are called tight, or facet, Bell inequalities. Facet inequalities give the minimal characterization of the polytope in terms of half-spaces in the sense that any other inequality that holds for the polytope can be written as a positive linear combination of the facet inequalities.

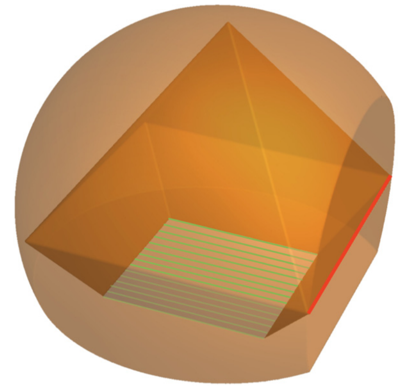

FIG. 1. Three-dimensional schematic of a local correlation polytope and the convex body of quantum correlations. Shared boundaries correspond to XOR games without quantum advantage. The green-striped region is a *facet* of the Bell polytope, while the red line is a *face* (not a facet). Theorem 1 excludes the former.

To state and prove our results, we use a convenient minimal parametrization of the no-signaling polytopes. For outcomes $a, b \in \{0, 1\}$, any no-signaling *behavior* $p(a, b|x, y)$ is fully characterized by the first moments $\alpha_x = \langle(-1)^a\rangle_{x,y} = \sum_{a,b}(-1)^a p(a, b|x, y)$ and $\beta_y = \langle(-1)^b\rangle_{x,y} = \sum_{a,b}(-1)^b p(a, b|x, y)$ (which, due to the no-signaling property, are independent of y and x, respectively); and the correlators $c_{xy} = \langle(-1)^{a+b}\rangle_{x,y} = \sum_{a,b}(-1)^{a+b} p(a, b|x, y)$. Indeed, from these $D = m_A m_B + m_A + m_B$ values we recover $4p(a, b|x, y) = 1 + (-1)^a\alpha_x + (-1)^b\beta_y + (-1)^{a+b}c_{xy}$. The polytope of $\mathcal{NS}$ distributions can hence be described by the tuple $(|\alpha\rangle, |\beta\rangle, C) \in \mathbb{R}^D$, where $|\alpha\rangle = \sum_x \alpha_x|x\rangle \in \mathbb{R}^{m_A}$ and $|\beta\rangle = \sum_y \beta_y|y\rangle \in \mathbb{R}^{m_B}$ are the local moment vectors, and $C = \sum_{x,y} c_{xy}|x\rangle\langle y|$ is the correlation matrix. The local, or Bell, polytope arises when restricting the strategies to convex combinations of local deterministic ones [31]. We use the subscript c to label such extremal classical strategies $|\alpha_c\rangle \in \{-1, 1\}^{m_A}$, $|\beta_c\rangle \in \{-1, 1\}^{m_B}$, for which we note that $c_{xy} = \alpha_x\beta_y$, that is, $C = |\alpha_c\rangle\langle\beta_c|$. The Bell polytope is then given by the convex hull

$$\mathcal{C} = \text{conv}\{(|\alpha_c\rangle, |\beta_c\rangle, |\alpha_c\rangle\langle\beta_c|)\}. \tag{1}$$

Finally, there are at least two definitions of sets of quantum behaviors that we have to consider: the most general setting is of a state $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, together with Alice's and Bob's observables $\hat{a}_x$ and $\hat{b}_y$, respectively, with eigenvalues 0 and 1 (i.e., they are projectors), and such that for all x, y, $[\hat{a}_x, \hat{b}_y] = 0$. Then, $p(a, b|x, y) = \langle\psi|\hat{a}_x\hat{b}_y|\psi\rangle$, and hence in the above parametrization for $\mathcal{NS}$, we have $\alpha_x = \langle\psi|(-1)^{\hat{a}_x}|\psi\rangle$, $\beta_x = \langle\psi|(-1)^{\hat{b}_y}|\psi\rangle$, and $c_{xy} = \langle\psi|(-1)^{\hat{a}_x}(-1)^{\hat{b}_y}|\psi\rangle$. The set of such behaviors is denoted $\mathcal{Q}_{\text{com}}$, the subscript standing for "commuting" strategies, and it is known to be a closed convex set contained in $\mathcal{NS}$. The other, traditionally considered setting is that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is a tensor product Hilbert space, and that $\hat{a}_x = \hat{a}_x^A \otimes \mathbb{1}_B$ and $\hat{b}_y = \mathbb{1}_A \otimes \hat{b}_y^B$, with observables $\hat{a}_x^A$ on $\mathcal{H}_A$ and $\hat{b}_y^B$ on $\mathcal{H}_B$. The

