# Resource theory of asymmetric distinguishability

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This paper systematically develops the resource theory of asymmetric distinguishability, as initiated roughly a decade ago [Matsumoto, Reverse test and characterization of quantum relative entropy, arXiv:1010.1030]. The key constituents of this resource theory are quantum boxes, consisting of a pair of quantum states, which can be manipulated for free by means of an arbitrary quantum channel. We introduce bits of asymmetric distinguishability as the basic currency in this resource theory, and we prove that it is a reversible resource theory in the asymptotic limit, with the quantum relative entropy being the fundamental rate of resource interconversion. The distillable distinguishability is the optimal rate at which a quantum box consisting of independent and identically distributed (i.i.d.) states can be converted to bits of asymmetric distinguishability, and the distinguishability cost is the optimal rate for the reverse transformation. Both of these quantities are equal to the quantum relative entropy. The exact one-shot distillable distinguishability is equal to the min-relative entropy, and the exact one-shot distinguishability cost is equal to the max-relative entropy. Generalizing these results, the approximate one-shot distillable distinguishability is equal to the smooth min-relative entropy, and the approximate one-shot distinguishability cost is equal to the smooth max-relative entropy. As a notable application of the former results, we prove that the optimal rate of asymptotic conversion from a pair of i.i.d. quantum states to another pair of i.i.d. quantum states is fully characterized by the ratio of their quantum relative entropies.

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## I. INTRODUCTION

Distinguishability plays a central role in all sciences. That is, the ability to distinguish one possibility from another is what allows us to discover new scientific laws and make predictions of future possibilities. In the process of scientific discovery, we form a hypothesis based on conjecture, which is to be tested against a conventional or null hypothesis by repeated trials or experiments. With sufficient statistical evidence, one can determine which hypothesis should be rejected in favor of the other. If the null hypothesis is accepted, one can form alternative hypotheses to test against the null hypothesis in future experiments.

What is essential in this approach is the ability to perform repeated trials. Repetition allows for increasing the distinguishability between the two hypotheses. A natural question in this context is to determine how many trials are required to reach a given conclusion. If the two different hypotheses are relatively distinguishable, then fewer trials are required to decide between the possibilities. In this sense, distinguishability can be understood as a *resource*, because it limits the amount of effort that we need to invest in order to make decisions.

One of the fundamental settings in which distinguishability can be studied in a mathematically rigorous manner is statistical hypothesis testing. The basic setup is that one draws a sample $x$ from one of two probability distributions $p \equiv \{p(x)\}_{x\in\mathcal{X}}$ or $q \equiv \{q(x)\}_{x\in\mathcal{X}}$, with common alphabet $\mathcal{X}$, with the goal being to decide from which distribution the sample $x$ has been drawn. Let $p$ be the null hypothesis and $q$ the alternative. A type-I error occurs if one decides $q$ when the distribution being sampled from is in fact $p$, and a type-II error occurs if one decides $p$ when the distribution being sampled from is in fact $q$. The goal of asymmetric hypothesis testing is to minimize the probability of the type-II error, subject to an upper bound constraint on the probability of committing the type-I error.

In the scientific spirit of repeated experiments, we can modify the above scenario to allow for independent and identically distributed (i.i.d.) samples from either the distribution $p$ or $q$. One of the fundamental results of asymptotic hypothesis testing is that, with a sufficiently large number of samples, it becomes possible to meet any upper bound constraint on the type-I error probability while having the type-II error probability decaying exponentially fast with the number of samples, with the optimal error exponent being given by the relative entropy [1,2]:

$$D(p\|q) = \sum_{x\in\mathcal{X}} p(x)\log_2[p(x)/q(x)]. \tag{1}$$

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That is, there exists a sequence of schemes that can achieve this error exponent for the type-II error probability while making the type-I error probability arbitrarily small in the limit of a large number of samples. At the same time, the strong converse property holds: any sequence of schemes that has a fixed constraint on the type-I error probability is such that its type-II error probability cannot decay any faster than the exponent $D(p\|q)$. This gives a fundamental operational meaning to the relative entropy and represents one core link between hypothesis testing and information theory [3], the latter being the fundamental mathematical theory of communication [4].

Another perspective on the above process of decision making in hypothesis testing, the *resource-theoretic perspective* [5,6] not commonly adopted in the literature on the topic, is that it is a process by which we *distill* distinguishability from the original distributions into a more standard form. That is, we can think of the distributions $p$ and $q$ being presented as a black box or ordered pair $(p, q)$. Given a sample $x \in \mathcal{X}$, we can perform a common transformation $\mathcal{T} : \mathcal{X} \to \{0, 1\}$ that outputs a single bit, "0" to decide $p$ and "1" to decide $q$. The common transformation $\mathcal{T}$ can even be stochastic. In this way, one transforms the initial box to a final box as

$$(p, q) \underset{\to}{\mathcal{T}} (p_f, q_f), \tag{2}$$

where $p_f \equiv \{p_f(y)\}_{y\in\{0,1\}}$ and $q_f \equiv \{q_f(y)\}_{y\in\{0,1\}}$ are binary distributions. Then the probability of a type-I error is $p_f(1)$, and the probability of a type-II error is $q_f(0)$. Since the goal is to extract or distill as much distinguishability as possible, we would like for $q_f(0)$ to be as small as possible given a constraint $\varepsilon \in [0, 1]$ on $p_f(1)$ (i.e., $p_f(1) \leqslant \varepsilon$).

Once we have adopted this resource-theoretic approach to distinguishability, it is natural to consider two other questions, the first of which is the question of the *reverse process* [5,6]. That is, we would like to start from initial binary distributions $p_i \equiv \{p_i(y)\}_{y\in\{0,1\}}$ and $q_i \equiv \{q_i(y)\}_{y\in\{0,1\}}$ having as little distinguishability as possible, and act on their samples with a common transformation $\mathcal{R} : \{0, 1\} \to \mathcal{X}$ in order to produce the distributions $p \equiv \{p(x)\}_{x\in\mathcal{X}}$ and $q \equiv \{q(x)\}_{x\in\mathcal{X}}$, while allowing for a slight error when reproducing $p$. That is, we would like to perform the *dilution* transformation

$$(p_i, q_i) \underset{\to}{\mathcal{R}} (\tilde{p}, q), \tag{3}$$

where $\tilde{p} \equiv \{\tilde{p}(x)\}_{x\in\mathcal{X}}$ is a distribution satisfying $d(p, \tilde{p}) \leqslant \varepsilon$, for some suitable metric $d$ of statistical distinguishability. In this way, we characterize the distinguishability of $p$ and $q$ in terms of the least distinguishable distributions $p_i$ and $q_i$ that can be diluted to prepare or simulate $p$ and $q$, respectively. This dilution question is motivated by related questions in the theory of quantum entanglement [7].

The second, more general question is regarding the existence of a common transformation $\mathcal{T} : \mathcal{X} \to \mathcal{Z}$ that converts initial distributions $p$ and $q$ into final distributions $r \equiv \{r(z)\}_{z\in\mathcal{Z}}$ and $t \equiv \{t(z)\}_{z\in\mathcal{Z}}$:

$$(p, q) \underset{\to}{\mathcal{T}} (\tilde{r}, t), \tag{4}$$

where $\tilde{r} \equiv \{\tilde{r}(z)\}_{z\in\mathcal{Z}}$ is a distribution satisfying $d(r, \tilde{r}) \leqslant \varepsilon$. One can then ask about the rate or efficiency at which it is possible to convert a pair of i.i.d. distributions to another pair of i.i.d. distributions.

This resource-theoretic approach to distinguishability offers a unique and powerful perspective on statistical hypothesis testing and distinguishability, similar to the perspective brought about by the seminal work on the resource theory of quantum entanglement [7], which has in turn inspired a flurry of activity on resource theories in quantum information and beyond [8]. Although the reverse process in Eq. (3) may seem nonsensical at first glance (why would one want to dilute fresh water to salt water? [9]), it plays a fundamental role in characterizing distinguishability as a resource, as well as for addressing the general question posed in Eq. (4). It is also natural from a thermodynamic or physical perspective to consider reversibility and cyclicity of processes. Another application for the reverse process is in understanding the minimal resources required for simulation in various quantum resource theories [8].

## II. MAIN RESULTS

The main goal of this paper is to develop systematically the resource-theoretic perspective on distinguishability, which was initiated in Refs. [5,6]. More precisely, the theory developed here is a *resource theory of asymmetric distinguishability*, given that approximation is allowed for the first distribution in all of the distillation, dilution, and general transformation tasks mentioned above. The theory that we develop applies in the more general setting of *quantum* distinguishability, as it did in Refs. [5,6], in particular when the distributions $p$ and $q$ are replaced by quantum states $\rho$ and $\sigma$, respectively, and the common transformations allowed on a quantum box $(\rho, \sigma)$ are quantum channels.

Some key findings of our work are as follows.

(1) We introduce the fundamental unit or currency of this resource theory, dubbed "bits of asymmetric distinguishability." Then the distinguishability distillation and dilution tasks amount to distilling bits of asymmetric distinguishability from a box $(\rho, \sigma)$ and diluting bits of asymmetric distinguishability to a box $(\rho, \sigma)$, respectively.

(2) We formally define the exact one-shot distinguishability distillation and dilution tasks, and we prove that the optimal number of bits of asymmetric distinguishability that can be distilled from a box $(\rho, \sigma)$ is equal to the min-relative entropy [10] [see (31)], while the optimal number of bits of asymmetric distinguishability that can be diluted to a box $(\rho, \sigma)$ is equal to the max-relative entropy [10] [see (35)], giving both of these quantities fundamental operational interpretations in the resource theory of asymmetric distinguishability.

(3) We define the approximate one-shot distinguishability distillation and dilution tasks, and we prove that the optimal number of bits of asymmetric distinguishability that can be distilled from a box $(\rho, \sigma)$ is equal to the smooth min-relative entropy [11–13] [see (44)], while the optimal number of bits of asymmetric distinguishability that can be diluted to a box $(\rho, \sigma)$ is equal to the smooth max-relative entropy [10] [see (48)], giving both of these quantities fundamental operational interpretations in the resource theory of asymmetric distinguishability.

(4) We prove that the optimization problems corresponding to one-shot distinguishability distillation and dilution, as well as the optimization corresponding to the quantum

generalization of the transformation problem considered in Eq. (4), are characterized by semidefinite programs (see Appendices B and C). Thus all of these quantities can be computed efficiently.

(5) We finally consider the asymptotic version of the resource theory and prove that it is reversible in this setting, with the optimal rate of distillation or dilution equal to the quantum relative entropy. The implication of this result is that the rate or efficiency at which a pair of i.i.d. quantum states can be converted to another pair of i.i.d. quantum states is fully characterized by the ratio of their quantum relative entropies [see (62)].

In what follows, we provide more details of the resource theory of asymmetric distinguishability and a full exposition of the main results stated above. We relegate details of mathematical proofs to several appendices, and we note here that some of the technical lemmas in the appendices may be of independent interest.

As far as we are aware, the first proposal for a resource theory of distinguishability was given in Refs. [5,6], which we have highlighted above. It appears that this aspect of the work [5,6] has gone largely unnoticed since its posting to the arXiv, given that there have been several subsequent proposals or calls to formalize a resource theory of distinguishability [14–16] that apparently were not aware of Refs. [5,6].

## III. RESOURCE THEORY OF ASYMMETRIC DISTINGUISHABILITY

We begin by establishing the basics of the resource theory of asymmetric distinguishability. The basics include the objects being manipulated, called "boxes," the fundamental units of resource, "bits of asymmetric distinguishability," and the free operations allowed, which are simply arbitrary quantum physical operations.

The basic object to manipulate in the resource theory of asymmetric distinguishability is the following "box" or ordered pair:

$$(\rho, \sigma), \tag{5}$$

where $\rho$ and $\sigma$ are quantum states acting on the same Hilbert space. The interpretation of the box $(\rho, \sigma)$ is that it corresponds to two different experiments or scenarios. In the first, the state $\rho$ is prepared, and in the second, the state $\sigma$ is prepared. The box is handed to another party, who is not aware of which experiment is being conducted (i.e., which state has been prepared).

One basic manipulation in this resource theory is to transform this box into another box by means of any quantum physical operation $\mathcal{N}$, as allowed by quantum mechanics. Such physical operations are mathematically described by completely positive, trace-preserving (CPTP) maps and are known as quantum channels. By acting on the box $(\rho, \sigma)$ with the common quantum channel $\mathcal{N}$, one obtains the transformed box $(\mathcal{N}(\rho), \mathcal{N}(\sigma))$. Observe that it is not necessary to know which experiment is being conducted in order to perform this transformation; one can perform it regardless of whether $\rho$ or $\sigma$ was prepared. For this reason, all quantum channels are allowed for free in this resource theory, so that the transformation

$$(\rho, \sigma) \underset{\to}{\mathcal{N}} (\mathcal{N}(\rho), \mathcal{N}(\sigma)) \tag{6}$$

is allowed for free.

If the channel being performed to transform the box in Eq. (5) is an isometric channel $\mathcal{U}(\omega) = U\omega U^\dagger$ (where $U$ is an isometry satisfying $U^\dagger U = I$ and $\omega$ is an arbitrary state), resulting in the box

$$(\mathcal{U}(\rho), \mathcal{U}(\sigma)), \tag{7}$$

then it is possible to invert this transformation and return to the original box in Eq. (5). A quantum channel that inverts the action of $\mathcal{U}$ is given by

$$\theta \to U^\dagger \theta U + \mathrm{Tr}[(I - UU^\dagger)\theta]\tau, \tag{8}$$

where $\theta$ is an arbitrary state and $\tau$ is some state.

Another kind of invertible transformation is the appending channel $\mathcal{A}_\tau(\omega) = \omega \otimes \tau$, which appends the state $\tau$ and has the following effect on the box:

$$(\mathcal{A}_\tau(\rho), \mathcal{A}_\tau(\sigma)) = (\rho \otimes \tau, \sigma \otimes \tau). \tag{9}$$

One can recover the original box $(\rho, \sigma)$ from (9) by discarding the second system (described mathematically by partial trace). Thus isometric channels and appending channels are perfectly reversible operations in this resource theory.

The fundamental goal of this resource theory is to determine how and whether it is possible to transform an initial box $(\rho, \sigma)$ to another box $(\tau, \omega)$ for states $\tau$ and $\omega$, by means of a common quantum channel $\mathcal{N}$. Mathematically, the question is to determine, for fixed states $\rho$, $\sigma$, $\tau$, and $\omega$, whether there exists a completely positive and trace-preserving map $\mathcal{N}$ such that $\mathcal{N}(\rho) = \tau$ and $\mathcal{N}(\sigma) = \omega$. As it turns out, various instantiations of this question have been studied considerably in prior work [17–30], and a variety of results are known regarding it. In this paper, we offer a fresh resource-theoretic perspective on this matter.

Motivated by practical concerns, one important variation of the aforementioned box transformation problem is to determine whether it is possible to accomplish the transformation *approximately* as

$$(\rho, \sigma) \underset{\to}{\mathcal{N}} (\tau_\varepsilon, \omega) \tag{10}$$

with some tolerance $\varepsilon \in [0, 1]$ allowed, such that the state $\tau_\varepsilon$ is $\varepsilon$-close to the desired $\tau$. The precise way in which we allow some tolerance is motivated exclusively by operational concerns. In a single run of the first experiment in which $\rho$ is prepared, the transformation $\mathcal{N}(\rho) = \tau_\varepsilon$ occurs. Then a third party would like to assess how accurate the conversion is. Such an individual can do so by performing a quantum measurement $\{\Lambda_x\}_x$ with outcomes $x$ (satisfying $\Lambda_x \geqslant 0$ for all $x$ and $\sum_x \Lambda_x = I$). The probability of obtaining a particular outcome $\Lambda_x$ is given by the Born rule $\mathrm{Tr}[\Lambda_x \tau_\varepsilon]$. What we demand is that the deviation between the actual probability $\mathrm{Tr}[\Lambda_x \tau_\varepsilon]$ and the ideal probability $\mathrm{Tr}[\Lambda_x \tau]$ be no larger than the tolerance $\varepsilon$. Since this should be the case for any possible measurement outcome, what we demand mathematically is that

$$\sup_{0 \leqslant \Lambda \leqslant I} |\mathrm{Tr}[\Lambda \tau_\varepsilon] - \mathrm{Tr}[\Lambda \tau]| \leqslant \varepsilon. \tag{11}$$

It is well known that

$$\sup_{0\leqslant\Lambda\leqslant I} |\mathrm{Tr}[\Lambda\tau_\varepsilon] - \mathrm{Tr}[\Lambda\tau]| = \tfrac{1}{2}\|\tau_\varepsilon - \tau\|_1, \tag{12}$$

indicating that our notion of approximation is most naturally quantified by the normalized trace distance $\frac{1}{2}\|\tau_\varepsilon - \tau\|_1$.

Thus, the mathematical formulation of the approximate box transformation problem is as follows:

$$\varepsilon((\rho,\sigma)\to(\tau,\omega)) := \inf_{\mathcal{N}\in\mathrm{CPTP}} \{\varepsilon\in[0,1] : \mathcal{N}(\rho)\approx_\varepsilon \tau,\ \mathcal{N}(\sigma)=\omega\}, \tag{13}$$

where the notation $\zeta\approx_\varepsilon\xi$ for states $\zeta$ and $\xi$ is a shorthand for $\frac{1}{2}\|\zeta-\xi\|_1\leqslant\varepsilon$, i.e.,

$$\zeta\approx_\varepsilon\xi \ \Leftrightarrow\ \tfrac{1}{2}\|\zeta-\xi\|_1\leqslant\varepsilon. \tag{14}$$

The fact that we allow for approximate conversion for the first state but not the second is related to the fact that the resource theory presented here is a resource theory of *asymmetric* distinguishability. In Appendix C, we show that (13) is equivalent to a semidefinite program (SDP), implying that it is efficiently computable with respect to the dimensions of the states involved. In the case that $\varepsilon((\rho,\sigma)\to(\tau,\omega))=0$, this means that it is possible to perform the desired transformation $(\rho,\sigma)\to(\tau,\omega)$ exactly, reproducing the previous result from Ref. [28].

We can also consider the asymptotic version of the box transformation problem, in which the box consists not just of a single copy of the states $\rho$ and $\sigma$ but many copies of them [i.e., the box $(\rho^{\otimes n},\sigma^{\otimes n})$ instead of the original $(\rho,\sigma)$]. By considering the asymptotic setting with approximation error, we can modify the original box transformation question as follows: what is the optimal rate $R$ at which the transformation

$$(\rho^{\otimes n},\sigma^{\otimes n})\to(\widetilde{\tau^{\otimes nR}},\omega^{\otimes nR}) \tag{15}$$

is possible, for large $n$ and arbitrarily small approximation error? In this setting, the SDP characterization of $\varepsilon((\rho^{\otimes n},\sigma^{\otimes n})\to(\tau^{\otimes nR},\omega^{\otimes nR}))$ is not particularly useful, due to the fact that the computational complexity of the optimization problem grows exponentially with increasing $n$, and so we resort to other, information-theoretic methods to address it.

### A. Bits of asymmetric distinguishability

One way of addressing the various formulations of the box transformation problem is to break the transformation down into two steps, in which we first *distill* a standard box and then *dilute* this standard box to the desired one. It turns out that the most natural way to do so is to consider the following basic unit of currency or fiducial box:

$$(|0\rangle\langle 0|,\pi), \tag{16}$$

where

$$\pi := \tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \tag{17}$$

is the maximally mixed qubit state. We also refer to the object in Eq. (16) as "one bit of asymmetric distinguishability."

As before, we should think of the box in Eq. (16) as being in correspondence with two different experiments. In the first experiment, the first state $\rho=|0\rangle\langle 0|$ ("null hypothesis") is prepared, and in the second experiment, the second state $\sigma=\pi$ ("alternative hypothesis") is prepared. A distinguisher presented with this box, and unaware of which experiment is being conducted, can try to determine which state $\rho$ or $\sigma$ has been prepared. Suppose that the distinguisher performs a measurement of the observable $\sigma_Z := |0\rangle\langle 0| - |1\rangle\langle 1|$ and assigns the outcome $+1$ to the decision "$\rho$ was prepared" and $-1$ to the decision "$\sigma$ was prepared." Then in the case that the state $\rho$ was prepared, he can determine this with zero chance of error; on the other hand, if the state $\sigma$ was prepared, then he can determine this with probability equal to 1/2. In other terms, with this strategy, he has zero chance of making a type-I error (misidentifying $\rho$) and he has a 50% chance of making a type-II error (misidentifying $\sigma$).

The above strategy of basing the decision rule on the outcome of a $\sigma_Z$ measurement is not the only strategy that the distinguisher can perform. By performing a quantum channel $\mathcal{N}$ that accepts a qubit as input and outputs another quantum system, the distinguisher can convert the box in Eq. (16) to the following box:

$$(\mathcal{N}(|0\rangle\langle 0|),\mathcal{N}(\pi)). \tag{18}$$

After doing so, the distinguisher can base his decision rule on the outcome of a general quantum measurement. However, if the goal is to have zero chance of making a type-I error, then it is intuitive and can be proven that no strategy can perform better than the $\sigma_Z$ measurement strategy given in the previous paragraph. Thus arbitrary channels acting on the box in Eq. (16) do not increase distinguishability.

One bit of asymmetric distinguishability is not a particularly strong resource. Indeed, with only one bit of asymmetric distinguishability, there is still a large chance of making a type-II error. However, the following box, consisting of $m$ bits of asymmetric distinguishability, improves the situation:

$$(|0\rangle\langle 0|^{\otimes m},\pi^{\otimes m}). \tag{19}$$

For such a box, there is a much smaller chance of making a type-II error. Indeed, by performing $m$ independent measurements of the observable $\sigma_Z$ on each qubit and assigning the outcome "$(+1,\dots,+1)$" to the decision "$|0\rangle\langle 0|^{\otimes m}$ was prepared" and the outcome "not $(+1,\dots,+1)$" to the decision "$\pi^{\otimes m}$ was prepared," the distinguisher still has zero chance of making a type-I error, but now has a one out of $2^m$ chance of making a type-II error. So with each extra bit of asymmetric distinguishability, the chance of making a type-II error decreases by a factor of two. This is the value of having more bits of asymmetric distinguishability.

Note that the following transformation is forbidden when $n>m$:

$$(|0\rangle\langle 0|^{\otimes m},\pi^{\otimes m})\not\to(|0\rangle\langle 0|^{\otimes n},\pi^{\otimes n}). \tag{20}$$

That is, one cannot increase bits of distinguishability by the action of a quantum channel; i.e., there is no quantum channel $\mathcal{N}$ that performs the map $\mathcal{N}(|0\rangle\langle 0|^{\otimes m})=|0\rangle\langle 0|^{\otimes n}$ and $\mathcal{N}(\pi^{\otimes m})=\pi^{\otimes n}$ for $n>m$. Quantum channels have a linear action on their inputs, and this linearity forbids such transformations, as shown in Appendix D.

A major goal of any resource theory is to quantify the amount of resource. For the simple boxes presented above, any Rényi relative entropy suffices as a good quantifier of the number of bits of asymmetric distinguishability contained in them. Two prominent examples of measures were put forward roughly a decade ago as measures of distinguishability and studied therein as quantum information-theoretic quantities of interest [10]. They are known as the min- and max-relative entropies, defined respectively as follows for states $\rho$ and $\sigma$:

$$D_{\min}(\rho\|\sigma) := -\log_2 \mathrm{Tr}[\Pi_\rho \sigma], \tag{21}$$

$$D_{\max}(\rho\|\sigma) := \inf\{\lambda \geqslant 0 : \rho \leqslant 2^\lambda \sigma\}, \tag{22}$$

where $\Pi_\rho$ denotes the projection onto the support of $\rho$. If $\rho$ is orthogonal to $\sigma$, then $D_{\min}(\rho\|\sigma) = \infty$, and if $\mathrm{supp}(\rho) \not\subseteq \mathrm{supp}(\sigma)$, then there is no finite $\lambda \geqslant 0$ such that $\rho \leqslant 2^\lambda\sigma$, implying that $D_{\max}(\rho\|\sigma) = \infty$. Evaluating these measures for the box given in Eq. (19), one finds that

$$D_{\min}(|0\rangle\langle 0|^{\otimes m}\|\pi^{\otimes m}) = mD_{\min}(|0\rangle\langle 0|\|\pi) = m, \tag{23}$$

$$D_{\max}(|0\rangle\langle 0|^{\otimes m}\|\pi^{\otimes m}) = mD_{\min}(|0\rangle\langle 0|\|\pi) = m, \tag{24}$$

consistent with the notion that the box in Eq. (19) contains $m$ bits of asymmetric distinguishability.

By performing the following quantum channel:

$$\omega \to \mathrm{Tr}[|0\rangle\langle 0|^{\otimes m}\omega]|0\rangle\langle 0| + \mathrm{Tr}[(I^{\otimes m} - |0\rangle\langle 0|^{\otimes m})\omega]|1\rangle\langle 1|, \tag{25}$$

one can convert the box in Eq. (19) to the following box:

$$(|0\rangle\langle 0|, 2^{-m}|0\rangle\langle 0| + (1 - 2^{-m})|1\rangle\langle 1|). \tag{26}$$

Furthermore, by performing the quantum channel

$$\theta \to \langle 0|\theta|0\rangle|0\rangle\langle 0|^{\otimes m} + \langle 1|\theta|1\rangle\frac{I^{\otimes m} - |0\rangle\langle 0|^{\otimes m}}{2^m - 1}, \tag{27}$$

one can convert the box in Eq. (26) back to the box in Eq. (19). For this reason, these boxes have an equivalent number of bits of asymmetric distinguishability, being equivalent by free operations. It also means that we can take the box in Eq. (26) to be the basic form of $m$ bits of asymmetric distinguishability. Once we have done that, it is then sensible to allow $m$ in Eq. (26) to be any non-negative real number, so that the box in Eq. (26) has $m$ bits of asymmetric distinguishability, with $m$ a non-negative real number. For this case, we still find that

$$D_{\min}(|0\rangle\langle 0|\|\sigma) = D_{\max}(|0\rangle\langle 0|\|\sigma) = m, \tag{28}$$

with $\sigma = 2^{-m}|0\rangle\langle 0| + (1 - 2^{-m})|1\rangle\langle 1|$.

Going forward from here, we take the box in Eq. (26) to be the basic form of *$m$ bits of asymmetric distinguishability, for $m$ any non-negative real number.*

### B. Exact distillation and dilution tasks

In any resource theory, the basic questions concern distillation and dilution tasks, and whether and in what senses the resource theory might be reversible [7,8]. In a distillation task, the goal is to process a general resource with free operations in order to distill as much of the basic resource as possible, while in the dilution task, the goal is to perform the opposite: process as little of the basic resource as possible, using free operations, in order to generate or dilute from it a more general resource. A prominent goal is to determine the ultimate rates at which these resource interconversions are possible and from there one can determine whether the resource theory is reversible.

In the resource theory of asymmetric distinguishability, the goal of *exact distinguishability distillation* is to process a general box $(\rho, \sigma)$ with an arbitrary quantum channel in order to distill as many bits of asymmetric distinguishability as possible. Mathematically, we can phrase this task as the following optimization problem:

$$D_d^0(\rho, \sigma) := \log_2 \sup_{\mathcal{P}\in\mathrm{CPTP}} \{M : \mathcal{P}(\rho) = |0\rangle\langle 0|,\ \mathcal{P}(\sigma) = \pi_M\}, \tag{29}$$

where the choice of $D_d$ in $D_d^0(\rho, \sigma)$ stands for *distillable distinguishability*, the “0” in $D_d^0(\rho, \sigma)$ indicates that we do not allow any error, CPTP denotes the set of CPTP maps (quantum channels), and

$$\pi_M := \frac{1}{M}|0\rangle\langle 0| + \left(1 - \frac{1}{M}\right)|1\rangle\langle 1|. \tag{30}$$

As we show in Appendix E 1, the following equality holds:

$$D_d^0(\rho, \sigma) = D_{\min}(\rho\|\sigma), \tag{31}$$

where $D_{\min}(\rho\|\sigma)$ is the min-relative entropy [10], as defined in Eq. (21). The equality in Eq. (31) thus assigns to $D_{\min}(\rho\|\sigma)$ a fundamental operational meaning as the exact distillable distinguishability in the resource theory of asymmetric distinguishability. A strongly related operational meaning for $D_{\min}(\rho\|\sigma)$ in quantum hypothesis testing was already given in Ref. [10].

In the case that $\rho$ is orthogonal to $\sigma$, then this means that the box $(\rho, \sigma)$ can be converted to the box $(|0\rangle\langle 0|, |1\rangle\langle 1|)$, by means of the quantum channel

$$\omega \to \mathrm{Tr}[\Pi_\rho\omega]|0\rangle\langle 0| + \mathrm{Tr}[(I - \Pi_\rho)\omega]|1\rangle\langle 1|. \tag{32}$$

From the latter box, one can obtain as many bits of asymmetric distinguishability as desired. Indeed by performing the channel

$$\mathcal{T}^m(\omega) = \langle 0|\omega|0\rangle\ |0\rangle\langle 0| + \langle 1|\omega|1\rangle\pi_{2^m}, \tag{33}$$

where $\pi_{2^m} := 2^{-m}|0\rangle\langle 0| + (1 - 2^{-m})|1\rangle\langle 1|$, one can obtain $m$ bits of asymmetric distinguishability from the box $(|0\rangle\langle 0|, |1\rangle\langle 1|)$. Since this is possible for any $m \geqslant 0$, it follows that the box $(|0\rangle\langle 0|, |1\rangle\langle 1|)$ has an infinite number of bits of asymmetric distinguishability, consistent with the fact that $D_{\min}(\rho\|\sigma) = \infty$ when $\rho$ is orthogonal to $\sigma$.

The goal of *exact distinguishability dilution* is the opposite: process as few bits of asymmetric distinguishability as possible, using free operations, in order to generate the box $(\rho, \sigma)$. Mathematically, we can phrase this task as the following optimization problem:

$$D_c^0(\rho, \sigma) := \log_2 \inf_{\mathcal{P}\in\mathrm{CPTP}} \{M : \mathcal{P}(|0\rangle\langle 0|) = \rho,\ \mathcal{P}(\pi_M) = \sigma\}, \tag{34}$$

where the choice of $D_c$ in $D_c^0(\rho, \sigma)$ stands for *distinguishability cost* and the “0” in $D_c^0(\rho, \sigma)$ again indicates that we do not allow any error. As we show in Appendix E 2, the following equality holds:

$$D_c^0(\rho, \sigma) = D_{\max}(\rho\|\sigma), \tag{35}$$

where $D_{\max}(\rho\|\sigma)$ is the max-relative entropy [10], as defined in Eq. (22). The equality in Eq. (35) thus assigns to the max-relative entropy $D_{\max}(\rho\|\sigma)$ a fundamental operational meaning as the exact distinguishability cost of the box $(\rho,\sigma)$.

In the case that the support of $\rho$ is not contained in the support of $\sigma$, then there is no finite value of $M$ nor any quantum channel $\mathcal{P}$ that performs the transformations $\mathcal{P}(|0\rangle\langle 0|)=\rho$ and $\mathcal{P}(\pi_M)=\sigma$. However, in the limit $M\to\infty$, the box $(|0\rangle\langle 0|,\pi_M)$ becomes the box $(|0\rangle\langle 0|,|1\rangle\langle 1|)$, which is interpreted as containing an infinite number of bits of asymmetric distinguishability, as discussed after (33). In this case, we can pick the channel $\mathcal{P}$ as $\mathcal{P}(\omega)=\langle 0|\omega|0\rangle\rho+\langle 1|\omega|1\rangle\sigma$, and then the transformation $\mathcal{P}(|0\rangle\langle 0|)=\rho$ and $\mathcal{P}(|1\rangle\langle 1|)=\sigma$ is easily achieved. Thus, in this sense, if the support of $\rho$ is not contained in the support of $\sigma$, then the distinguishability cost $D_c^0(\rho,\sigma)=\infty$, consistent with the fact that $D_{\max}(\rho\|\sigma)=\infty$ in this case.

An important case to consider in any resource theory is the case of independent and identically distributed (i.i.d.) resources. For our case, this means that we should analyze the box $(\rho^{\otimes n},\sigma^{\otimes n})$ for arbitrary $n\geqslant 1$. Due to the additivity of $D_{\min}(\rho\|\sigma)$ and $D_{\max}(\rho\|\sigma)$, it follows that

$$D_d^0(\rho^{\otimes n},\sigma^{\otimes n})=nD_{\min}(\rho\|\sigma), \tag{36}$$

$$D_c^0(\rho^{\otimes n},\sigma^{\otimes n})=nD_{\max}(\rho\|\sigma), \tag{37}$$

so that the number of bits of asymmetric distinguishability distilled and required in each respective task scales precisely linearly with $n$.

Due to the fact that we generally have $D_{\min}(\rho\|\sigma)\neq D_{\max}(\rho\|\sigma)$ for states $\rho$ and $\sigma$, it follows that the resource theory of asymmetric distinguishability is not reversible if we demand exact conversions from one box to another. In fact, the irreversibility in the exact case can be as extreme as desired. By picking $\rho=|0\rangle\langle 0|$ and $\sigma=|\psi\rangle\langle\psi|$ for $|\psi\rangle=\sqrt{1-\delta}|0\rangle+\sqrt{\delta}|1\rangle$ and $\delta\in(0,1)$, we have that $D_{\min}(\rho\|\sigma)=-\log_2(1-\delta)$ while $D_{\max}(\rho\|\sigma)=\infty$ for all $\delta\in(0,1)$, so that the exact distillable distinguishability can be arbitrarily close to zero while the exact distinguishability cost is always infinite in this case.

### C. Approximate distillation and dilution tasks

In realistic experimental scenarios, it is typically not possible to perform transformations exactly, thus motivating the need to consider approximate transformations and approximations of the ideal resources. For the resource theory of asymmetric distinguishability, we define an $\varepsilon$-approximate bit of asymmetric distinguishability as

$$(\widetilde{0}_\varepsilon,\pi), \tag{38}$$

where $\varepsilon\in[0,1]$ and

$$\widetilde{0}_\varepsilon:=(1-\varepsilon)|0\rangle\langle 0|+\varepsilon|1\rangle\langle 1|, \tag{39}$$

so that $\widetilde{0}_\varepsilon\approx_\varepsilon|0\rangle\langle 0|$. The motivation for this choice is operational as before [see the discussion before (13)]. Also, since the maximally mixed state $\pi$ is diagonal in any basis, it suffices to consider (38) as the basic definition of an $\varepsilon$-approximate bit of asymmetric distinguishability, because one could simply perform the diagonalizing unitary for a general qubit state $\tau$ to bring a general box $(\tau,\pi)$ into the form of (38).

Generalizing (26) and (38), the following box represents $m$ approximate bits of asymmetric distinguishability:

$$(\widetilde{0}_\varepsilon,2^{-m}|0\rangle\langle 0|+(1-2^{-m})|1\rangle\langle 1|). \tag{40}$$

If $m$ is an integer, then this box is equivalent by the transformation in Eq. (27) to the following one:

$$(\widetilde{0}_\varepsilon^m,\pi^{\otimes m}), \tag{41}$$

where

$$\widetilde{0}_\varepsilon^m:=(1-\varepsilon)|0\rangle\langle 0|^{\otimes m}+\varepsilon\frac{I^{\otimes m}-|0\rangle\langle 0|^{\otimes m}}{2^m-1}, \tag{42}$$

so that $\widetilde{0}_\varepsilon^m\approx_\varepsilon|0\rangle\langle 0|^{\otimes m}$.

With such a notion in place, we can now generalize exact distillation of asymmetric distinguishability to its approximate version. The goal of *$\varepsilon$-approximate distinguishability distillation* is to distill as many $\varepsilon$-approximate bits of asymmetric distinguishability as possible from a given box $(\rho,\sigma)$. Mathematically, it corresponds to the following optimization for $\varepsilon\in[0,1]$:

$$D_d^\varepsilon(\rho,\sigma):=\log_2\sup_{\mathcal{P}\in\mathrm{CPTP}}\{M:\mathcal{P}(\rho)\approx_\varepsilon|0\rangle\langle 0|,\ \mathcal{P}(\sigma)=\pi_M\}. \tag{43}$$

As we show in Appendix F 1, the following equality holds

$$D_d^\varepsilon(\rho,\sigma)=D_{\min}^\varepsilon(\rho\|\sigma), \tag{44}$$

where $D_{\min}^\varepsilon(\rho\|\sigma)$ is the smooth min-relative entropy [11–13], defined as

$$D_{\min}^\varepsilon(\rho\|\sigma):=-\log_2\inf_{0\leqslant\Lambda\leqslant I}\{\mathrm{Tr}[\Lambda\sigma]:\mathrm{Tr}[\Lambda\rho]\geqslant 1-\varepsilon\}. \tag{45}$$

Thus the equality in Eq. (44) assigns to the smooth min-relative entropy an operational meaning as the $\varepsilon$-approximate distillable distinguishability of the box $(\rho,\sigma)$. This operational interpretation is directly linked to the role of $D_{\min}^\varepsilon(\rho\|\sigma)$ in quantum hypothesis testing [13,31–35]. Note that $D_{\min}^\varepsilon(\rho\|\sigma)$ is also known as "hypothesis testing relative entropy" in the literature, which is terminology introduced in Ref. [13]. This quantity can be computed efficiently by means of a semidefinite program [36], the proof of which we recall in Appendix B.

Note that by combining (31), (44), and the fact that $\lim_{\varepsilon\to 0}D_d^\varepsilon(\rho,\sigma)=D_d^0(\rho,\sigma)$, we conclude the following limit:

$$\lim_{\varepsilon\to 0}D_{\min}^\varepsilon(\rho\|\sigma)=D_{\min}(\rho\|\sigma). \tag{46}$$

We provide an alternative proof in Appendix A 3.

We can also generalize the distinguishability dilution task to the approximate case. In this case, we define the *$\varepsilon$-approximate distinguishability cost* of the box $(\rho,\sigma)$ to be the least number of ideal bits of asymmetric distinguishability that are needed to generate the box $(\rho_\varepsilon,\sigma)$, where $\rho_\varepsilon\approx_\varepsilon\rho$. This notion of approximate distinguishability cost is fully operational and consistent with the more general problem in

Eq. (13). The precise definition of the $\varepsilon$-approximate distinguishability cost of the box $(\rho, \sigma)$ is as follows:

$$D_c^\varepsilon(\rho, \sigma) := \log_2 \inf_{\mathcal{P}\in\text{CPTP}} \{M : \mathcal{P}(|0\rangle\langle 0|) \approx_\varepsilon \rho,\ \mathcal{P}(\pi_M) = \sigma\}. \tag{47}$$

As we show in Appendix F 2, the following equality holds:

$$D_c^\varepsilon(\rho, \sigma) = D_{\max}^\varepsilon(\rho\|\sigma), \tag{48}$$

where $D_{\max}^\varepsilon(\rho\|\sigma)$ is the smooth max-relative entropy [10], defined as

$$D_{\max}^\varepsilon(\rho\|\sigma) := \inf_{\widetilde{\rho}:\frac{1}{2}\|\widetilde{\rho}-\rho\|_1\leqslant\varepsilon} D_{\max}(\widetilde{\rho}\|\sigma). \tag{49}$$

Thus the equality in Eq. (48) assigns to the smooth max-relative entropy a fundamental operational meaning as the $\varepsilon$-approximate distinguishability cost of the box $(\rho, \sigma)$. The smooth max-relative entropy can also be efficiently calculated by means of a semidefinite program, the proof of which we give in Appendix B.

Note that by combining (35), (48), and the fact that $\lim_{\varepsilon\to 0} D_c^\varepsilon(\rho, \sigma) = D_c^0(\rho, \sigma)$, we conclude the following limit:

$$\lim_{\varepsilon\to 0} D_{\max}^\varepsilon(\rho\|\sigma) = D_{\max}(\rho\|\sigma). \tag{50}$$

We provide an alternative proof in Appendix A 3.