FIG. 2. Representation of a XOR game. The goal is that Alice and Bob output a and b such that $a \oplus b = f(x, y)$.

corresponding set of behaviors is convex, but recently has been shown not to be closed [32] for $m_A, m_B \geqslant 5$ [33], which is why we define $\mathcal{Q}_\otimes$ to be its closure. By definition, $\mathcal{Q}_\otimes \subseteq \mathcal{Q}_{\text{com}}$, and while it is open whether the two sets are equal, this would be equivalent to Connes' long-standing embedding problem in the theory of von Neumann algebras [34,35]. The sets of quantum behaviors are convex sets, but unlike the classical and no-signaling sets, they are not polytopes: they have uncountably many extreme points, and part of their boundary is curved.

The study of nonlocal correlations is often carried out in a simplified scenario, the so-called *correlation polytope*, which is given by the set of correlators C (without including the local terms). The corresponding linear criteria that define the set of classical/local correlations $\mathcal{C}_0$ are called *correlation Bell inequalities* [31,36,37]. The projection of quantum and no-signaling behaviors onto the correlator subspace are the quantum $\mathcal{Q}_0$ and no-signaling $\mathcal{NS}_0$ correlations, respectively. Note that by Tsirelson's results [14], both $\mathcal{Q}_{\text{com}}$ and $\mathcal{Q}_\otimes$ give rise to the same quantum correlator set, realized in fact with local Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of bounded dimension. See also [38–40] for recent developments on the geometry of the sets $\mathcal{Q}_0$ and $\mathcal{Q}_\otimes$.

XOR games. Nonlocal games provide an intuitive operational setting in which to cast Bell inequalities, and relate those to the well-established field of interactive proofs in computer science. Here we will focus on the particular class of two-player (Alice and Bob) XOR games [20], where the outcomes of each party are binary and the winning condition depends on the exclusive disjunction (XOR) of the outcomes. XOR games have a prominent role in nonlocality: the paradigmatic CHSH inequality [2], the Greenberger-Horne-Zeilinger paradox [41], and NLC [23] can all be phrased as XOR games; and most importantly, they provide a characterization of the correlation Bell polytope as will become apparent below.

In an XOR game (see Fig. 2), the referee provides queries $x \in [m_A]$ to Alice and $y \in [m_B]$ to Bob, sampled from a prior probability distribution $q(x, y)$ known to both players. In order to win the game, upon receiving their inputs x and y, Alice and Bob must produce a binary output $a, b \in \{0, 1\}$, respectively, such that $a \oplus b = f(x, y)$, where f is a given Boolean function also known to both players. The performance of their strategy is quantified by the average winning probability

$$\omega = \sum_{x,y} q(x, y) p(a \oplus b = f(x, y)|x, y) = \frac{1}{2}(1 + \xi), \qquad (2)$$

where $\xi = \sum_{x,y} q(x, y)(-1)^{f(x,y)} c_{xy}$ is the gain (or bias). Note that XOR games can always be won with at least probability $\frac{1}{2}$ if Alice (or Bob) produces a random output independently of the input. Since $q(x, y)$ and $f(x, y)$ are a given, we can characterize the game by the so-called game matrix $\Phi = \sum_{x,y} (-1)^{f(x,y)} q(x, y) |x\rangle\langle y|$, so that the gain can be written in terms of the correlation matrix as $\xi = \operatorname{tr} C\Phi^T$. Every correlation Bell inequality, as it is based on a linear function of the correlators C, can be written in the form $\operatorname{tr} C\Phi^T \leqslant \xi$, and by rescaling if necessary, Φ can be chosen as the game matrix of a suitable XOR game. The optimal classical success probability can always be attained by extremal (i.e., deterministic) strategies $|\alpha_c\rangle$ and $|\beta_c\rangle$, with $C = |\alpha_c\rangle\langle\beta_c|$. Hence the gain of the local classical average winning probability can be written as

$$\xi_c = \max_{\alpha_c, \beta_c} \langle \alpha_c | \Phi | \beta_c \rangle. \qquad (3)$$

In the SM (A) we present various useful ways to write the quantum gain, which are employed in the proofs of our main results.

It is easy to see that no-signaling behaviors allow one to win XOR games with $\omega_{NS} = 1$ [42], and therefore any XOR game with $\omega_c < 1$ will correspond to a nontrivial Bell inequality, i.e., one that can potentially be violated quantumly. See the SM (B) for further discussion of this point.