An application of the operational approach to distinguishability taken here is the following bound relating $D_{\min}^\varepsilon$ and $D_{\max}^\varepsilon$:

$$D_{\min}^{\varepsilon_1}(\rho\|\sigma) \leqslant D_{\max}^{\varepsilon_2}(\rho\|\sigma) + \log_2\left(\frac{1}{1-\varepsilon_1-\varepsilon_2}\right), \tag{51}$$

where $\varepsilon_1, \varepsilon_2 \geqslant 0$, and $\varepsilon_1 + \varepsilon_2 < 1$. The bound in Eq. (51) is most closely related to the upper bound in theorem 11 in Ref. [37], but we employ a different notion of smoothing for the smooth max-relative entropy. It also generalizes the bound from Eq. (47) in Ref. [36] (by appropriately working through the different conventions here and in Ref. [36]) and is in the same spirit as proposition 5.5 in Ref. [38] and Eq. (22) in Ref. [39].

The main idea for arriving at the bound in Eq. (51) follows from resource-theoretic reasoning. Any approximate distillation protocol performed on the box $(|0\rangle\langle 0|, \pi_M)$ that leads to the box $(\widetilde{0}_\varepsilon, \pi_K)$, for $\varepsilon \in [0, 1)$, is required to obey the bound

$$\log_2 K \leqslant \log_2 M + \log_2(1/[1-\varepsilon]), \tag{52}$$

which follows as a consequence of the fundamental limitation in Eq. (44). One way to realize the transformation $(|0\rangle\langle 0|, \pi_M) \to (\widetilde{0}_\varepsilon, \pi_K)$ is to proceed in two steps: first perform an optimal dilution protocol $(|0\rangle\langle 0|, \pi_M) \to (\rho_{\varepsilon_2}, \sigma)$ such that $\log_2 M = D_{\max}^{\varepsilon_2}(\rho\|\sigma)$ and then perform an optimal distillation protocol $(\rho, \sigma) \to (\widetilde{0}_{\varepsilon_1}, \pi_K)$ such that $\log_2 K = D_{\min}^{\varepsilon_1}(\rho\|\sigma)$. By employing the triangle inequality, the error of the overall transformation is no larger than $\varepsilon_1 + \varepsilon_2$. Since the fundamental limitation in Eq. (52) applies to any protocol, the bound in Eq. (51) follows. We give a detailed proof in Appendix G.

### D. Asymptotic distillable distinguishability and distinguishability cost

We can now reconsider the i.i.d. case of a box $(\rho^{\otimes n}, \sigma^{\otimes n})$ in the context of approximate distillation and dilution. Recall that the quantum relative entropy $D(\rho\|\sigma)$ is defined as [40]

$$D(\rho\|\sigma) := \text{Tr}[\rho(\log_2 \rho - \log_2 \sigma)], \tag{53}$$

if $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ and $D(\rho\|\sigma) = \infty$ otherwise. By defining the asymptotic distillable distinguishability and asymptotic distinguishability cost of the box $(\rho, \sigma)$ as follows:

$$D_d(\rho, \sigma) := \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} D_d^\varepsilon(\rho^{\otimes n}, \sigma^{\otimes n}), \tag{54}$$

$$D_c(\rho, \sigma) := \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} D_c^\varepsilon(\rho^{\otimes n}, \sigma^{\otimes n}), \tag{55}$$

respectively, we conclude from the quantum Stein's lemma [31,32] and the asymptotic equipartition property for the smooth max-relative entropy [41] that

$$D_d(\rho, \sigma) = D_c(\rho, \sigma) = D(\rho\|\sigma), \tag{56}$$

thus demonstrating the fundamental operational interpretation of the quantum relative entropy in the resource theory of asymmetric distinguishability. It is worthwhile to note that we can conclude the stronger statement

$$D_d^\varepsilon(\rho^{\otimes n}, \sigma^{\otimes n}) = nD(\rho\|\sigma) + O(\sqrt{n}), \tag{57}$$

$$D_c^\varepsilon(\rho^{\otimes n}, \sigma^{\otimes n}) = nD(\rho\|\sigma) + O(\sqrt{n}), \tag{58}$$

from Refs. [38,39,42] (see Appendix H). Thus the equality of approximate distillable distinguishability and approximate distinguishability cost in the i.i.d. case holds in the leading order term, with a difference in sublinear in $n$ terms. As discussed in Appendix L, the second-order term in Eq. (57) can be identified exactly by appealing to Refs. [39,42]. The second-order term in Eq. (58) can be identified also by appealing to Ref. [39], but there is a need in this case to change the quantification of error in the resource theory of asymmetric distinguishability from normalized trace distance to infidelity.

As a consequence of the fundamental equality in Eq. (56), we conclude that the resource theory of asymmetric distinguishability is reversible in the asymptotic setting. That is, for large $n$, by starting with the box $(\rho^{\otimes n}, \sigma^{\otimes n})$ one can distill it approximately to $nD(\rho\|\sigma)$ bits of asymmetric distinguishability, and then one can dilute these $nD(\rho\|\sigma)$ bits of asymmetric distinguishability back to the box $(\rho^{\otimes n}, \sigma^{\otimes n})$ approximately.

### E. Asymptotic box transformations

We can also solve the asymptotic box transformation problem stated around (15). Before doing so, let us formalize the problem. Let $n, m \in \mathbb{Z}^+$ and $\varepsilon \in [0, 1]$. An $(n, m, \varepsilon)$ box transformation protocol for the boxes $(\rho, \sigma)$ and $(\tau, \omega)$ consists of a channel $\mathcal{N}^{(n)}$ such that

$$\mathcal{N}^{(n)}(\rho^{\otimes n}) \approx_\varepsilon \tau^{\otimes m}, \tag{59}$$

$$\mathcal{N}^{(n)}(\sigma^{\otimes n}) = \omega^{\otimes m}. \tag{60}$$

A rate $R$ is *achievable* if for all $\varepsilon \in (0, 1]$, $\delta > 0$, and sufficiently large $n$, there exists an $(n, n[R-\delta], \varepsilon)$ box transformation protocol. The optimal box transformation rate $R((\rho,\sigma)\to(\tau,\omega))$ is then equal to the supremum of all achievable rates.

On the other hand, a rate $R$ is a *strong converse rate* if for all $\varepsilon\in[0,1)$, $\delta>0$, and sufficiently large $n$, there does not exist an $(n, n[R+\delta], \varepsilon)$ box transformation protocol. The strong converse box transformation rate $\widetilde{R}((\rho,\sigma)\to(\tau,\omega))$ is then equal to the infimum of all strong converse rates.

Note that the following inequality is a consequence of the definitions:

$$R((\rho,\sigma)\to(\tau,\omega))\leqslant\widetilde{R}((\rho,\sigma)\to(\tau,\omega)). \tag{61}$$

The final result of our paper is the following fundamental equality for the resource theory of asymmetric distinguishability:

$$R((\rho,\sigma)\to(\tau,\omega))=\widetilde{R}((\rho,\sigma)\to(\tau,\omega))=\frac{D(\rho\|\sigma)}{D(\tau\|\omega)}, \tag{62}$$

indicating that the quantum relative entropy plays a central role as the optimal conversion rate between boxes.

We should clarify (62) a bit further. It holds whenever $\mathrm{supp}(\rho)\subseteq\mathrm{supp}(\sigma)$ and $\mathrm{supp}(\tau)\subseteq\mathrm{supp}(\omega)$. If $\mathrm{supp}(\rho)\subseteq\mathrm{supp}(\sigma)$ but $\mathrm{supp}(\tau)\nsubseteq\mathrm{supp}(\omega)$, then $\frac{D(\rho\|\sigma)}{D(\tau\|\omega)}=0$ and it is not possible to perform the transformation at a non-negligible rate. If $\mathrm{supp}(\rho)\nsubseteq\mathrm{supp}(\sigma)$ but $\mathrm{supp}(\tau)\subseteq\mathrm{supp}(\omega)$, then $\frac{D(\rho\|\sigma)}{D(\tau\|\omega)}=\infty$ and it is possible to produce as many copies of $\tau$ and $\omega$ as desired.

The proof of this result consists of two parts: achievability and optimality. For the achievability part, i.e., the bound

$$R((\rho,\sigma)\to(\tau,\omega))\geqslant\frac{D(\rho\|\sigma)}{D(\tau\|\omega)}, \tag{63}$$

we first distill bits of asymmetric distinguishability from $(\rho^{\otimes n},\sigma^{\otimes n})$ at the rate $\approx D(\rho\|\sigma)$. After doing so, we then dilute these $\approx nD(\rho\|\sigma)$ bits of asymmetric distinguishability to the box $(\tau^{\otimes m},\omega^{\otimes m})$, such that $m\approx n[D(\rho\|\sigma)/D(\tau\|\omega)]$, establishing that $R((\rho,\sigma)\to(\tau,\omega))\geqslant\frac{D(\rho\|\sigma)}{D(\tau\|\omega)}$. For the optimality part, i.e., the strong converse bound

$$\widetilde{R}((\rho,\sigma)\to(\tau,\omega))\leqslant\frac{D(\rho\|\sigma)}{D(\tau\|\omega)}, \tag{64}$$

we suppose that there exists a sequence of $(n,m,\varepsilon)$ box transformation protocols and then employ a pseudocontinuity inequality for sandwiched Rényi relative entropy (lemma 1) and its data processing inequality to conclude that $\widetilde{R}((\rho,\sigma)\to(\tau,\omega))\leqslant\frac{D(\rho\|\sigma)}{D(\tau\|\omega)}$. Alternatively, we can employ a pseudocontinuity inequality for the Petz-Rényi relative entropy (lemma 3) and its data processing inequality. See Appendix J for details. We note here that the bounds in propositions 1 and 2 are *exponential strong converse bounds*, demonstrating that the error in the transformation converges to one exponentially fast if the rate of conversion is strictly larger than $\frac{D(\rho\|\sigma)}{D(\tau\|\omega)}$.

## IV. CONCLUSION

In this paper, we have developed the resource theory of asymmetric distinguishability. The main constituents consist of boxes as the objects of manipulation, all quantum channels as the free operations, and bits of asymmetric distinguishability as the fundamental currency of interconversion. The resource theory is reversible in the asymptotic case, and the quantum relative entropy emerges as the fundamental rate at which boxes can be converted. Our one-shot results can be compactly summarized as follows. (1) The min-relative entropy is equal to the exact one-shot distillable distinguishability. (2) The max-relative entropy is equal to the exact one-shot distinguishability cost. (3) The smooth min-relative entropy is equal to the approximate one-shot distillable distinguishability. (4) The smooth max-relative entropy is equal to the approximate one-shot distinguishability cost.

Thus each of these one-shot entropies are fundamentally operational quantities. Finally, the ratio of quantum relative entropies of two pairs of quantum states is equal to the optimal rate of asymptotic box transformations between them.

Going forward from here, there are many interesting directions to pursue. The resource theory of asymmetric distinguishability for quantum channels has recently been developed in Ref. [43]. The main constituents consist of a channel box $(\mathcal{N},\mathcal{M})$, for quantum channels $\mathcal{N}$ and $\mathcal{M}$, as the basic objects of manipulation, superchannels [44] as the free operations, and bits of asymmetric distinguishability as the fundamental currency. Some basic results are that the one-shot distillable distinguishability of a channel box is equal to the smooth channel min-relative entropy [45], and the one-shot distinguishability cost is equal to the smooth channel max-relative entropy [46,47]. The theory reduces to the theory for quantum states in the case that the channels are environment-seizable, as defined in Ref. [48].

It remains open to determine optimal error exponents and strong converse exponents for the distinguishability dilution task, as well as for the more general box transformation problem. These quantities have been established for distinguishability distillation (i.e., hypothesis testing) [49–53], and so there is a strong possibility that these operational quantities could be determined for the dual task. Some of the bounds in Appendix K could be useful for this purpose. The same questions remain open for second-order asymptotics.

In Appendix L, we explore a variation of the resource theory of asymmetric distinguishability in which the infidelity is employed as a measure of approximation, rather than the normalized trace distance. There are similar interesting questions regarding this variation, in particular, whether error exponents and strong converse exponents for distinguishability dilution could be proven to be optimal.

One could also consider the case in which the boxes consist of not just two states but multiple states, connecting with the theory of quantum state discrimination [54,55]. The boxes could even consist of a continuum of states or channels, connecting with quantum estimation theory [56,57] and the resource theoretic approach put forward in Ref. [58]. The boxes could also consist of a state and a set of states, with the set of free operations restricted, which allows for connecting with general resource theories [8,59]. Extending this, the boxes could consist of a channel and set of channels, with restricted free operations, allowing to connect with general resource theories of quantum channels [47,60].

A particularly interesting direction would be to consider reversibility of the resource theory of asymmetric distinguishability beyond the first order and investigate resource resonance effects. For this direction, the recent results of Refs. [61–64] are quite relevant. Related to this, one could investigate more fine-grained questions related to asymptotic reversibility along the lines of Ref. [65], where we expect similar findings to hold in the resource theory of asymmetric distinguishability.

In the Appendices, we provide detailed proofs of all claims in the main text. As a resource, we have included derivations of some of the dual semidefinite programs listed below as an ancillary file available for download with the arXiv posting of this paper. Appendix A begins by providing some background facts, some of which can be found in Ref. [67].

*Note added in proof.* Recently, we learned about the independent and related work of Ref. [66].

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## APPENDIX A: BACKGROUND

### 1. Normalized trace distance

A quantum state is described mathematically by a positive semidefinite operator with trace equal to one. The normalized trace distance between two quantum states $\rho$ and $\sigma$ is given by $\frac{1}{2}\|\rho-\sigma\|_1$, where the trace norm of an operator $A$ is defined as $\|A\|_1 = \mathrm{Tr}[\sqrt{A^\dagger A}]$. The following variational characterization of the normalized trace distance is well known [56]:

$$\tfrac{1}{2}\|\rho-\sigma\|_1 = \sup_{\Lambda \geqslant 0} \{\mathrm{Tr}[\Lambda(\rho-\sigma)] : \Lambda \leqslant I\}, \tag{A1}$$

endowing the normalized trace distance with its operational meaning as the largest probability difference that a single POVM element can assign to two quantum states. The right-hand side of (A1) is a semidefinite program as written, with the following dual:

$$\inf_{Y \geqslant 0} \{\mathrm{Tr}[Y] : Y \geqslant \rho-\sigma\} = \tfrac{1}{2}\|\rho-\sigma\|_1, \tag{A2}$$

where the equality holds from strong duality.

### 2. Choi isomorphism

The Choi isomorphism is a standard way of characterizing quantum channels that is suitable for optimizing over them in semidefinite programs. For a quantum channel $\mathcal{N}_{A\to B}$, its Choi operator is given by

$$J^{\mathcal{N}}_{RB} := \mathcal{N}_{A\to B}(\Gamma_{RA}), \tag{A3}$$

where $\Gamma_{RA} = |\Gamma\rangle\langle\Gamma|_{RA}$ and

$$|\Gamma\rangle_{RA} := \sum_i |i\rangle_R |i\rangle_A, \tag{A4}$$

with $\{|i\rangle_R\}_i$ and $\{|i\rangle_A\}_i$ orthonormal bases. The Choi operator is positive semidefinite $J^{\mathcal{N}}_{RB} \geqslant 0$, corresponding to $\mathcal{N}_{A\to B}$ being completely positive, and satisfies $\mathrm{Tr}_B[J^{\mathcal{N}}_{RB}] = I_R$, the latter corresponding to $\mathcal{N}_{A\to B}$ being trace preserving.

On the other hand, given an operator $J^{\mathcal{M}}_{RB}$ satisfying $J^{\mathcal{M}}_{RB} \geqslant 0$ and $\mathrm{Tr}_B[J^{\mathcal{M}}_{RB}] = I_R$, one realizes via postselected teleportation [68] the following quantum channel:

$$\mathcal{M}_{A\to B}(\rho_A) = \langle\Gamma|_{SR}\big(\rho_S \otimes J^{\mathcal{M}}_{RB}\big)|\Gamma\rangle_{SR} \tag{A5}$$

$$= \mathrm{Tr}_R\left[T_R(\rho_R) J^{\mathcal{M}}_{RB}\right], \tag{A6}$$

where systems $S$, $R$, and $A$ are isomorphic and the last line employs the facts that $(M_S \otimes I_R)|\Gamma\rangle_{SR} = (I_S \otimes T_R(M_R))|\Gamma\rangle_{SR}$ for $T_R$ the transpose map, defined as

$$T_R(\rho_R) = \sum_{i,j} |i\rangle\langle j|_R \rho_R |i\rangle\langle j|_R, \tag{A7}$$

and $\langle\Gamma|_{SR}(I_S \otimes X_{RB})|\Gamma\rangle_{SR} = \mathrm{Tr}_R[X_{RB}]$. We often abbreviate the transpose map simply as

$$\rho_R^T = T_R(\rho_R). \tag{A8}$$

Since the constraints $J^{\mathcal{M}}_{RB} \geqslant 0$ and $\mathrm{Tr}_B[J^{\mathcal{M}}_{RB}] = I_R$ are semidefinite, this is a useful way of incorporating optimizations over quantum channels into semidefinite programs.

### 3. Relative entropies and data processing

The Petz-Rényi relative entropy is defined for states $\rho$ and $\sigma$ as [69]

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log_2 \mathrm{Tr}[\rho^\alpha \sigma^{1-\alpha}] \tag{A9}$$

$$= \frac{2}{\alpha-1}\log_2 \|\rho^{\alpha/2}\sigma^{(1-\alpha)/2}\|_2, \tag{A10}$$

if $\alpha \in (0,1)$ or $\alpha \in (1,\infty)$ and $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$. If $\alpha \in (1,\infty)$ and $\mathrm{supp}(\rho) \not\subseteq \mathrm{supp}(\sigma)$, then $D_\alpha(\rho\|\sigma) = \infty$ [41]. The Petz-Rényi relative entropy obeys the following data processing inequality [41,69,70] for a quantum channel $\mathcal{N}$ and $\alpha \in (0,1) \cup (1,2]$:

$$D_\alpha(\rho\|\sigma) \geqslant D_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \tag{A11}$$

The following limits hold:

$$\lim_{\alpha\to 1} D_\alpha(\rho\|\sigma) = D(\rho\|\sigma), \tag{A12}$$

$$\lim_{\alpha\to 0} D_\alpha(\rho\|\sigma) = D_{\min}(\rho\|\sigma), \tag{A13}$$

where $D(\rho\|\sigma)$ is the quantum relative entropy defined in Eq. (53) and $D_{\min}(\rho\|\sigma)$ is defined in Eq. (21). The Petz-Rényi relative entropies are ordered in the following sense [41,70]:

$$D_\alpha(\rho\|\sigma) \geqslant D_\beta(\rho\|\sigma), \tag{A14}$$

for $\alpha \geqslant \beta > 0$.

The sandwiched Rényi relative entropy is defined for states $\rho$ and $\sigma$ as [71,72]

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log_2 \mathrm{Tr}[(\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha})^\alpha]$$

$$= \frac{\alpha}{\alpha-1}\log_2 \|\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha}\|_\alpha$$

$$= \frac{2\alpha}{\alpha-1}\log_2 \|\sigma^{(1-\alpha)/2\alpha}\rho^{1/2}\|_{2\alpha}, \tag{A15}$$

if $\alpha \in (0,1)$ or $\alpha \in (1,\infty)$ and $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$. If $\alpha \in (1,\infty)$ and $\mathrm{supp}(\rho) \not\subseteq \mathrm{supp}(\sigma)$, then $\widetilde{D}_\alpha(\rho\|\sigma) = \infty$. The sandwiched Rényi relative entropy obeys the following data processing inequality [73] for a quantum channel $\mathcal{N}$ and $\alpha \in [1/2,1) \cup (1,\infty)$:

$$\widetilde{D}_\alpha(\rho\|\sigma) \geqslant \widetilde{D}_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \tag{A16}$$

[See [74] for an alternative proof of (A16).] The following limits hold:

$$\lim_{\alpha\to 1} \widetilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma), \tag{A17}$$

$$\lim_{\alpha\to \infty} \widetilde{D}_\alpha(\rho\|\sigma) = D_{\max}(\rho\|\sigma), \tag{A18}$$

$$\lim_{\alpha\to 1/2} \widetilde{D}_\alpha(\rho\|\sigma) = -\log F(\rho,\sigma), \tag{A19}$$

where

$$F(\rho,\sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2 \tag{A20}$$

is the quantum fidelity [75]. The sandwiched Rényi relative entropies are ordered in the following sense [71]:

$$\widetilde{D}_\alpha(\rho\|\sigma) \geqslant \widetilde{D}_\beta(\rho\|\sigma), \tag{A21}$$

for $\alpha \geqslant \beta > 0$.

Note that the following inequality holds:

$$D_{\min}(\rho\|\sigma) \leqslant \widetilde{D}_{1/2}(\rho\|\sigma), \tag{A22}$$

as a consequence of the equality [76]

$$F(\rho,\sigma) = \left(\inf_{\{\Lambda_x\}_x} \sum_x \sqrt{\mathrm{Tr}[\Lambda_x\rho]\,\mathrm{Tr}[\Lambda_x\sigma]}\right)^2, \tag{A23}$$

where the optimization is with respect to POVMs $\{\Lambda_x\}_x$, and by choosing this POVM suboptimally as $\{\Pi_\rho, I-\Pi_\rho\}$.

The min-relative entropy obeys the data processing inequality for states $\rho$ and $\sigma$ and a quantum channel $\mathcal{N}$:

$$D_{\min}(\rho\|\sigma) \geqslant D_{\min}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \tag{A24}$$

This inequality was proved in Ref. [10] by utilizing its relation to the Petz-Rényi relative entropies. For an alternative proof, first note that the inequality in Eq. (A24) is equivalent to

$$\mathrm{Tr}[\Pi_\rho\sigma] \leqslant \mathrm{Tr}[\Pi_{\mathcal{N}(\rho)}\mathcal{N}(\sigma)]. \tag{A25}$$

To see the latter, let $U$ be an isometric extension of the channel $\mathcal{N}$, so that

$$\mathcal{N}_{A\to B}(\omega_A) = \mathrm{Tr}_E[U_{A\to BE}\,\omega_A(U_{A\to BE})^\dagger]. \tag{A26}$$

Then we find that

$$\mathrm{Tr}[\Pi_\rho\sigma] = \mathrm{Tr}[U\Pi_\rho U^\dagger U\sigma U^\dagger] \tag{A27}$$

$$= \mathrm{Tr}[\Pi_{U\rho U^\dagger} U\sigma U^\dagger] \tag{A28}$$

$$\leqslant \mathrm{Tr}[(\Pi_{\mathcal{N}(\rho)}\otimes I_E)U\sigma U^\dagger] \tag{A29}$$

$$= \mathrm{Tr}[\Pi_{\mathcal{N}(\rho)}\mathcal{N}(\sigma)]. \tag{A30}$$

The first equality follows because $U\Pi_\rho U^\dagger = \Pi_{U\rho U^\dagger}$. The inequality follows because the support of $U\rho U^\dagger$ is contained in the support of $\mathcal{N}(\rho)\otimes I_E$, see Appendix B in Ref. [77].

The smooth min-relative entropy obeys the data processing inequality as well, in fact for any trace nonincreasing positive map $\mathcal{N}$ and for all $\varepsilon \in (0,1)$:

$$D^\varepsilon_{\min}(\rho\|\sigma) \geqslant D^\varepsilon_{\min}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \tag{A31}$$

This follows from the definition. Let $\Lambda$ be an arbitrary operator such that $\mathrm{Tr}[\Lambda\mathcal{N}(\rho)] \geqslant 1-\varepsilon$ and $0 \leqslant \Lambda \leqslant I$. Then it follows that $\mathrm{Tr}[\mathcal{N}^\dagger(\Lambda)\rho] = \mathrm{Tr}[\Lambda\mathcal{N}(\rho)] \geqslant 1-\varepsilon$ and $0 \leqslant \mathcal{N}^\dagger(\Lambda) \leqslant \mathcal{N}^\dagger(I) \leqslant I$, the latter inequalities following because $\mathcal{N}^\dagger$ is a positive map if $\mathcal{N}$ is and $\mathcal{N}^\dagger$ is subunital if $\mathcal{N}$ is trace non-increasing. So then $\mathcal{N}^\dagger(\Lambda)$ is a candidate for $D^\varepsilon_{\min}(\rho\|\sigma)$ and thus $D^\varepsilon_{\min}(\rho\|\sigma) \geqslant -\log\mathrm{Tr}[\mathcal{N}^\dagger(\Lambda)\sigma] = -\log\mathrm{Tr}[\Lambda\mathcal{N}(\sigma)]$. Since the argument holds for an arbitrary $\Lambda$ satisfying $\mathrm{Tr}[\Lambda\mathcal{N}(\rho)] \geqslant 1-\varepsilon$ and $0 \leqslant \Lambda \leqslant I$, we conclude (A31).

The max-relative entropy also obeys the data processing inequality for an arbitrary positive map $\mathcal{N}$:

$$D_{\max}(\rho\|\sigma) \geqslant D_{\max}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \tag{A32}$$

To see this, let $\lambda$ be such that $\rho \leqslant 2^\lambda\sigma$. Then from the fact that $\mathcal{N}$ is positive, it follows that $\mathcal{N}(\rho) \leqslant 2^\lambda\mathcal{N}(\sigma)$. It then follows that

$$\lambda \geqslant \inf\{\mu : \mathcal{N}(\rho) \leqslant 2^\mu\mathcal{N}(\sigma)\} \tag{A33}$$

$$= D_{\max}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \tag{A34}$$

Since this is true for arbitrary $\lambda$ satisfying $\rho \leqslant 2^\lambda\sigma$, we conclude (A32).

The smooth max-relative entropy obeys the data processing inequality for a positive, trace-preserving map $\mathcal{N}$ and for all $\varepsilon \in (0,1)$:

$$D^\varepsilon_{\max}(\rho\|\sigma) \geqslant D^\varepsilon_{\max}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \tag{A35}$$

To see this, let $\widetilde{\rho}$ be an arbitrary state such that

$$\tfrac{1}{2}\|\widetilde{\rho}-\rho\|_1 \leqslant \varepsilon. \tag{A36}$$

Then from the data processing inequality for normalized trace distance under positive trace-preserving maps, it follows that

$$\tfrac{1}{2}\|\mathcal{N}(\widetilde{\rho})-\mathcal{N}(\rho)\|_1 \leqslant \varepsilon. \tag{A37}$$

So it follows that

$$D_{\max}(\widetilde{\rho}\|\sigma) \geqslant D_{\max}(\mathcal{N}(\widetilde{\rho})\|\mathcal{N}(\sigma)) \tag{A38}$$

$$\geqslant D^\varepsilon_{\max}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \tag{A39}$$

Since the inequality holds for an arbitrary state $\widetilde{\rho}$ satisfying (A36), we conclude (A35).

Since all of the above quantities obey the data processing inequality for quantum channels, we conclude that they are invariant under the action of an isometric channel $\mathcal{U}(\cdot) = U(\cdot)U^\dagger$:

$$D_{\min}(\rho\|\sigma) = D_{\min}(\mathcal{U}(\rho)\|\mathcal{U}(\sigma)), \tag{A40}$$

$$D^\varepsilon_{\min}(\rho\|\sigma) = D^\varepsilon_{\min}(\mathcal{U}(\rho)\|\mathcal{U}(\sigma)), \tag{A41}$$

$$D_{\max}(\rho\|\sigma) = D_{\max}(\mathcal{U}(\rho)\|\mathcal{U}(\sigma)), \tag{A42}$$

$$D^\varepsilon_{\max}(\rho\|\sigma) = D^\varepsilon_{\max}(\mathcal{U}(\rho)\|\mathcal{U}(\sigma)), \tag{A43}$$

which follows because $\mathcal{U}$ is a channel and the channel in Eq. (8) perfectly reverses the action of $\mathcal{U}$.

As stated in Eq. (46), the following limit holds:

$$\lim_{\varepsilon\to 0} D^{\varepsilon}_{\min}(\rho\|\sigma) = D_{\min}(\rho\|\sigma). \tag{A44}$$

In the main text, we provided an operational proof of this limit. An alternative proof goes as follows. Consider that the following inequality holds for all $\varepsilon \in (0,1)$:

$$D^{\varepsilon}_{\min}(\rho\|\sigma) \geqslant D_{\min}(\rho\|\sigma), \tag{A45}$$

because the measurement operator $\Pi_\rho$ (projection onto support of $\rho$) satisfies $\mathrm{Tr}[\Pi_\rho \rho] \geqslant 1-\varepsilon$ for all $\varepsilon \in (0,1)$. So we conclude that

$$\liminf_{\varepsilon\to 0} D^{\varepsilon}_{\min}(\rho\|\sigma) \geqslant D_{\min}(\rho\|\sigma). \tag{A46}$$

Alternatively, suppose that $\Lambda$ is a measurement operator satisfying $\mathrm{Tr}[\Lambda\rho] = 1-\varepsilon$ (note that when optimizing $D^{\varepsilon}_{\min}$, it suffices to optimize over measurement operators satisfying the constraint $\mathrm{Tr}[\Lambda\rho] \geqslant 1-\varepsilon$ with equality [78]). Then applying the data processing inequality for $D_\alpha(\rho\|\sigma)$ under the measurement $\{\Lambda, I-\Lambda\}$, which holds for $\alpha \in (0,1)$, we find that

$$\begin{aligned} D_\alpha(\rho\|\sigma) \geqslant \frac{1}{\alpha-1}\log_2[&(1-\varepsilon)^\alpha\,\mathrm{Tr}[\Lambda\sigma]^{1-\alpha} \\ &+\varepsilon^\alpha(1-\mathrm{Tr}[\Lambda\sigma])^{1-\alpha}]. \end{aligned} \tag{A47}$$

Since this bound holds for all measurement operators $\Lambda$ satisfying $\mathrm{Tr}[\Lambda\rho] = 1-\varepsilon$, we conclude the following bound for all $\alpha \in (0,1)$:

$$\begin{aligned} D_\alpha(\rho\|\sigma) \geqslant \frac{1}{\alpha-1}\log_2\Big[&(1-\varepsilon)^\alpha\big(2^{-D^{\varepsilon}_{\min}(\rho\|\sigma)}\big)^{1-\alpha} \\ &+\varepsilon^\alpha\big(1-2^{-D^{\varepsilon}_{\min}(\rho\|\sigma)}\big)^{1-\alpha}\Big]. \end{aligned} \tag{A48}$$

Now taking the limit of the right-hand side as $\varepsilon \to 0$, we find that the following bound holds for all $\alpha \in (0,1)$:

$$D_\alpha(\rho\|\sigma) \geqslant \limsup_{\varepsilon\to 0} D^{\varepsilon}_{\min}(\rho\|\sigma). \tag{A49}$$

Since the bound holds for all $\alpha \in (0,1)$, we can take the limit on the left-hand side to arrive at

$$\lim_{\alpha\to 0} D_\alpha(\rho\|\sigma) = D_{\min}(\rho\|\sigma) \geqslant \limsup_{\varepsilon\to 0} D^{\varepsilon}_{\min}(\rho\|\sigma). \tag{A50}$$

Now putting together (A46) and (A50), we conclude (A44).

As stated in Eq. (50), the following limit holds:

$$\lim_{\varepsilon\to 0} D^{\varepsilon}_{\max}(\rho\|\sigma) = D_{\max}(\rho\|\sigma). \tag{A51}$$

In the main text, we provided an operational proof of this limit. An alternative proof goes as follows. Consider that the following bound holds for all $\varepsilon \in (0,1)$:

$$D^{\varepsilon}_{\max}(\rho\|\sigma) \leqslant D_{\max}(\rho\|\sigma), \tag{A52}$$

which follows as a simple consequence of the fact that we can always set $\widetilde{\rho} = \rho$. Then the following limit holds

$$\limsup_{\varepsilon\to 0} D^{\varepsilon}_{\max}(\rho\|\sigma) \leqslant D_{\max}(\rho\|\sigma). \tag{A53}$$

To see the other inequality, let $\widetilde{\rho}$ be a state satisfying $\frac{1}{2}\|\widetilde{\rho}-\rho\|_1 \leqslant \varepsilon$. Then this means that $\|\widetilde{\rho}-\rho\|_\infty \leqslant 2\varepsilon$. Consider that

$$\begin{aligned} &D_{\max}(\widetilde{\rho}\|\sigma) \\ &= \log_2\|\sigma^{-1/2}\widetilde{\rho}\sigma^{-1/2}\|_\infty \\ &\geqslant \log_2(\|\sigma^{-1/2}\rho\sigma^{-1/2}\|_\infty - \|\sigma^{-1/2}(\widetilde{\rho}-\rho)\sigma^{-1/2}\|_\infty) \\ &\geqslant \log_2\big(\|\sigma^{-1/2}\rho\sigma^{-1/2}\|_\infty - \|\sigma^{-1/2}\|^2_\infty\|\widetilde{\rho}-\rho\|_\infty\big) \\ &\geqslant \log_2(\|\sigma^{-1/2}\rho\sigma^{-1/2}\|_\infty - 2\varepsilon\|\sigma^{-1}\|_\infty). \end{aligned} \tag{A54}$$

Since this bound holds for all $\widetilde{\rho}$ satisfying $\frac{1}{2}\|\widetilde{\rho}-\rho\|_1 \leqslant \varepsilon$, we conclude that

$$D^{\varepsilon}_{\max}(\rho\|\sigma) \geqslant \log_2(\|\sigma^{-1/2}\rho\sigma^{-1/2}\|_\infty - 2\varepsilon\|\sigma^{-1}\|_\infty). \tag{A55}$$

Then taking the limit $\varepsilon \to 0$, we find that

$$\begin{aligned} \liminf_{\varepsilon\to 0} D^{\varepsilon}_{\max}(\rho\|\sigma) &\geqslant \log_2\|\sigma^{-1/2}\rho\sigma^{-1/2}\|_\infty \\ &= D_{\max}(\rho\|\sigma). \end{aligned} \tag{A56}$$

Putting together (A53) and (A56), we conclude (A51).

## APPENDIX B: SDPs FOR SMOOTH MIN- AND MAX-RELATIVE ENTROPIES

Here we show that the smooth min- and max-relative entropies are characterized by semidefinite programs. We also give the dual programs for convenience.

Consider that

$$D^{\varepsilon}_{\min}(\rho\|\sigma) = -\log_2 \inf_{\Lambda\geqslant 0}\left\{\begin{matrix}\mathrm{Tr}[\Lambda\sigma] : \Lambda \leqslant I, \\ \mathrm{Tr}[\Lambda\rho] \geqslant 1-\varepsilon\end{matrix}\right\}, \tag{B1}$$

which is an SDP as written. The dual SDP is given by

$$-\log_2 \sup_{\mu, X\geqslant 0}\left\{\begin{matrix}\mu(1-\varepsilon) - \mathrm{Tr}[X] : \\ \mu\rho \leqslant \sigma + X\end{matrix}\right\}, \tag{B2}$$

and is equal to $D^{\varepsilon}_{\min}(\rho\|\sigma)$ by strong duality. See [36] in this context.

By employing the definition of the smooth max-relative entropy in Eq. (49) and the dual characterization of the normalized trace distance in Eq. (A2), we find that

$$D^{\varepsilon}_{\max}(\rho\|\sigma) = \log\inf\left\{\begin{matrix}\lambda : \\ \widetilde{\rho} \leqslant \lambda\sigma,\ \mathrm{Tr}[Y] \leqslant \varepsilon, \\ \mathrm{Tr}[\widetilde{\rho}] = 1,\ Y \geqslant \rho - \widetilde{\rho}, \\ \widetilde{\rho},\ Y \geqslant 0\end{matrix}\right\}. \tag{B3}$$

The dual SDP is given by

$$\log\sup\left\{\begin{matrix}\mathrm{Tr}[Q\rho] + \mu - \varepsilon t : \\ \mathrm{Tr}[X\sigma] \leqslant 1,\ Q \leqslant tI, \\ Q + \mu I \leqslant X, \\ X, Q, t \geqslant 0,\ \mu \in \mathbb{R}\end{matrix}\right\}, \tag{B4}$$

and is equal to $D^{\varepsilon}_{\max}(\rho\|\sigma)$ by strong duality.

## APPENDIX C: APPROXIMATE BOX TRANSFORMATION IS AN SDP

We prove that the approximate box transformation problem can be computed by a semidefinite program. First, recall that the problem is characterized by

$$\varepsilon((\rho,\sigma)\to(\tau,\omega)) := \inf_{\mathcal{N}\in\mathrm{CPTP}}\{\varepsilon\in[0,1]:\mathcal{N}(\rho)\approx_\varepsilon \tau,\ \mathcal{N}(\sigma)=\omega\}, \quad \text{(C1)}$$

for states $\rho$, $\sigma$, $\tau$, and $\omega$. By employing the dual form of the trace distance from (A2), we find that

$$\varepsilon((\rho,\sigma)\to(\tau,\omega)) = \inf_{Y_B,J^{\mathcal{N}}_{RB}}\left\{\begin{array}{c}\mathrm{Tr}[Y_B]:\\ Y_B\geqslant \tau_B-\mathrm{Tr}_R\left[\rho_R^T J^{\mathcal{N}}_{RB}\right],\\ \mathrm{Tr}_R\left[\sigma_R^T J^{\mathcal{N}}_{RB}\right]=\omega_B,\\ \mathrm{Tr}_B\left[J^{\mathcal{N}}_{RB}\right]=I_R,\ Y_B, J^{\mathcal{N}}_{RB}\geqslant 0\end{array}\right\}. \quad \text{(C2)}$$

The dual program is given by

$$\varepsilon((\rho,\sigma)\to(\tau,\omega)) = \sup_{X_B,W_B,Z_R}\left\{\begin{array}{c}\mathrm{Tr}[\tau_B X_B]+\mathrm{Tr}[\omega_B W_B]+\mathrm{Tr}[Z_R]:\\ X_B\leqslant I_B,\\ \rho_R^T\otimes X_B+\sigma_R^T\otimes W_B+Z_R\otimes I_B\leqslant 0,\\ X_B\geqslant 0,\ W_B, Z_R\in\mathrm{Herm}\end{array}\right\}, \quad \text{(C3)}$$

with the equality holding from strong duality.

## APPENDIX D: IMPOSSIBILITY OF DISTINGUISHABILITY INCREASING TRANSFORMATIONS

It is impossible for a quantum channel $\mathcal{N}$ to increase the distinguishability of a box $(\rho,\sigma)$. That is, it impossible for the transformation $(\rho,\sigma)\xrightarrow{\mathcal{N}}(\mathcal{N}(\rho),\mathcal{N}(\sigma))$ to be such that the distinguishability of $(\mathcal{N}(\rho),\mathcal{N}(\sigma))$ is strictly larger than the distinguishability of $(\rho,\sigma)$. This follows as a direct consequence of the data processing inequality for quantum relative entropy [79]:

$$D(\rho\|\sigma)\geqslant D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)), \quad \text{(D1)}$$

when using quantum relative entropy as a quantifier of distinguishability.

For the specific transformation in Eq. (20), we find that

$$m = D(|0\rangle\langle 0|^{\otimes m}\|\pi^{\otimes m}), \quad \text{(D2)}$$

$$n = D(|0\rangle\langle 0|^{\otimes n}\|\pi^{\otimes n}), \quad \text{(D3)}$$

so that if the transformation in Eq. (20) existed, it would violate (D1), due to the assumption $n>m$.