Observe finally that without loss of generality, we may restrict ourselves to XOR games with game matrices Φ that have no all-zero rows or columns. Indeed, because such a row or column of zeros implies that the marginal $q(x)$ or $q(y)$ is zero for some inputs, we can redefine the set of possible queries (decreasing m_A or m_B accordingly) to obtain an equivalent game without all-zero rows or columns in its game matrix. We refer to such games with $q(x) > 0$ and $q(y) > 0$ for all $x \in [m_A]$ and $y \in [m_B]$ as *exhaustive games*.

Results. For a long time, it was implicitly assumed that strategies using entangled strategies can attain a greater success probability than those limited to classical resources, for any nontrivial Bell inequality. As explained in the Introduction, it took a while to find examples of nontrivial games that do not show any quantum advantage.

Here we show that XOR games (which characterize the correlation polytope) without quantum advantage never define a facet of the Bell polytope (full behaviors or correlations). This in turn implies that all (nontrivial) tight correlation Bell inequalities have quantum violations.

In Ref. [42] Ramanathan *et al.* derived a necessary and sufficient condition for a two-player XOR game to have no quantum advantage, which will turn out to be fundamental for the proof of our first result.

Theorem 1. If an exhaustive XOR game has no quantum advantage, the corresponding Bell inequality does not define a facet of the Bell polytope, or of the correlation Bell polytope.

The *proof* [see the SM (C) for full details] proceeds by bounding the dimension of the face $\mathcal{F}$ in the Bell polytope corresponding to the maximum classical bias ξ_c of the given XOR game:

$$\begin{aligned}\mathcal{F} &= \{(|\alpha\rangle, |\beta\rangle, C) \in \mathcal{C} : \operatorname{tr} C\Phi^T = \xi_c\} \\ &= \operatorname{conv}\{(|\alpha_c\rangle, |\beta_c\rangle, |\alpha_c\rangle\langle\beta_c|) : \langle\alpha_c|\Phi|\beta_c\rangle = \xi_c\}.\end{aligned}$$

The first ingredient is the characterization of the maximum quantum bias ξ_Q by semidefinite programming (SDP) [43], which by SDP duality leads to strong constraints on any optimal strategy via complementary slackness. From the assumption that $\xi_Q = \xi_c$, this leads to the second, and key, insight of the proof, namely, that in any pair $(|\alpha_c\rangle, |\beta_c\rangle)$ of optimal classical strategies, Alice's and Bob's local answers uniquely determine each other linearly: as we show in the proof, $|\beta_c\rangle = F|\alpha_c\rangle$ for a certain matrix F. Assuming without loss of generality $m_A \leqslant m_B$, we thus have

$$\mathcal{F} = \text{conv}\{(|\alpha_c\rangle, F|\alpha_c\rangle, |\alpha_c\rangle\langle\alpha_c|F^T) : |\alpha_c\rangle \text{opt.}\},$$

and its dimension can be upper bounded by that of

$$\text{aff}\{(|\alpha_c\rangle, F|\alpha_c\rangle, |\alpha_c\rangle\langle\alpha_c|F^T) : |\alpha_c\rangle \in \{\pm 1\}^{m_A}\},$$

which is $m_A + \frac{1}{2}m_A(m_A - 1) < D - 1$. In the case of the correlation polytope, the dimension is similarly upper bounded by $\frac{1}{2}m_A(m_A - 1) < m_A m_B$.

Theorem 2. All nontrivial tight correlation Bell inequalities for bipartite systems with binary outcomes have a quantum violation.

Proof. Consider a (nonexhaustive) XOR game with M_A (Alice) and M_B (Bob) inputs. Without loss of generality, the first m_A (m_B) inputs of Alice (Bob) have nonzero probability; the rest are never asked, so we can apply Theorem 1 to the reduced exhaustive game, which relates the optimal strategies for the indices $x \in [m_A]$ and $y \in [m_B]$, but leaves completely unconstrained the remaining ones. Hence, given a strategy by Alice $|\alpha_c\rangle \oplus |\alpha'_c\rangle$, Bob's strategy must be $(F|\alpha_c\rangle) \oplus |\beta'_c\rangle$, where $|\alpha'_c\rangle \in \{\pm 1\}^{M_A - m_A}$ and $|\beta'_c\rangle \in \{\pm 1\}^{M_B - m_B}$.