The fact that the transformation in Eq. (20) does not exist can also be seen as a consequence of the linearity of quantum channels. Let us first suppose that the boxes $(|0\rangle\langle 0|^{\otimes m},\pi^{\otimes m})$ and $(|0\rangle\langle 0|^{\otimes n},\pi^{\otimes n})$ have been reversibly transformed to their standard form as

$$(|0\rangle\langle 0|,\pi_{2^m}), \quad \text{(D4)}$$

$$(|0\rangle\langle 0|,\pi_{2^n}), \quad \text{(D5)}$$

respectively, where we recall that $\pi_{2^m}=2^{-m}|0\rangle\langle 0|+(1-2^{-m})|1\rangle\langle 1|$. Then the original question is equivalent to the question of whether there exists a channel $\mathcal{N}$ that takes the first box to the second for $n>m$. Such a channel would then perform the transformations:

$$\mathcal{N}(|0\rangle\langle 0|)=|0\rangle\langle 0|, \quad \text{(D6)}$$

$$\mathcal{N}(2^{-m}|0\rangle\langle 0|+(1-2^{-m})|1\rangle\langle 1|) = 2^{-n}|0\rangle\langle 0|+(1-2^{-n})|1\rangle\langle 1|. \quad \text{(D7)}$$

By linearity of the channel, consider that we can conclude the action of the channel on the orthogonal state $|1\rangle\langle 1|$:

$$\mathcal{N}(|1\rangle\langle 1|)=\mathcal{N}((1-2^{-m})^{-1}(\pi_{2^m}-2^{-m}|0\rangle\langle 0|)) \quad \text{(D8)}$$

$$=(1-2^{-m})^{-1}[\mathcal{N}(\pi_{2^m})-2^{-m}\mathcal{N}(|0\rangle\langle 0|)] \quad \text{(D9)}$$

$$=(1-2^{-m})^{-1}\pi_{2^n}-(1-2^{-m})^{-1}2^{-m}|0\rangle\langle 0| \quad \text{(D10)}$$

$$=(1-2^{-m})^{-1}(2^{-n}|0\rangle\langle 0|+(1-2^{-n})|1\rangle\langle 1|) -(1-2^{-m})^{-1}2^{-m}|0\rangle\langle 0| \quad \text{(D11)}$$

$$=\frac{2^{-n}-2^{-m}}{1-2^{-m}}|0\rangle\langle 0|+\frac{1-2^{-n}}{1-2^{-m}}|1\rangle\langle 1|. \quad \text{(D12)}$$

If $n>m$, then we have that $\frac{2^{-n}-2^{-m}}{(1-2^{-m})}<0$, so that $\mathcal{N}(|1\rangle\langle 1|)$ is not a quantum state. Thus there cannot exist a quantum channel performing the transformation in Eq. (20) whenever $n>m$.

By the same reasoning, we have that $(|0\rangle\langle 0|,\pi_M)\not\to(|0\rangle\langle 0|,\pi_N)$ whenever $N>M$.

## APPENDIX E: ENTROPIC CHARACTERIZATIONS OF EXACT DISTINGUISHABILITY DISTILLATION AND DILUTION

### 1. Exact distillable distinguishability

We prove the equality in Eq. (31):

$$D_d^0(\rho,\sigma)=D_{\min}(\rho\|\sigma). \quad \text{(E1)}$$

Recall that

$$D_d^0(\rho,\sigma):=\log_2\sup_{\mathcal{P}\in\mathrm{CPTP}}\{M:\mathcal{P}(\rho)=|0\rangle\langle 0|,\ \mathcal{P}(\sigma)=\pi_M\}, \quad \text{(E2)}$$

First suppose that $\mathrm{Tr}[\Pi_\rho\sigma]\neq 0$. Consider that the measurement channel

$$\mathcal{M}(\omega)=\mathrm{Tr}[\Pi_\rho\omega]|0\rangle\langle 0|+\mathrm{Tr}[(I-\Pi_\rho)\omega]|1\rangle\langle 1| \quad \text{(E3)}$$

achieves

$$\mathcal{M}(\rho)=|0\rangle\langle 0|, \quad \text{(E4)}$$

$$\mathcal{M}(\sigma)=\mathrm{Tr}[\Pi_\rho\sigma]|0\rangle\langle 0|+\mathrm{Tr}[(I-\Pi_\rho)\sigma]|1\rangle\langle 1| \quad \text{(E5)}$$

$$=\pi_{M=1/\mathrm{Tr}[\Pi_\rho\sigma]}, \quad \text{(E6)}$$

so that

$$D_d^0(\rho,\sigma)\geqslant\log_2(1/\mathrm{Tr}[\Pi_\rho\sigma]) \quad \text{(E7)}$$

$$=-\log_2\mathrm{Tr}[\Pi_\rho\sigma] \quad \text{(E8)}$$

$$=D_{\min}(\rho\|\sigma). \quad \text{(E9)}$$

Now let $\mathcal{P}$ be a particular quantum channel such that $\mathcal{P}(\rho)=|0\rangle\langle 0|$ and $\mathcal{P}(\sigma)=\pi_M$. Then by the data processing inequality for $D_{\min}$ as recalled in Eq. (A24), we find that

$$D_{\min}(\rho\|\sigma) \geqslant D_{\min}(\mathcal{P}(\rho)\|\mathcal{P}(\sigma)) \tag{E10}$$
$$= D_{\min}(|0\rangle\langle 0|\|\pi_M) \tag{E11}$$
$$= \log_2 M. \tag{E12}$$

Since the inequality $D_{\min}(\rho\|\sigma) \geqslant \log_2 M$ holds for all channels $\mathcal{P}$ satisfying the constraints in Eq. (E2), we conclude that

$$D_{\min}(\rho\|\sigma) \geqslant D_d^0(\rho, \sigma). \tag{E13}$$

Combining (E7)–(E9) and (E13), we conclude the equality in Eq. (31), i.e., $D_{\min}(\rho\|\sigma) = D_d^0(\rho, \sigma)$.

In the case that $\mathrm{Tr}[\Pi_\rho \sigma] = 0$, then this means that the measurement channel above is such that $\mathcal{M}(\rho) = |0\rangle\langle 0|$ and $\mathcal{M}(\sigma) = |1\rangle\langle 1|$. In this case, as stated in the main text, the interpretation is that the box $(\rho, \sigma)$ contains an infinite number of bits of asymmetric distinguishability, so that $D_d^0(\rho, \sigma) = \infty$. This is consistent with $D_{\min}(\rho\|\sigma) = \infty$ in this case.

### 2. Exact distinguishability cost

We now prove the equality in Eq. (35):

$$D_c^0(\rho, \sigma) = D_{\max}(\rho\|\sigma). \tag{E14}$$

First recall that

$$D_c^0(\rho, \sigma) := \log_2 \inf_{\mathcal{P}\in\mathrm{CPTP}} \{M : \mathcal{P}(|0\rangle\langle 0|) = \rho,\ \mathcal{P}(\pi_M) = \sigma\}. \tag{E15}$$

Let us first suppose that $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ and $D_{\max}(\rho\|\sigma) = 0$. By definition, this means that the condition $\rho \leqslant \sigma$ holds, which in turn implies that $\sigma - \rho \geqslant 0$. Given the characterization of the normalized trace distance in Eq. (A2), this means that we can set $Y = \sigma - \rho$. Since $\mathrm{Tr}[Y] = 0$, we conclude that $\frac{1}{2}\|\sigma - \rho\|_1 = 0$. Since $\|\cdot\|_1$ is a norm, this means that $\rho = \sigma$. So in this trivial case, it follows that we can take $\mathcal{P}$ in Eq. (E15) to be the replacer channel $\mathrm{Tr}[\cdot]\rho$ and it follows that we can achieve the dilution task with zero bits of asymmetric distinguishability. So then $D_c^0(\rho, \sigma) = 0$ if $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ and $D_{\max}(\rho\|\sigma) = 0$.

Now suppose that $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ and $D_{\max}(\rho\|\sigma) > 0$. Let $\lambda > 0$ be such that $2^\lambda \sigma \geqslant \rho$. This then means that $2^\lambda \sigma - \rho \geqslant 0$, so that $\omega := \frac{2^\lambda\sigma - \rho}{2^\lambda - 1}$ is a quantum state. Furthermore, we have that

$$\sigma = 2^{-\lambda}\rho + (1 - 2^{-\lambda})\omega. \tag{E16}$$

Then by means of the following channel:

$$\mathcal{P}(\tau) = \langle 0|\tau|0\rangle\rho + \langle 1|\tau|1\rangle\omega, \tag{E17}$$

we have that

$$\mathcal{P}(|0\rangle\langle 0|) = \rho, \tag{E18}$$

$$\mathcal{P}(\pi_{2^\lambda}) = 2^{-\lambda}\rho + (1 - 2^{-\lambda})\omega = \sigma, \tag{E19}$$

so that this protocol accomplishes the distinguishability dilution task. This means that

$$D_c^0(\rho, \sigma) \leqslant \lambda. \tag{E20}$$

Now taking the infimum over all $\lambda$ satisfying $2^\lambda \sigma \geqslant \rho$, we conclude that

$$D_c^0(\rho, \sigma) \leqslant D_{\max}(\rho\|\sigma). \tag{E21}$$

Now consider an arbitrary channel $\mathcal{P}$ that accomplishes the transformation $(|0\rangle\langle 0|, \pi) \to (\rho, \sigma)$. By the data processing inequality for the max-relative entropy as recalled in Eq. (A32), we have that

$$\log_2 M = D_{\max}(|0\rangle\langle 0|\|\pi_M) \tag{E22}$$
$$\geqslant D_{\max}(\mathcal{P}(|0\rangle\langle 0|)\|\mathcal{P}(\pi_M)) \tag{E23}$$
$$= D_{\max}(\rho\|\sigma). \tag{E24}$$

Taking an infimum over all such protocols, we conclude that

$$D_c^0(\rho, \sigma) \geqslant D_{\max}(\rho\|\sigma). \tag{E25}$$

Putting together (E21) and (E25), we conclude the equality in Eq. (35), i.e., $D_c^0(\rho, \sigma) = D_{\max}(\rho\|\sigma)$.

In the case that $\mathrm{supp}(\rho) \not\subseteq \mathrm{supp}(\sigma)$, we have that $\mathrm{Tr}[\Pi_\sigma \rho] < 1$ and by definition $D_{\max}(\rho\|\sigma) = \infty$. This is consistent with the fact that, in such a case, there is no finite $\lambda \geqslant 0$ such that $2^\lambda \sigma - \rho \geqslant 0$. For if there were, then we would have that

$$2^\lambda - 1 = \mathrm{Tr}[(2^\lambda \sigma - \rho)] \tag{E26}$$
$$= \mathrm{Tr}[\{2^\lambda \sigma \geqslant \rho\}(2^\lambda \sigma - \rho)] \tag{E27}$$
$$\geqslant \mathrm{Tr}[\Pi_\sigma (2^\lambda \sigma - \rho)] \tag{E28}$$
$$= \mathrm{Tr}[2^\lambda \sigma] - \mathrm{Tr}[\Pi_\sigma \rho] \tag{E29}$$
$$= 2^\lambda - \mathrm{Tr}[\Pi_\sigma \rho], \tag{E30}$$

where the inequality follows from $\mathrm{Tr}[\{A \geqslant 0\}A] \geqslant \mathrm{Tr}[\Pi A]$ for any Hermitian operator $A$, projector $\Pi$, and $\{A \geqslant 0\}$ denoting the projection onto the positive eigenspace of $A$. The above implies that

$$\mathrm{Tr}[\Pi_\sigma \rho] \geqslant 1, \tag{E31}$$

contradicting the fact that $\mathrm{Tr}[\Pi_\sigma \rho] < 1$ when $\mathrm{supp}(\rho) \not\subseteq \mathrm{supp}(\sigma)$.

As explained in the main text, when $\mathrm{supp}(\rho) \not\subseteq \mathrm{supp}(\sigma)$, there is no finite value of $M$ nor any quantum channel $\mathcal{P}$ such that $\mathcal{P}(|0\rangle\langle 0|) = \rho$ and $\mathcal{P}(\pi_M) = \sigma$. If there were, then by the general fact that, for a quantum channel $\mathcal{N}$ and states $\tau$ and $\omega$, $\mathrm{supp}(\mathcal{N}(\tau)) \subseteq \mathrm{supp}(\mathcal{N}(\omega))$ if $\mathrm{supp}(\tau) \subseteq \mathrm{supp}(\omega)$ (see Appendix B in Ref. [77] and the fact that $\mathrm{supp}(|0\rangle\langle 0|) \subseteq \mathrm{supp}(\pi_M)$ for all $M < \infty$, the existence of such a channel $\mathcal{P}$ would contradict the assumption that $\mathrm{supp}(\rho) \not\subseteq \mathrm{supp}(\sigma)$. The interpretation then is as stated in the main text: that $D_c^0(\rho, \sigma) = \infty$ when $\mathrm{supp}(\rho) \not\subseteq \mathrm{supp}(\sigma)$, which is consistent with the fact that $D_{\max}(\rho\|\sigma) = \infty$ in such a case.

## APPENDIX F: ENTROPIC CHARACTERIZATIONS OF APPROXIMATE DISTINGUISHABILITY DISTILLATION AND DILUTION

### 1. Approximate distillable distinguishability

We prove the equality in Eq. (44):

$$D_d^\varepsilon(\rho, \sigma) = D_{\min}^\varepsilon(\rho\|\sigma). \tag{F1}$$

First recall that

$$D_d^\varepsilon(\rho,\sigma) := \log_2 \sup_{\mathcal{P}\in\mathrm{CPTP}} \{M : \mathcal{P}(\rho) \approx_\varepsilon |0\rangle\langle 0|,\ \mathcal{P}(\sigma) = \pi_M\}. \tag{F2}$$

Let $\Lambda$ be an arbitrary measurement operator satisfying $0 \leqslant \Lambda \leqslant I$ and $\mathrm{Tr}[\Lambda\rho] \geqslant 1-\varepsilon$. Then we can take the channel $\mathcal{P}$ to be as follows:

$$\mathcal{P}(\omega) = \mathrm{Tr}[\Lambda\omega]|0\rangle\langle 0| + \mathrm{Tr}[(I-\Lambda)\omega]|1\rangle\langle 1|, \tag{F3}$$

and we find that

$$\begin{aligned}
&\tfrac{1}{2}\|\mathcal{P}(\rho) - |0\rangle\langle 0|\|_1 \\
&= \tfrac{1}{2}\|\mathrm{Tr}[\Lambda\rho]|0\rangle\langle 0| + \mathrm{Tr}[(I-\Lambda)\rho]|1\rangle\langle 1| - |0\rangle\langle 0|\|_1 && \text{(F4)}\\
&= \tfrac{1}{2}\|-\mathrm{Tr}[(I-\Lambda)\rho]|0\rangle\langle 0| + \mathrm{Tr}[(I-\Lambda)\rho]|1\rangle\langle 1|\|_1 && \text{(F5)}\\
&= (\mathrm{Tr}[(I-\Lambda)\rho])\tfrac{1}{2}\||1\rangle\langle 1| - |0\rangle\langle 0|\|_1 && \text{(F6)}\\
&= \mathrm{Tr}[(I-\Lambda)\rho] \leqslant \varepsilon. && \text{(F7)}
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
\mathcal{P}(\sigma) &= \mathrm{Tr}[\Lambda\sigma]|0\rangle\langle 0| + \mathrm{Tr}[(I-\Lambda)\sigma]|1\rangle\langle 1| && \text{(F8)}\\
&= \pi_{M=1/\mathrm{Tr}[\Lambda\sigma]}. && \text{(F9)}
\end{aligned}$$

So this means that

$$\begin{aligned}
D_d^\varepsilon(\rho,\sigma) &\geqslant \log_2(1/\mathrm{Tr}[\Lambda\sigma]) && \text{(F10)}\\
&= -\log_2 \mathrm{Tr}[\Lambda\sigma]. && \text{(F11)}
\end{aligned}$$

Now maximizing the right-hand side with respect to all $\Lambda$ satisfying $0 \leqslant \Lambda \leqslant I$ and $\mathrm{Tr}[\Lambda\rho] \geqslant 1-\varepsilon$, we conclude that

$$D_d^\varepsilon(\rho,\sigma) \geqslant D_{\min}^\varepsilon(\rho\|\sigma). \tag{F12}$$

To see the other inequality, let $\mathcal{P}$ be an arbitrary channel satisfying $\mathcal{P}(\rho) \approx_\varepsilon |0\rangle\langle 0|$ and $\mathcal{P}(\sigma) = \pi_M$. Then by the data processing inequality for $D_{\min}^\varepsilon$, we have that

$$\begin{aligned}
D_{\min}^\varepsilon(\rho\|\sigma) &\geqslant D_{\min}^\varepsilon(\mathcal{P}(\rho)\|\mathcal{P}(\sigma)) && \text{(F13)}\\
&= D_{\min}^\varepsilon(\mathcal{P}(\rho)\|\pi_M) && \text{(F14)}\\
&\geqslant \log_2 M. && \text{(F15)}
\end{aligned}$$

The last inequality above is a consequence of the following reasoning: Let $\Delta(\cdot) = |0\rangle\langle 0|(\cdot)|0\rangle\langle 0| + |1\rangle\langle 1|(\cdot)|1\rangle\langle 1|$ denote the completely dephasing channel. Since $\mathcal{P}(\rho) \approx_\varepsilon |0\rangle\langle 0|$, we find from applying the data processing inequality for normalized trace distance that

$$\begin{aligned}
\varepsilon &\geqslant \tfrac{1}{2}\|\mathcal{P}(\rho) - |0\rangle\langle 0|\|_1 \\
&\geqslant \tfrac{1}{2}\|(\Delta\circ\mathcal{P})(\rho) - \Delta(|0\rangle\langle 0|)\|_1 \\
&= \tfrac{1}{2}\|(\Delta\circ\mathcal{P})(\rho) - |0\rangle\langle 0|\|_1 \\
&= \tfrac{1}{2}\|\langle 0|\mathcal{P}(\rho)|0\rangle|0\rangle\langle 0| + \langle 1|\mathcal{P}(\rho)|1\rangle|1\rangle\langle 1| - |0\rangle\langle 0|\|_1 \\
&= 1 - \langle 0|\mathcal{P}(\rho)|0\rangle, && \text{(F16)}
\end{aligned}$$

which implies that $\langle 0|\mathcal{P}(\rho)|0\rangle \geqslant 1-\varepsilon$. Thus we can take $\Lambda = |0\rangle\langle 0|$ in the definition of $D_{\min}^\varepsilon(\mathcal{P}(\rho)\|\pi_M)$, and we have that $\mathrm{Tr}[\Lambda\mathcal{P}(\rho)] \geqslant 1-\varepsilon$ while $\mathrm{Tr}[\Lambda\pi_M] = 1/M$. Since $D_{\min}^\varepsilon(\mathcal{P}(\rho)\|\pi_M)$ involves an optimization over all measurement operators $\Lambda$ satisfying $\mathrm{Tr}[\Lambda\mathcal{P}(\rho)] \geqslant 1-\varepsilon$, we conclude the inequality in Eq. (F15).

Since the inequality $D_{\min}^\varepsilon(\rho\|\sigma) \geqslant \log_2 M$ holds for all possible distinguishability distillation protocols, we conclude that

$$D_{\min}^\varepsilon(\rho\|\sigma) \geqslant D_d^\varepsilon(\rho,\sigma). \tag{F17}$$

By combining the inequalities in Eqs. (F12) and (F17), we conclude the equality in Eq. (44), i.e., $D_{\min}^\varepsilon(\rho\|\sigma) = D_d^\varepsilon(\rho,\sigma)$.

It is worthwhile to mention a somewhat singular case. In the case that $\mathrm{supp}(\rho) \not\subseteq \mathrm{supp}(\sigma)$, we have that $\mathrm{Tr}[\Pi_\sigma\rho] < 1$, which means that $\mathrm{Tr}[(I-\Pi_\sigma)\rho] > 0$. If we also have that $\mathrm{Tr}[(I-\Pi_\sigma)\rho] \geqslant 1-\varepsilon$, then we can take the channel $\mathcal{P}$ to be as follows:

$$\mathcal{P}(\omega) = \mathrm{Tr}[(I-\Pi_\sigma)\omega]|0\rangle\langle 0| + \mathrm{Tr}[\Pi_\sigma\omega]|1\rangle\langle 1|. \tag{F18}$$

In such a case, we have that $\mathcal{P}(\rho) \approx_\varepsilon |0\rangle\langle 0|$, while $\mathcal{P}(\sigma) = |1\rangle\langle 1| = \lim_{M\to\infty}\pi_M$, implying that $D_{\min}^\varepsilon(\rho\|\sigma) = D_d^\varepsilon(\rho,\sigma) = \infty$ in this case.

### 2. Approximate distinguishability cost

Here we prove the equality in Eq. (48):

$$D_c^\varepsilon(\rho,\sigma) = D_{\max}^\varepsilon(\rho\|\sigma). \tag{F19}$$

Recall that

$$D_c^\varepsilon(\rho,\sigma) := \log_2 \inf_{\mathcal{P}\in\mathrm{CPTP}} \{M : \mathcal{P}(|0\rangle\langle 0|) \approx_\varepsilon \rho,\ \mathcal{P}(\pi_M) = \sigma\}. \tag{F20}$$

Let $\widetilde{\rho}$ be a state such that $\frac{1}{2}\|\rho - \widetilde{\rho}\|_1 \leqslant \varepsilon$. Then by executing the protocol in Eqs. (E18) and (E19), but replacing $\rho$ with $\widetilde{\rho}$, we find that

$$D_c^\varepsilon(\rho,\sigma) \leqslant D_{\max}(\widetilde{\rho}\|\sigma). \tag{F21}$$

Since this is possible for any state $\widetilde{\rho}$ satisfying $\frac{1}{2}\|\rho - \widetilde{\rho}\|_1 \leqslant \varepsilon$, we conclude that

$$D_c^\varepsilon(\rho,\sigma) \leqslant D_{\max}^\varepsilon(\rho\|\sigma). \tag{F22}$$

To see the other inequality, consider an arbitrary channel $\mathcal{P}$ performing the transformation $\mathcal{P}(|0\rangle\langle 0|) \approx_\varepsilon \rho$ and $\mathcal{P}(\pi_M) = \sigma$. Then from the data processing inequality for the max-relative entropy, as recalled in Eq. (A32), and its definition, we conclude that

$$\begin{aligned}
\log_2 M &= D_{\max}(|0\rangle\langle 0|\|\pi_M) && \text{(F23)}\\
&\geqslant D_{\max}(\mathcal{P}(|0\rangle\langle 0|)\|\mathcal{P}(\pi_M)) && \text{(F24)}\\
&= D_{\max}(\mathcal{P}(|0\rangle\langle 0|)\|\sigma) && \text{(F25)}\\
&\geqslant D_{\max}^\varepsilon(\rho\|\sigma). && \text{(F26)}
\end{aligned}$$

Since the inequality holds for an arbitrary channel $\mathcal{P}$ performing the transformation $\mathcal{P}(|0\rangle\langle 0|) \approx_\varepsilon \rho$ and $\mathcal{P}(\pi_M) = \sigma$, we conclude that

$$D_c^\varepsilon(\rho,\sigma) \geqslant D_{\max}^\varepsilon(\rho\|\sigma). \tag{F27}$$

By combining the inequalities in Eqs. (F22) and (F27), we conclude the equality in Eq. (48), i.e., $D_c^\varepsilon(\rho,\sigma) = D_{\max}^\varepsilon(\rho\|\sigma)$.

## APPENDIX G: OPERATIONAL PROOF FOR INEQUALITY RELATING SMOOTH MIN- AND MAX-RELATIVE ENTROPIES

Here we prove the inequality in Eq. (51):

$$D_{\min}^{\varepsilon_1}(\rho\|\sigma) \leqslant D_{\max}^{\varepsilon_2}(\rho\|\sigma) + \log_2\left(\frac{1}{1-\varepsilon_1-\varepsilon_2}\right), \quad (G1)$$

for $\varepsilon_1, \varepsilon_2 \geqslant 0$ and $\varepsilon_1 + \varepsilon_2 < 1$.

First, consider that an arbitrary protocol performing the transformation $(|0\rangle\langle 0|, \pi_M) \to (\widetilde{0}_\varepsilon, \pi_K)$ is required to obey the following inequality:

$$\log_2 K \leqslant D_{\min}^{\varepsilon}(|0\rangle\langle 0|\|\pi_M) \quad (G2)$$

$$= \log_2 M + \log_2(1/[1-\varepsilon]). \quad (G3)$$

To see the equality in Eq. (G3), consider that $\Lambda = (1-\varepsilon)|0\rangle\langle 0|$ is a measurement operator achieving $\mathrm{Tr}[\Lambda|0\rangle\langle 0|] \geqslant 1-\varepsilon$, while $\mathrm{Tr}[\Lambda\pi_M] = (1-\varepsilon)/M$, implying that

$$D_{\min}^{\varepsilon}(|0\rangle\langle 0|\|\pi_M) \geqslant \log_2 M + \log_2(1/[1-\varepsilon]). \quad (G4)$$

To see the other inequality, suppose that $\mathrm{Tr}[\Lambda|0\rangle\langle 0|] \geqslant 1-\varepsilon$. Then we have that

$$\mathrm{Tr}[\Lambda\pi_M] = \frac{1}{M}\mathrm{Tr}[\Lambda|0\rangle\langle 0|] + \left(1-\frac{1}{M}\right)\mathrm{Tr}[\Lambda|1\rangle\langle 1|] \quad (G5)$$

$$\geqslant \frac{1}{M}\mathrm{Tr}[\Lambda|0\rangle\langle 0|] \quad (G6)$$

$$\geqslant \frac{1-\varepsilon}{M}. \quad (G7)$$

Since this is a uniform bound holding for all measurement operators $\Lambda$ satisfying $\mathrm{Tr}[\Lambda|0\rangle\langle 0|] \geqslant 1-\varepsilon$, we conclude that

$$D_{\min}^{\varepsilon}(|0\rangle\langle 0|\|\pi_M) \leqslant -\log_2\left(\frac{1-\varepsilon}{M}\right) \quad (G8)$$

$$= \log_2 M + \log_2(1/[1-\varepsilon]), \quad (G9)$$

completing the proof of the equality in Eq. (G3).

Given that the bound $\log_2 K \leqslant \log_2 M + \log_2(1/[1-\varepsilon])$ holds for an arbitrary channel performing the transformation $(|0\rangle\langle 0|, \pi_M) \to (\widetilde{0}_\varepsilon, \pi_K)$, we can consider a particular way of completing it in two steps. Fix $\varepsilon_1, \varepsilon_2 \geqslant 0$ such that $\varepsilon_1 + \varepsilon_2 < 1$. In the first step, we perform the dilution transformation $(|0\rangle\langle 0|, \pi_M) \to (\rho_{\varepsilon_2}, \sigma)$ optimally and in the second, we perform the distillation transformation $(\rho, \sigma) \to (\widetilde{0}_{\varepsilon_1}, \pi_K)$ optimally. For the dilution part, we have that $\log_2 M = D_{\max}^{\varepsilon_2}(\rho\|\sigma)$ and there exists a channel $\mathcal{P}_1$ such that $\mathcal{P}_1(|0\rangle\langle 0|) = \rho_{\varepsilon_2} \approx_{\varepsilon_2} \rho$ and $\mathcal{P}_1(\pi_M) = \sigma$. For the distillation part, we have that $\log_2 K = D_{\min}^{\varepsilon_1}(\rho\|\sigma)$ and there exists a channel $\mathcal{P}_2$ such that $\mathcal{P}_2(\rho) = \widetilde{0}_{\varepsilon_1} \approx_{\varepsilon_1} |0\rangle\langle 0|$ and $\mathcal{P}_2(\sigma) = \pi_K$. By composing the two channels, we have that

$$(\mathcal{P}_2 \circ \mathcal{P}_1)(\pi_M) = \pi_K, \quad (G10)$$

while

$$\tfrac{1}{2}\|(\mathcal{P}_2 \circ \mathcal{P}_1)(|0\rangle\langle 0|) - |0\rangle\langle 0|\|_1$$

$$\leqslant \tfrac{1}{2}\|(\mathcal{P}_2 \circ \mathcal{P}_1)(|0\rangle\langle 0|) - \mathcal{P}_2(\rho)\|_1 + \tfrac{1}{2}\|\mathcal{P}_2(\rho) - |0\rangle\langle 0|\|_1 \quad (G11)$$

$$\leqslant \tfrac{1}{2}\|\mathcal{P}_1(|0\rangle\langle 0|) - \rho\|_1 + \varepsilon_1 \quad (G12)$$

$$\leqslant \varepsilon_2 + \varepsilon_1. \quad (G13)$$

So this means that we have a protocol $(|0\rangle\langle 0|, \pi_M) \to (\widetilde{0}_{\varepsilon_1+\varepsilon_2}, \pi_K)$ with $\log_2 M = D_{\max}^{\varepsilon_2}(\rho\|\sigma)$ and $\log_2 K = D_{\min}^{\varepsilon_1}(\rho\|\sigma)$. By (G3), we then conclude the inequality in Eq. (51), as restated in Eq. (G1).

## APPENDIX H: ASYMPTOTIC DISTILLABLE DISTINGUISHABILITY AND DISTINGUISHABILITY COST

As a direct consequence of (44) and results from Refs. [39,42], the following expansion holds for sufficiently large $n$:

$$D_d^{\varepsilon}(\rho^{\otimes n}, \sigma^{\otimes n}) = nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon) + O(\ln n), \quad (H1)$$

where $D(\rho\|\sigma)$ is the quantum relative entropy. The relative entropy variance $V(\rho\|\sigma)$ [39,42] is defined as

$$V(\rho\|\sigma) := \mathrm{Tr}[\rho(\log_2\rho - \log_2\sigma - D(\rho\|\sigma))^2], \quad (H2)$$

if $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ and is otherwise undefined. Furthermore, $\Phi^{-1}(\varepsilon)$ is the inverse of the cumulative normal distribution function, defined as

$$\Phi^{-1}(\varepsilon) = \sup\{a \in \mathbb{R} \mid \Phi(a) \leqslant \varepsilon\}, \quad (H3)$$

where

$$\Phi(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} dx\, \exp\left(\frac{-x^2}{2}\right). \quad (H4)$$

Based on the inequality in Eq. (51), we have that

$$D_{\min}^{1-\varepsilon-\delta}(\rho\|\sigma) \leqslant D_{\max}^{\varepsilon}(\rho\|\sigma) + \log_2\left(\frac{1}{\delta}\right).$$

Then by picking $\delta = 1/\sqrt{n}$, and applying (48), (44), (H1), and the fact that $\Phi^{-1}(1-\varepsilon) = -\Phi^{-1}(\varepsilon)$, we find that

$$D_c^{\varepsilon}(\rho^{\otimes n}, \sigma^{\otimes n}) \geqslant nD(\rho\|\sigma) - \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon) + O(\log n). \quad (H5)$$

By following the proof of Eq. (21) in Ref. [39], but instead using the normalized trace distance as the metric for smooth max-relative entropy, we find that

$$D_{\max}^{\varepsilon}(\rho\|\sigma) \leqslant D_{\min}^{1-\varepsilon^2}(\rho\|\sigma) + \log_2|\mathrm{spec}(\sigma)| + \log_2\left(\frac{1}{1-\varepsilon^2}\right), \quad (H6)$$

where $\varepsilon \in (0,1)$ and $|\mathrm{spec}(\sigma)|$ is equal to the number of distinct eigenvalues of $\sigma$. We give a detailed proof of (H6) in Appendix I. By the operational interpretations of $D_{\max}^{\varepsilon}$ and $D_{\min}^{1-\varepsilon^2}$, the inequality in Eq. (H6) can equivalently be written as

$$D_c^{\varepsilon}(\rho, \sigma) \leqslant D_d^{1-\varepsilon^2}(\rho, \sigma) + \log_2|\mathrm{spec}(\sigma)| + \log_2\left(\frac{1}{1-\varepsilon^2}\right). \quad (H7)$$

Now accounting for the fact that $|\mathrm{spec}(\sigma^{\otimes n})| = O(\ln n)$ and applying (H1), we conclude that

$$D_c^{\varepsilon}(\rho^{\otimes n}, \sigma^{\otimes n}) \leqslant nD(\rho\|\sigma) - \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon^2) + O(\ln n). \quad (H8)$$

Thus we have that

$$D_c^\varepsilon(\rho^{\otimes n}, \sigma^{\otimes n}) = nD(\rho\|\sigma) + O(\sqrt{n}). \tag{H9}$$

## APPENDIX I: BOUND RELATING SMOOTH MAX- AND MIN-RELATIVE ENTROPIES

Here we prove the following bound:

$$D_{\max}^\varepsilon(\rho\|\sigma) \leqslant D_{\min}^{1-\varepsilon^2}(\rho\|\sigma) + \log_2|\mathrm{spec}(\sigma)| + \log_2\left(\frac{1}{1-\varepsilon^2}\right), \tag{I1}$$

where $|\mathrm{spec}(\sigma)|$ is equal to the number of distinct eigenvalues of $\sigma$.

The proof follows the proof of Eq. (21) in Ref. [39] closely, but instead using the normalized trace distance as the metric for smooth max-relative entropy and accounting for a minor typo present in the proof of Eq. (39) in Ref. [39].

Let the eigendecomposition of $\sigma$ be $\sigma = \sum_x \lambda_x^\sigma \Pi_x^\sigma$, where $\Pi_x^\sigma$ is the projection onto the eigenspace of $\sigma$ with eigenvalue $\lambda_x^\sigma$. Let $\mathcal{E}_\sigma(\cdot) = \sum_x \Pi_x^\sigma(\cdot)\Pi_x^\sigma$ denote the pinching quantum channel. In what follows, we make use of the pinching inequality [80]:

$$\rho \leqslant |\mathrm{spec}(\sigma)|\mathcal{E}_\sigma(\rho). \tag{I2}$$

Let $\mu$ be the largest value such that $\mathrm{Tr}[Q\mathcal{E}_\sigma(\rho)] = 1-\varepsilon^2$, where $Q = \{\mathcal{E}_\sigma(\rho) \leqslant 2^\mu\sigma\}$. Due to the fact that $Q$ commutes with $\sigma$, we have that $\mathcal{E}_\sigma(Q) = Q$, which implies that

$$\begin{aligned}\mathrm{Tr}[Q\mathcal{E}_\sigma(\rho)] &= \mathrm{Tr}[\mathcal{E}_\sigma(Q)\rho] &\text{(I3)}\\ &= \mathrm{Tr}[Q\rho] &\text{(I4)}\\ &= 1-\varepsilon^2. &\text{(I5)}\end{aligned}$$

Then we set

$$\widetilde{\rho} = \frac{Q\rho Q}{\mathrm{Tr}[Q\rho]}, \tag{I6}$$

for which we have that

$$F(\rho,\widetilde{\rho}) \geqslant 1-\varepsilon^2, \tag{I7}$$

by applying lemma 9.4.1 in Ref. [67]. This in turn implies that

$$\tfrac{1}{2}\|\rho-\widetilde{\rho}\|_1 \leqslant \varepsilon, \tag{I8}$$

via the inequality $\frac{1}{2}\|\rho-\widetilde{\rho}\|_1 \leqslant \sqrt{1-F(\rho,\widetilde{\rho})}$ [81], so that $\widetilde{\rho}$ is a candidate for the optimization involved in $D_{\max}^\varepsilon(\rho\|\sigma)$. Now consider that

$$\begin{aligned}\widetilde{\rho} &= \frac{Q\rho Q}{\mathrm{Tr}[Q\rho]} &\text{(I9)}\\ &\leqslant \frac{Q\rho Q}{1-\varepsilon^2} &\text{(I10)}\\ &\leqslant \frac{|\mathrm{spec}(\sigma)|}{1-\varepsilon^2}Q\mathcal{E}_\sigma(\rho)Q &\text{(I11)}\\ &\leqslant \frac{2^\mu|\mathrm{spec}(\sigma)|}{1-\varepsilon^2}Q\sigma Q &\text{(I12)}\\ &\leqslant \frac{2^\mu|\mathrm{spec}(\sigma)|}{1-\varepsilon^2}\sigma. &\text{(I13)}\end{aligned}$$

So it follows that

$$\begin{aligned}D_{\max}^\varepsilon(\rho\|\sigma) &\leqslant D_{\max}(\widetilde{\rho}\|\sigma)\\ &\leqslant \mu + \log_2|\mathrm{spec}(\sigma)| + \log_2\left(\frac{1}{1-\varepsilon^2}\right). &\text{(I14)}\end{aligned}$$

Now consider that $\mathrm{Tr}[(I-Q)\rho] = \varepsilon^2$ and $I - Q = \{\mathcal{E}_\sigma(\rho) > 2^\mu\sigma\}$, for which we have that

$$\mathrm{Tr}[\{\mathcal{E}_\sigma(\rho) > 2^\mu\sigma\}(\mathcal{E}_\sigma(\rho) - 2^\mu\sigma)] \geqslant 0, \tag{I15}$$

implying that

$$\begin{aligned}\mathrm{Tr}[(I-Q)\sigma] &= \mathrm{Tr}[\{\mathcal{E}_\sigma(\rho) > 2^\mu\sigma\}\sigma] &\text{(I16)}\\ &\leqslant 2^{-\mu}\,\mathrm{Tr}[\{\mathcal{E}_\sigma(\rho) > 2^\mu\sigma\}\mathcal{E}_\sigma(\rho)] &\text{(I17)}\\ &\leqslant 2^{-\mu}. &\text{(I18)}\end{aligned}$$

Taking a negative logarithm, this gives

$$-\ln\mathrm{Tr}[(I-Q)\sigma] \geqslant \mu. \tag{I19}$$

Since $\mathrm{Tr}[(I-Q)\rho] = \varepsilon^2$, this means that $I-Q$ is a candidate for $\Lambda$ in the definition of smooth min-relative entropy, from which we conclude that

$$\begin{aligned}\mu &\leqslant D_{\min}^{1-\varepsilon^2}(\mathcal{E}_\sigma(\rho)\|\sigma) &\text{(I20)}\\ &\leqslant D_{\min}^{1-\varepsilon^2}(\rho\|\sigma), &\text{(I21)}\end{aligned}$$

where the latter inequality follows from the data processing inequality in Eq. (A31). Putting together (I14) and (I21), we arrive at (I1).

## APPENDIX J: ASYMPTOTIC BOX TRANSFORMATIONS

We now provide a proof of Eq. (62), i.e.,

$$R((\rho,\sigma)\to(\tau,\omega)) = \widetilde{R}((\rho,\sigma)\to(\tau,\omega)) = \frac{D(\rho\|\sigma)}{D(\tau\|\omega)}, \tag{J1}$$

so that the quantum relative entropy gives the optimal conversion rate for boxes. We prove this result in two steps, called the direct part and strong converse part.

### 1. Achievability: direct part

We begin with the direct part. The goal is to show that for all $\varepsilon \in (0,1]$, $\delta > 0$, and sufficiently large $n$, there exists an $(n, n[R-\delta], \varepsilon)$ box transformation protocol

$$(\rho^{\otimes n}, \sigma^{\otimes n}) \to (\widetilde{\tau^{\otimes n[R-\delta]}}, \omega^{\otimes n[R-\delta]}) \tag{J2}$$

with $R = \frac{D(\rho\|\sigma)}{D(\tau\|\omega)}$. The approach we take here is related to an approach from Ref. [82].

Fix $\varepsilon \in (0,1)$ and $\delta > 0$. Suppose that $\varepsilon = \varepsilon_1 + \varepsilon_2$, so that $\varepsilon_1, \varepsilon_2 \in (0,1)$ and $\varepsilon_1 + \varepsilon_2 < 1$. Also, suppose that $\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4$, such that $\delta_1, \delta_2, \delta_3, \delta_4 > 0$.