We thus arrive at a codimension $\Delta = D - \dim \mathcal{F}$ of the face of $\Delta \geqslant m_B + M_A(m_B - m_A) + \frac{m_A}{2}(m_A + 1) > 1$. That is, XOR games with quantum equal to classical value do not define a facet of the full Bell polytope. Following the same argument for the correlation polytope leads to the codimension $\Delta_0 \geqslant M_A(m_B - m_A) + \frac{1}{2}m_A(m_A + 1)$, and this is greater than 1 unless $m_A = m_B = 1$, corresponding precisely to the trivial inequalities $|c_{xy}| \leqslant 1$. ∎

In the SM (D) we give a different proof of the above result for nonlocal computation games, showing also that the dimension bounds are asymptotically attained for nontrivial games.

From the proof of Theorem 1 we learn that the optimal extremal behaviors in an exhaustive XOR game with no quantum advantage are fully determined by the strategy of one of the parties. We will now show that this feature actually extends to all optimal quantum behaviors of arbitrary XOR games.

To understand the following theorem, we recall the definition of a face $\mathcal{F} \subset \mathcal{Q}$ of a general compact convex set $\mathcal{Q}$: namely, that whenever $\mathcal{F} \ni \vec{p} = t\vec{q} + (1-t)\vec{r}$, $0 < t < 1$, then both $\vec{q}, \vec{r} \in \mathcal{F}$. An *exposed face* is obtained as $\mathcal{F} = \mathcal{Q} \cap H$ with a supporting hyperplane H of $\mathcal{Q}$; all exposed faces are faces of $\mathcal{Q}$, but not vice versa [38,44]. However, for polytopes every face is an exposed face [30]. Also, facets, and more generally maximal faces, are always exposed.

Note that this result explains the previous two theorems on the classical behaviors as being due to broader properties of the quantum sets.

Theorem 3. Nontrivial XOR games, or equivalently nontrivial correlation Bell inequalities, never define a facet of the quantum sets of behaviors $\mathcal{Q}_{\text{com}}$ and $\mathcal{Q}_{\otimes}$. As a consequence, the set $\mathcal{Q}_0$ of quantum correlations has no nontrivial facets.

See the SM (C) for the complete proof. To give the broad outline, we start with an exhaustive XOR game. The complementary slackness condition in the proof of Theorem 1 for the optimal quantum strategy leads to $|\beta_y\rangle = \sum_{x'} F_{yx'}|\alpha_{x'}\rangle$, with the same matrix F as before. In other words, once again Alice's optimal quantum strategy uniquely determines Bob's, and vice versa.

Thus, we get for an optimal quantum correlation matrix $C_{xy} = \langle\alpha_x|\beta_y\rangle = \sum_{x'} F_{yx'}\langle\alpha_x|\alpha_{x'}\rangle$, and hence the dimension of their affine span is bounded by that of the Gram matrices $[\langle\alpha_x|\alpha_{x'}\rangle]_{xx'}$, with dimension $\frac{1}{2}m_A(m_A - 1)$, leading to the same dimension bound as in the proof of Theorem 1. Nonexhaustive XOR games are treated as in the proof of Theorem 2.

Discussion and outlook. We have shown that a two-party correlation Bell inequality (XOR game) with no quantum violation (quantum advantage) cannot define a facet of the Bell polytope. The contrapositive of this statement has deep physical implications: all tight correlation Bell inequalities exhibit a quantum violation. In fact, we have proven lower bounds on the codimension of the defined face (increasing with the number of inputs). As a consequence, when the codimension is lower bounded by $\Delta > 1$, not only all tight correlation inequalities will have quantum violations, but also those corresponding to faces $\mathcal{F}$ with $\dim \mathcal{F} \geqslant D - \Delta$.

On the way, we have proved that this in fact is due to a broader property of the convex set of quantum correlations, namely, that it does not have any nontrivial facets, only lower-dimensional faces. It remains to be seen what the physical meaning of this curious geometric observation is, which complements recent insights into the the geometry of the quantum set, such as the existence of nontrivial faces, of nonexposed extreme points, and of "quantum voids" (postquantum faces of $\mathcal{NS}$) [38–40].

While two-player, binary outcome XOR games cover a large class of Bell inequalities, including CHSH and many other classic examples, there are of course more general settings. For instance, we leave open the interesting questions whether XOR games for more than two players can define common facets of the quantum and classical sets (note that GYNI defines such a facet, but it is not an XOR game), or whether for two players there are any tight Bell inequalities at all without quantum violations. On the other hand, it might be possible to extend our results at least to two-player MOD-q games, where each player has a q-ary outcome.

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