Consider that we can perform the transformation $(\rho^{\otimes n}, \sigma^{\otimes n}) \to (\widetilde{0}_{\varepsilon_1}, \pi_M)$ such that

$$\log_2 M = D_{\min}^{\varepsilon_1}(\rho^{\otimes n}\|\sigma^{\otimes n}). \tag{J3}$$

Then applying the following inequality from proposition 3.2 of Ref. [83] (see also proposition 3 of Ref. [84]):

$$D_{\min}^\varepsilon(\rho\|\sigma) \geqslant D_\alpha(\rho\|\sigma) + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{\varepsilon}\right), \tag{J4}$$

we find that

$$\log_2 M \geqslant nD_\alpha(\rho\|\sigma) + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{\varepsilon_1}\right). \tag{J5}$$

Set $\alpha \in (0, 1)$ such that

$$\delta_1 \cdot D(\tau\|\omega) \geqslant D(\rho\|\sigma) - D_\alpha(\rho\|\sigma), \tag{J6}$$

which is possible due to (A12) and (A14), and for this choice of $\alpha$, take $n$ large enough so that

$$\delta_2 \cdot D(\tau\|\omega) \geqslant \frac{\alpha}{n(1-\alpha)}\log_2\left(\frac{1}{\varepsilon_1}\right). \tag{J7}$$

Then we have that

$$\log_2 M \geqslant nD(\rho\|\sigma) - nD(\tau\|\omega)[\delta_1 + \delta_2]. \tag{J8}$$

Also, consider that we can perform the transformation $(|0\rangle\langle 0|, \pi) \to (\widetilde{\tau^{\otimes m}, \omega^{\otimes m}})$ (with error $\varepsilon_2$), for fixed $M$, by taking $m$ as large as possible so that the following inequality still holds

$$\log_2 M \geqslant D_{\max}^{\varepsilon_2}(\tau^{\otimes m}\|\omega^{\otimes m}). \tag{J9}$$

If it is not possible to find an $m$ to saturate the inequality, then one can find states $\tau'$ and $\omega'$ with just enough distinguishability such that

$$\log_2 M = D_{\max}^{\varepsilon_2}(\tau^{\otimes m}\otimes\tau'\|\omega^{\otimes m}\otimes\omega'), \tag{J10}$$

while having a negligible impact on the final parameters of the protocol. The resulting protocol then produces the states $\approx_\varepsilon \tau^{\otimes m}\otimes\tau'$ and $\omega^{\otimes m}\otimes\omega'$, and the final step is to perform a partial trace over the extra ancilla system. By applying the following inequality from proposition K

$$D_{\max}^{\varepsilon_2}(\rho\|\sigma) \leqslant \widetilde{D}_\beta(\rho\|\sigma) + \log_2\left(1/[1-\varepsilon_2^2]\right) + \frac{1}{\beta-1}\log_2\left(1/\varepsilon_2^2\right), \tag{J11}$$

proved in Appendix K, we find that

$$\log_2 M \leqslant m\widetilde{D}_\beta(\tau\|\omega) + \widetilde{D}_\beta(\tau'\|\omega') + \log_2\left(1/[1-\varepsilon_2^2]\right) + \frac{1}{\beta-1}\log_2\left(1/\varepsilon_2^2\right). \tag{J12}$$

Now set $\beta > 1$ such that

$$\delta_3 nD(\tau\|\omega) \geqslant m[\widetilde{D}_\beta(\tau\|\omega) - D(\tau\|\omega)], \tag{J13}$$

which is possible due to (A17) and (A21), and for this choice of $\beta$, take $n$ sufficiently large so that

$$\delta_4 \cdot D(\tau\|\omega) \geqslant \frac{1}{n}\widetilde{D}_\beta(\tau'\|\omega') + \frac{1}{n}\log_2\left(1/[1-\varepsilon_2^2]\right) + \frac{1}{n(\beta-1)}\log_2\left(1/\varepsilon_2^2\right). \tag{J14}$$

[Note that we require $n$ large enough so that both (J7) and (J14) hold.] Then we have that

$$\log_2 M \leqslant mD(\tau\|\omega) + nD(\tau\|\omega)[\delta_3 + \delta_4]. \tag{J15}$$

Putting together (J8) and (J15), we find that

$$nD(\rho\|\sigma) - nD(\tau\|\omega)[\delta_1 + \delta_2] \leqslant mD(\tau\|\omega) + nD(\tau\|\omega)[\delta_3 + \delta_4]. \tag{J16}$$

Now dividing both sides by $nD(\tau\|\omega)$, we find that

$$\frac{m}{n} \geqslant \frac{D(\rho\|\sigma)}{D(\tau\|\omega)} - [\delta_1 + \delta_2 + \delta_3 + \delta_4] \tag{J17}$$

$$= \frac{D(\rho\|\sigma)}{D(\tau\|\omega)} - \delta. \tag{J18}$$

The rate of this scheme is equal to $m/n$. The error of the protocol is no larger then $\varepsilon_1 + \varepsilon_2 = \varepsilon$, following from an application of the triangle inequality.

Thus we have shown that for all $\varepsilon \in (0, 1)$, $\delta > 0$, there exists an $(n, n[R-\delta], \varepsilon)$ box transformation protocol with $R = \frac{D(\rho\|\sigma)}{D(\tau\|\omega)}$, concluding the proof of the achievability part.

### 2. Strong converse via sandwiched Rényi relative entropy

Before proving the strong converse, we establish the following lemma as a generalization of proposition 85 of Ref. [85]. In fact, the proof of the following lemma is contained in the proof of proposition 2.8 of Ref. [85]. The following lemma serves as a pseudo-continuity inequality for the sandwiched Rényi relative entropies.

*Lemma 1.* Let $\rho_0$, $\rho_1$, and $\sigma$ be quantum states such that $\mathrm{supp}(\rho_0) \subseteq \mathrm{supp}(\sigma)$. Fix $\alpha \in (1/2, 1)$ and $\beta \equiv \beta(\alpha) := \alpha/(2\alpha-1) > 1$. Then

$$\widetilde{D}_\beta(\rho_0\|\sigma) - \widetilde{D}_\alpha(\rho_1\|\sigma) \geqslant \frac{\alpha}{1-\alpha}\log_2 F(\rho_0, \rho_1). \tag{J19}$$

*Proof.* Consider that

$$\widetilde{D}_\beta(\rho_0\|\sigma) - \widetilde{D}_\alpha(\rho_1\|\sigma) = \frac{2\beta}{\beta-1}\log_2\left\|\rho_0^{1/2}\sigma^{(1-\beta)/2\beta}\right\|_{2\beta} - \frac{2\alpha}{\alpha-1}\log_2\left\|\sigma^{(1-\alpha)/2\alpha}\rho_1^{1/2}\right\|_{2\alpha} \tag{J20}$$

$$= \frac{2\alpha}{1-\alpha}\log_2\left\|\rho_0^{1/2}\sigma^{(1-\beta)/2\beta}\right\|_{2\beta} + \frac{2\alpha}{1-\alpha}\log_2\left\|\sigma^{(1-\alpha)/2\alpha}\rho_1^{1/2}\right\|_{2\alpha} \tag{J21}$$

$$= \frac{2\alpha}{1-\alpha}\log_2\left[\left\|\rho_0^{1/2}\sigma^{(1-\beta)/2\beta}\right\|_{2\beta}\left\|\sigma^{(1-\alpha)/2\alpha}\rho_1^{1/2}\right\|_{2\alpha}\right] \tag{J22}$$

$$\geqslant \frac{2\alpha}{1-\alpha}\log_2\left\|\rho_0^{1/2}\sigma^{(1-\beta)/2\beta}\sigma^{(1-\alpha)/2\alpha}\rho_1^{1/2}\right\|_1 \tag{J23}$$

$$= \frac{2\alpha}{1-\alpha}\log_2\left\|\rho_0^{1/2}\rho_1^{1/2}\right\|_1 \tag{J24}$$

$$= \frac{\alpha}{1-\alpha}\log_2 F(\rho_0, \rho_1). \tag{J25}$$

The sole inequality follows from the Hölder inequality. ■

The following is an auxiliary lemma that serves as a one-shot converse for any approximate box transformation $(\rho, \sigma) \xrightarrow{\mathcal{N}} (\tau, \omega)$ where $\omega = \mathcal{N}(\sigma)$:

*Lemma 2.* Let $\rho$, $\sigma$, $\tau$, and $\omega$ be quantum states and $\mathcal{N}$ a quantum channel such that $\mathcal{N}(\sigma) = \omega$. Then for $\alpha \in (1/2, 1)$ and $\beta \equiv \beta(\alpha) := \alpha/(2\alpha-1)$, we have that

$$\widetilde{D}_\beta(\rho\|\sigma) \geqslant \widetilde{D}_\alpha(\tau\|\omega) + \frac{\alpha}{1-\alpha}\log_2 F(\mathcal{N}(\rho), \tau). \tag{J26}$$

*Proof.* Consider that

$$\widetilde{D}_\beta(\rho\|\sigma) \geqslant \widetilde{D}_\beta(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \tag{J27}$$

$$= \widetilde{D}_\beta(\mathcal{N}(\rho)\|\omega) \tag{J28}$$

$$\geqslant \widetilde{D}_\alpha(\tau\|\omega) + \frac{\alpha}{1-\alpha}\log_2 F(\mathcal{N}(\rho),\tau). \tag{J29}$$

The first inequality follows from the quantum data processing inequality in Eq. (A16) and the other from lemma 1. ■

*Proposition 1.* Let $n, m \in \mathbb{Z}^+$ and $\varepsilon \in [0,1)$. Let $\rho$, $\sigma$, $\tau$, and $\omega$ be quantum states and $\mathcal{N}^{(n)}$ a quantum channel constituting an $(n, m, \varepsilon)$ box transformation protocol (i.e., so that $\mathcal{N}^{(n)}(\rho^{\otimes n}) \approx_\varepsilon \tau^{\otimes m}$ and $\mathcal{N}^{(n)}(\sigma^{\otimes n}) = \omega^{\otimes m}$). Then for $\alpha \in (1/2, 1)$ and $\beta \equiv \beta(\alpha) := \alpha/(2\alpha-1)$, we have that

$$\frac{\widetilde{D}_\beta(\rho\|\sigma)}{\widetilde{D}_\alpha(\tau\|\omega)} \geqslant \frac{m}{n} + \frac{2\alpha}{n(1-\alpha)\widetilde{D}_\alpha(\tau\|\omega)}\log_2(1-\varepsilon). \tag{J30}$$

Alternatively, if we set $R = m/n$, then the above bound can be written as

$$-\frac{1}{n}\log_2(1-\varepsilon) \geqslant \frac{1}{2}\left(\frac{1-\alpha}{\alpha}\right)(R\,\widetilde{D}_\alpha(\tau\|\omega) - \widetilde{D}_\beta(\rho\|\sigma)) \tag{J31}$$

$$= \frac{1}{2}\left(\frac{\beta-1}{\beta}\right)(R\,\widetilde{D}_\alpha(\tau\|\omega) - \widetilde{D}_\beta(\rho\|\sigma)). \tag{J32}$$

*Proof.* Consider that

$$n\widetilde{D}_\beta(\rho\|\sigma)$$

$$= \widetilde{D}_\beta(\rho^{\otimes n}\|\sigma^{\otimes n}) \tag{J33}$$

$$\geqslant \widetilde{D}_\alpha(\tau^{\otimes m}\|\omega^{\otimes m}) + \frac{\alpha}{1-\alpha}\log_2 F(\mathcal{N}^{(n)}(\rho^{\otimes n}),\tau^{\otimes m}) \tag{J34}$$

$$= m\widetilde{D}_\alpha(\tau\|\omega) + \frac{\alpha}{1-\alpha}\log_2 F(\mathcal{N}^{(n)}(\rho^{\otimes n}),\tau^{\otimes m}) \tag{J35}$$

$$= m\widetilde{D}_\alpha(\tau\|\omega) + \frac{2\alpha}{1-\alpha}\log_2 \sqrt{F}(\mathcal{N}^{(n)}(\rho^{\otimes n}),\tau^{\otimes m}) \tag{J36}$$

$$\geqslant m\widetilde{D}_\alpha(\tau\|\omega) + \frac{2\alpha}{1-\alpha}\log_2(1-\varepsilon), \tag{J37}$$

where to get the last inequality, we used the fact that [81]

$$\tfrac{1}{2}\|\rho_0-\rho_1\|_1 \geqslant 1-\sqrt{F}(\rho_0,\rho_1). \tag{J38}$$

Dividing by $n$, we find that

$$\widetilde{D}_\beta(\rho\|\sigma) \geqslant \frac{m}{n}\widetilde{D}_\alpha(\tau\|\omega) + \frac{2\alpha}{n(1-\alpha)}\log_2(1-\varepsilon), \tag{J39}$$

which concludes the proof. ■

We now give a proof for the strong converse statement in Eq. (62). Our proof is related to the approach from Ref. [82]. Fix $\varepsilon \in [0,1)$ and $\delta > 0$. We need to show that there is an $n$ large enough such that there does not exist an $(n, n[R+\delta], \varepsilon)$ box transformation protocol, with $R$ set as follows:

$$R = \frac{D(\rho\|\sigma)}{D(\tau\|\omega)}. \tag{J40}$$

From proposition 3, the following bound holds for an arbitrary $(n, m, \varepsilon)$ protocol, $\alpha \in (1/2, 1)$, and $\beta \equiv \beta(\alpha) := \alpha/(2\alpha-1)$:

$$n\widetilde{D}_\beta(\rho\|\sigma) + \frac{2\alpha}{1-\alpha}\log_2(1/[1-\varepsilon]) \geqslant m\widetilde{D}_\alpha(\tau\|\omega). \tag{J41}$$

Set $\delta_2$ such that $0 < \delta_2 < \delta D(\tau\|\omega)$. Then set $\delta_1 > 0$ such that the following equation is satisfied

$$\frac{D(\rho\|\sigma)+\delta_1+\delta_2}{D(\tau\|\omega)-\delta_1} = \frac{D(\rho\|\sigma)}{D(\tau\|\omega)} + \delta, \tag{J42}$$

i.e.,

$$\delta_1 = \frac{D(\tau\|\omega)[\delta D(\tau\|\omega)-\delta_2]}{D(\rho\|\sigma)+D(\tau\|\omega)[1+\delta]}. \tag{J43}$$

Set $\alpha \in (1/2, 1)$ such that

$$\delta_1 > \max\{D(\tau\|\omega) - \widetilde{D}_\alpha(\tau\|\omega), \widetilde{D}_\beta(\rho\|\sigma) - D(\rho\|\sigma)\}, \tag{J44}$$

which is possible due to (A12), (A14), (A17), (A21), and the fact that $\beta = \alpha/(2\alpha-1)$, and for this choice of $\alpha$, pick $n$ large enough so that

$$\delta_2 > \frac{2\alpha}{n(1-\alpha)}\log_2(1/[1-\varepsilon]). \tag{J45}$$

For these choices, we then have that

$$\widetilde{D}_\beta(\rho\|\sigma) + \frac{2\alpha}{n(1-\alpha)}\log_2(1/[1-\varepsilon]) < D(\rho\|\sigma)+\delta_1+\delta_2, \tag{J46}$$

and we also have that

$$\frac{m}{n}\widetilde{D}_\alpha(\tau\|\omega) > \frac{m}{n}[D(\tau\|\omega)-\delta_1]. \tag{J47}$$

Putting these inequalities together, we find that

$$\frac{m}{n} < \frac{D(\rho\|\sigma)+\delta_1+\delta_2}{D(\tau\|\omega)-\delta_1} = \frac{D(\rho\|\sigma)}{D(\tau\|\omega)} + \delta. \tag{J48}$$

Thus the rate of the protocol $\frac{m}{n}$ is strictly less than $\frac{D(\rho\|\sigma)}{D(\tau\|\omega)} + \delta$, so that an $(n, n[R+\delta], \varepsilon)$ box transformation protocol cannot exist for the choice of $n$ taken nor any $n$ larger than that [for the latter statement, note that (J45) still holds for larger $n$].

### 3. Strong converse via Petz-Rényi relative entropy

We now discuss an alternative proof of the strong converse by going through the Petz-Rényi relative entropy. We begin with a pseudo-continuity inequality for the Petz-Rényi relative entropy. The proof of lemma 3 below follows the spirit of the proof of proposition 85 of Ref. [85], but this time some steps are different.

*Lemma 3.* Let $\rho_0$, $\rho_1$, and $\sigma$ be quantum states such that $\text{supp}(\rho_0) \subseteq \text{supp}(\sigma)$. Fix $\alpha \in (0,1)$ and $\beta \equiv \beta(\alpha) := 2-\alpha \in (1,2)$. Then

$$D_\beta(\rho_0\|\sigma) - D_\alpha(\rho_1\|\sigma) \geqslant \frac{2}{1-\alpha}\log_2\left[1-\frac{1}{2}\|\rho_0-\rho_1\|_1\right]. \tag{J49}$$

*Proof.* Consider that $\alpha - 1 = 1 - \beta$, so that

$$D_\beta(\rho_0\|\sigma) - D_\alpha(\rho_1\|\sigma)$$

$$= \frac{1}{\beta-1}\log_2 \text{Tr}\left[\rho_0^\beta\sigma^{1-\beta}\right] - \frac{1}{\alpha-1}\log_2 \text{Tr}\left[\rho_1^\alpha\sigma^{1-\alpha}\right] \tag{J50}$$

$$= \frac{1}{\beta-1}\log_2 \mathrm{Tr}\left[\rho_0^{\beta}\sigma^{1-\beta}\right] + \frac{1}{\beta-1}\log_2 \mathrm{Tr}\left[\rho_1^{\alpha}\sigma^{1-\alpha}\right] \tag{J51}$$

$$= \frac{1}{\beta-1}\log_2\left(\mathrm{Tr}\left[\rho_0^{\beta}\sigma^{1-\beta}\right]\mathrm{Tr}\left[\rho_1^{\alpha}\sigma^{1-\alpha}\right]\right) \tag{J52}$$

$$= \frac{1}{\beta-1}\log_2\left(\left\|\rho_0^{\beta/2}\sigma^{(1-\beta)/2}\right\|_2^2\left\|\sigma^{(1-\alpha)/2}\rho_1^{\alpha/2}\right\|_2^2\right) \tag{J53}$$

$$\geqslant \frac{1}{\beta-1}\log_2\left\|\rho_0^{\beta/2}\sigma^{(1-\beta)/2}\sigma^{(1-\alpha)/2}\rho_1^{\alpha/2}\right\|_1^2 \tag{J54}$$

$$= \frac{2}{\beta-1}\log_2\left\|\rho_0^{\beta/2}\rho_1^{\alpha/2}\right\|_1 \tag{J55}$$

$$\geqslant \frac{2}{\beta-1}\log_2 \mathrm{Tr}\left[\rho_0^{\beta/2}\rho_1^{\alpha/2}\right] \tag{J56}$$

$$= \frac{2}{\beta-1}\log_2 \mathrm{Tr}\left[\rho_0^{\beta/2}\rho_1^{(2-\beta)/2}\right] \tag{J57}$$

$$= \frac{2}{\beta-1}\log_2 \mathrm{Tr}\left[\rho_0^{\beta/2}\rho_1^{1-\beta/2}\right] \tag{J58}$$

$$\geqslant \frac{2}{\beta-1}\log_2\left[1-\frac{1}{2}\|\rho_0-\rho_1\|_1\right] \tag{J59}$$

$$= \frac{2}{1-\alpha}\log_2\left[1-\frac{1}{2}\|\rho_0-\rho_1\|_1\right]. \tag{J60}$$

The fourth equality follows from a rewriting of the Petz-Rényi relative entropy in terms of the Schatten 2-norm, as given in Eq. (3.10) in Ref. [86]. The first inequality follows from an application of the Cauchy-Schwarz inequality. The second inequality follows from the variational characterization of the trace norm as $\|A\|_1 = \sup_U |\mathrm{Tr}[AU]|$, where the optimization is over all unitaries and we pick $U = I$ to get the inequality. The last inequality follows from theorem 1 of Ref. [87] and because $\beta/2 \in (1/2, 1)$. ■

*Remark.* We note here that the bound from lemma 3 can be used to obtain pseudo-continuity bounds for information quantities derived from the Petz-Rényi relative entropy, such as mutual information and conditional entropy, much like what is done in proposition 2.8 of Ref. [85].

The following is another auxiliary lemma that serves as a one-shot converse for any approximate box transformation $(\rho,\sigma)\underset{\rightarrow}{\mathcal{N}}(\tau,\omega)$ where $\omega=\mathcal{N}(\sigma)$:

*Lemma 4.* Let $\rho$, $\sigma$, $\tau$, and $\omega$ be quantum states and $\mathcal{N}$ a quantum channel such that $\mathcal{N}(\sigma)=\omega$. Then for $\alpha\in(0,1)$ and $\beta\equiv\beta(\alpha):=2-\alpha$, we have that

$$D_\beta(\rho\|\sigma) \geqslant D_\alpha(\tau\|\omega) + \frac{2}{1-\alpha}\log_2\left[1-\frac{1}{2}\|\mathcal{N}(\rho)-\tau\|_1\right]. \tag{J61}$$

*Proof.* Consider that

$$D_\beta(\rho\|\sigma) \geqslant D_\beta(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \tag{J62}$$

$$= D_\beta(\mathcal{N}(\rho)\|\omega) \tag{J63}$$

$$\geqslant D_\alpha(\tau\|\omega) + \frac{2}{1-\alpha}\log_2\left[1-\frac{1}{2}\|\mathcal{N}(\rho)-\tau\|_1\right]. \tag{J64}$$

The first inequality follows from the quantum data processing inequality in Eq. (A11) and the other from lemma 3. ■

*Proposition 2.* Let $n,m\in\mathbb{Z}^+$ and $\varepsilon\in[0,1]$. Let $\rho$, $\sigma$, $\tau$, and $\omega$ be quantum states and $\mathcal{N}^{(n)}$ a quantum channel constituting an $(n,m,\varepsilon)$ box transformation protocol (i.e., so that $\mathcal{N}^{(n)}(\rho^{\otimes n})\approx_\varepsilon \tau^{\otimes m}$ and $\mathcal{N}^{(n)}(\sigma^{\otimes n})=\omega^{\otimes m}$). Then for $\alpha\in(0,1)$ and $\beta\equiv\beta(\alpha):=2-\alpha$, we have that

$$\frac{D_\beta(\rho\|\sigma)}{D_\alpha(\tau\|\omega)} \geqslant \frac{m}{n} + \frac{2}{n(1-\alpha)D_\alpha(\tau\|\omega)}\log_2(1-\varepsilon). \tag{J65}$$

Alternatively, if we set $R=m/n$, then the above bound can be written as

$$-\frac{1}{n}\log_2(1-\varepsilon)$$
$$\geqslant \left(\frac{1-\alpha}{2}\right)(R\,D_\alpha(\tau\|\omega)-D_\beta(\rho\|\sigma)) \tag{J66}$$

$$= \left(\frac{\beta-1}{2}\right)(R\,D_\alpha(\tau\|\omega)-D_\beta(\rho\|\sigma)). \tag{J67}$$

*Proof.* Consider that

$$nD_\beta(\rho\|\sigma) = D_\beta(\rho^{\otimes n}\|\sigma^{\otimes n}) \tag{J68}$$

$$\geqslant D_\alpha(\tau^{\otimes m}\|\omega^{\otimes m}) + \frac{2}{1-\alpha}\log_2\left[1-\frac{1}{2}\left\|\mathcal{N}^{(n)}(\rho^{\otimes n})-\tau^{\otimes m}\right\|_1\right] \tag{J69}$$

$$\geqslant mD_\alpha(\tau\|\omega) + \frac{2}{1-\alpha}\log_2(1-\varepsilon), \tag{J70}$$

Dividing by $n$, we find that

$$D_\beta(\rho\|\sigma) \geqslant \frac{m}{n}D_\alpha(\tau\|\omega) + \frac{2}{n(1-\alpha)}\log_2(1-\varepsilon), \tag{J71}$$

which concludes the proof. ■

We note here that one could arrive at the strong converse statement by going through steps similar to those in Eqs. (J41)–(J48), but using proposition 2 instead.

## APPENDIX K: BOUNDING THE SMOOTH MAX-RELATIVE ENTROPY WITH QUANTUM RELATIVE ENTROPIES

In this Appendix, we establish lower and upper bounds for the smooth max-relative entropy in terms of the Rényi relative entropies. We begin with the following lower bound:

*Proposition 3.* Let $\rho$ and $\sigma$ be quantum states. The following bound holds for all $\alpha\in[1/2,1)$ and $\varepsilon\in[0,1)$:

$$D_{\max}^{\varepsilon}(\rho\|\sigma) \geqslant \widetilde{D}_\alpha(\rho\|\sigma) + \frac{2\alpha}{\alpha-1}\log_2\left(\frac{1}{1-\varepsilon}\right). \tag{K1}$$

*Proof.* First fix $\alpha\in(1/2,1)$. Let $\widetilde{\rho}$ be a state such that $\frac{1}{2}\|\widetilde{\rho}-\rho\|_1\leqslant\varepsilon$. Then for $\beta\equiv\beta(\alpha):=\alpha/(2\alpha-1)$, we find that

$$D_{\max}(\widetilde{\rho}\|\sigma) \geqslant \widetilde{D}_\beta(\widetilde{\rho}\|\sigma) \tag{K2}$$

$$\geqslant \widetilde{D}_\alpha(\rho\|\sigma) + \frac{\alpha}{1-\alpha}\log_2 F(\widetilde{\rho},\rho) \tag{K3}$$

$$= \widetilde{D}_\alpha(\rho\|\sigma) + \frac{2\alpha}{1-\alpha}\log_2\sqrt{F}(\widetilde{\rho},\rho) \tag{K4}$$

$$\geqslant \widetilde{D}_\alpha(\rho\|\sigma) + \frac{2\alpha}{1-\alpha}\log_2(1-\varepsilon). \tag{K5}$$

The first inequality follows from (A18) and (A21). The second inequality follows from Lemma 1. The final inequality follows because [81]

$$1-\sqrt{F(\widetilde{\rho},\rho)} \leqslant \tfrac{1}{2}\|\widetilde{\rho}-\rho\|_1. \tag{K6}$$

Since the bound holds for an arbitrary $\widetilde{\rho}$ satisfying $\frac{1}{2}\|\widetilde{\rho}-\rho\|_1 \leqslant \varepsilon$, we conclude (K1).

The inequality in Eq. (K1) for $\alpha = 1/2$ follows since (K1) holds for all $\alpha \in (1/2, 1)$ and by taking the limit as $\alpha \to 1/2$. ■

Another lower bound on the smooth max-relative entropy is as follows:

*Proposition 4.* Let $\rho$ and $\sigma$ be quantum states. The following bound holds for all $\alpha \in [0, 1)$ and $\varepsilon \in [0, 1)$:

$$D^{\varepsilon}_{\max}(\rho\|\sigma) \geqslant D_\alpha(\rho\|\sigma) + \frac{2}{\alpha-1}\log_2\left(\frac{1}{1-\varepsilon}\right). \tag{K7}$$

*Proof.* First fix $\alpha \in (0, 1)$. Let $\widetilde{\rho}$ be a state such that $\frac{1}{2}\|\widetilde{\rho}-\rho\|_1 \leqslant \varepsilon$. Then for $\beta \equiv \beta(\alpha) := 2-\alpha$, we find that

$$D_{\max}(\widetilde{\rho}\|\sigma) \geqslant D_\beta(\widetilde{\rho}\|\sigma) \tag{K8}$$

$$\geqslant D_\alpha(\rho\|\sigma) + \frac{2}{1-\alpha}\log_2\left[1-\frac{1}{2}\|\widetilde{\rho}-\rho\|_1\right] \tag{K9}$$

$$\geqslant D_\alpha(\rho\|\sigma) + \frac{2}{1-\alpha}\log_2(1-\varepsilon). \tag{K10}$$

The first inequality follows from (A14) and Eqs. (43)–(46) in Ref. [88], the latter of which we repeat below:

$$\begin{aligned}
D_2(\widetilde{\rho}\|\sigma) &= \log_2 \mathrm{Tr}[\widetilde{\rho}^2\sigma^{-1}] && \text{(K11)}\\
&= \log_2 \mathrm{Tr}[\widetilde{\rho}\widetilde{\rho}^{1/2}\sigma^{-1}\widetilde{\rho}^{1/2}] && \text{(K12)}\\
&\leqslant \log_2 \sup_\tau \mathrm{Tr}[\tau\widetilde{\rho}^{1/2}\sigma^{-1}\widetilde{\rho}^{1/2}] && \text{(K13)}\\
&= \log_2 \|\widetilde{\rho}^{1/2}\sigma^{-1}\widetilde{\rho}^{1/2}\|_\infty && \text{(K14)}\\
&= D_{\max}(\widetilde{\rho}\|\sigma). && \text{(K15)}
\end{aligned}$$

Note that the optimization above is over quantum states $\tau$. The second inequality in Eq. (K9) follows from lemma 3. Since the bound holds for an arbitrary state $\widetilde{\rho}$ satisfying $\frac{1}{2}\|\widetilde{\rho}-\rho\|_1 \leqslant \varepsilon$, we conclude (K7).

The inequality in Eq. (K7) for $\alpha = 0$ follows since (K7) holds for all $\alpha \in (0, 1)$ and by taking the limit as $\alpha \to 0$. ■

We now give some upper bounds on the smooth max-relative entropy in terms of the quantum relative entropy and the sandwiched Rényi relative entropy. The method for doing so follows the proof approach of theorem 1 in Ref. [89] very closely. The upper bound in proposition 5 is very similar to theorem 1 of Ref. [89], but it is expressed in terms of quantum relative entropy rather than observational divergence.

*Proposition 5.* Given states $\rho$ and $\sigma$, the following bound holds for all $\varepsilon \in (0, 1)$:

$$D^{\varepsilon}_{\max}(\rho\|\sigma) \leqslant \frac{1}{\varepsilon^2}\left[D(\rho\|\sigma) + \frac{1}{2\ln 2}\|\rho-\sigma\|_1\right] + \log_2\left(\frac{1}{1-\varepsilon^2}\right). \tag{K16}$$

*Proof.* The statement is trivially true if $\rho = \sigma$ or if $\mathrm{supp}(\rho) \not\subseteq \mathrm{supp}(\sigma)$. So going forward, we assume that $\rho \neq \sigma$ and $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$. The SDP dual of $D_{\max}(\tau\|\omega)$ is given by

$$D_{\max}(\tau\|\omega) = \log_2 \sup_{\Lambda\geqslant 0}\{\mathrm{Tr}[\Lambda\tau] : \mathrm{Tr}[\Lambda\omega] \leqslant 1\}, \tag{K17}$$

implying that

$$D^{\varepsilon}_{\max}(\rho\|\sigma) = \log_2 \inf_{\widetilde{\rho}:\frac{1}{2}\|\widetilde{\rho}-\rho\|_1\leqslant\varepsilon} \ \sup_{\substack{\Lambda\,:\,\Lambda\geqslant 0,\\ \mathrm{Tr}[\Lambda\sigma]\leqslant 1}} \mathrm{Tr}[\Lambda\widetilde{\rho}]. \tag{K18}$$

Since the objective function $\mathrm{Tr}[\Lambda\widetilde{\rho}]$ is linear in $\Lambda$ and $\widetilde{\rho}$, the set $\{\Lambda : \Lambda \geqslant 0, \mathrm{Tr}[\Lambda\sigma] \leqslant 1\}$ is compact and concave, and the set

$$\left\{\widetilde{\rho} : \tfrac{1}{2}\|\widetilde{\rho}-\rho\|_1 \leqslant \varepsilon,\ \widetilde{\rho} \geqslant 0,\ \mathrm{Tr}[\widetilde{\rho}] = 1\right\} \tag{K19}$$

is compact and convex (due to convexity of normalized trace distance), the minimax theorem applies and we find that

$$D^{\varepsilon}_{\max}(\rho\|\sigma) = \log_2 \sup_{\substack{\Lambda\,:\,\Lambda\geqslant 0,\\ \mathrm{Tr}[\Lambda\sigma]\leqslant 1}} \ \inf_{\widetilde{\rho}:\frac{1}{2}\|\widetilde{\rho}-\rho\|_1\leqslant\varepsilon} \mathrm{Tr}[\Lambda\widetilde{\rho}]. \tag{K20}$$

For a fixed operator $\Lambda \geqslant 0$ with spectral decomposition

$$\Lambda = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|, \tag{K21}$$

let us define the following set, for a choice of $\lambda > 0$ to be specified later:

$$\mathcal{S} := \{i : \langle\phi_i|\rho|\phi_i\rangle > 2^\lambda\langle\phi_i|\sigma|\phi_i\rangle\}. \tag{K22}$$

Let

$$\Pi = \sum_{i\in\mathcal{S}} |\phi_i\rangle\langle\phi_i|. \tag{K23}$$

Then from the definition, we find that

$$\mathrm{Tr}[\Pi\rho] > 2^\lambda\,\mathrm{Tr}[\Pi\sigma], \tag{K24}$$

which implies that

$$\frac{\mathrm{Tr}[\Pi\rho]}{\mathrm{Tr}[\Pi\sigma]} > 2^\lambda. \tag{K25}$$

Now consider from the data processing inequality under the channel

$$\Delta(\omega) := \mathrm{Tr}[\Pi\omega]|0\rangle\langle 0| + \mathrm{Tr}[\hat{\Pi}\omega]|1\rangle\langle 1|, \tag{K26}$$

where

$$\hat{\Pi} := I - \Pi, \tag{K27}$$

that

$$\begin{aligned}
&D(\rho\|\sigma)\\
&\geqslant D(\Delta(\rho)\|\Delta(\sigma))\\
&= \mathrm{Tr}[\Pi\rho]\log_2\left(\frac{\mathrm{Tr}[\Pi\rho]}{\mathrm{Tr}[\Pi\sigma]}\right) + \mathrm{Tr}[\hat{\Pi}\rho]\log_2\left(\frac{\mathrm{Tr}[\hat{\Pi}\rho]}{\mathrm{Tr}[\hat{\Pi}\sigma]}\right)\\
&= \mathrm{Tr}[\Pi\rho]\log_2\left(\frac{\mathrm{Tr}[\Pi\rho]}{\mathrm{Tr}[\Pi\sigma]}\right) + \frac{1}{\ln 2}(\mathrm{Tr}[\Pi\sigma]-\mathrm{Tr}[\Pi\rho])\\
&\quad + \mathrm{Tr}[\hat{\Pi}\rho]\log_2\left(\frac{\mathrm{Tr}[\hat{\Pi}\rho]}{\mathrm{Tr}[\hat{\Pi}\sigma]}\right) + \frac{1}{\ln 2}(\mathrm{Tr}[\hat{\Pi}\sigma]-\mathrm{Tr}[\hat{\Pi}\rho])\\
&\geqslant \mathrm{Tr}[\Pi\rho]\log_2\left(\frac{\mathrm{Tr}[\Pi\rho]}{\mathrm{Tr}[\Pi\sigma]}\right) + \frac{1}{\ln 2}(\mathrm{Tr}[\Pi\sigma]-\mathrm{Tr}[\Pi\rho])\\
&\geqslant \lambda\,\mathrm{Tr}[\Pi\rho] + \frac{1}{\ln 2}(\mathrm{Tr}[\Pi\sigma]-\mathrm{Tr}[\Pi\rho]),
\end{aligned} \tag{K28}$$

where the second inequality follows because

$$x\log_2(x/y)+\frac{1}{\ln 2}(y-x) = \frac{1}{\ln 2}[x\ln(x/y)+y-x]\geqslant 0, \quad \text{(K29)}$$

for all $x, y \geqslant 0$, and the last inequality follows from (K25). Then we find that

$$\mathrm{Tr}[\Pi\rho]\leqslant\lambda^{-1}\left(D(\rho\|\sigma)+\frac{1}{\ln 2}\mathrm{Tr}[\Pi\rho]-\mathrm{Tr}[\Pi\sigma]\right) \quad \text{(K30)}$$

$$\leqslant\lambda^{-1}\left(D(\rho\|\sigma)+\frac{1}{2\ln 2}\|\rho-\sigma\|_1\right). \quad \text{(K31)}$$

Pick

$$\lambda=\frac{1}{\varepsilon^2}\left[D(\rho\|\sigma)+\frac{1}{2\ln 2}\|\rho-\sigma\|_1\right], \quad \text{(K32)}$$

and we conclude from the above that

$$\mathrm{Tr}[\Pi\rho]\leqslant\varepsilon^2. \quad \text{(K33)}$$

So this means that

$$\mathrm{Tr}[\hat{\Pi}\rho]\geqslant 1-\varepsilon^2. \quad \text{(K34)}$$

Thus the state

$$\rho':=\frac{\hat{\Pi}\rho\hat{\Pi}}{\mathrm{Tr}[\hat{\Pi}\rho]} \quad \text{(K35)}$$

is such that (see lemma 9.4.1 in Ref. [67])

$$F(\rho,\rho')\geqslant 1-\varepsilon^2, \quad \text{(K36)}$$

and in turn that [81]

$$\tfrac{1}{2}\|\rho-\rho'\|_1\leqslant\varepsilon. \quad \text{(K37)}$$

We also have that

$$\rho'\leqslant\frac{\hat{\Pi}\rho\hat{\Pi}}{1-\varepsilon^2}. \quad \text{(K38)}$$

Now let $\Lambda$ be an arbitrary operator satisfying $\Lambda\geqslant 0$ and $\mathrm{Tr}[\Lambda\sigma]\leqslant 1$, and let $\Pi$ be the projection defined in Eq. (K23) for this choice of $\Lambda$. Then we find that

$$(1-\varepsilon^2)\,\mathrm{Tr}[\Lambda\rho']\leqslant\mathrm{Tr}[\Lambda\hat{\Pi}\rho\hat{\Pi}] \quad \text{(K39)}$$

$$=\mathrm{Tr}[\hat{\Pi}\Lambda\hat{\Pi}\rho] \quad \text{(K40)}$$

$$=\sum_{i\notin\mathcal{S}}\lambda_i\langle\phi_i|\rho|\phi_i\rangle \quad \text{(K41)}$$

$$\leqslant 2^{\lambda}\sum_{i\notin\mathcal{S}}\lambda_i\langle\phi_i|\sigma|\phi_i\rangle \quad \text{(K42)}$$

$$\leqslant 2^{\lambda}\,\mathrm{Tr}[\Lambda\sigma] \quad \text{(K43)}$$

$$\leqslant 2^{\lambda}. \quad \text{(K44)}$$

Thus we have found the following uniform bound for any operator $\Lambda$ satisfying $\Lambda\geqslant 0$ and $\mathrm{Tr}[\Lambda\sigma]\leqslant 1$, with $\rho'$ the state in Eq. (K35) depending on $\Lambda$ and satisfying $\frac{1}{2}\|\rho-\rho'\|_1\leqslant\varepsilon$:

$$\mathrm{Tr}[\Lambda\rho']\leqslant 2^{\lambda+\log_2(\frac{1}{1-\varepsilon^2})}. \quad \text{(K45)}$$

Then it follows that

$$D^{\varepsilon}_{\max}(\rho\|\sigma)=\log_2\sup_{\substack{\Lambda\,:\,\Lambda\geqslant 0,\\ \mathrm{Tr}[\Lambda\sigma]\leqslant 1}}\ \inf_{\widetilde{\rho}:\frac{1}{2}\|\widetilde{\rho}-\rho\|_1\leqslant\varepsilon}\mathrm{Tr}[\Lambda\widetilde{\rho}] \quad \text{(K46)}$$

$$\leqslant\log_2\sup_{\substack{\Lambda\,:\,\Lambda\geqslant 0,\\ \mathrm{Tr}[\Lambda\sigma]\leqslant 1}}\mathrm{Tr}[\Lambda\rho'] \quad \text{(K47)}$$

$$\leqslant\lambda+\log_2\left(\frac{1}{1-\varepsilon^2}\right). \quad \text{(K48)}$$

This concludes the proof. ■

The proof of the following proposition follows the same proof approach of theorem 1 in Ref. [89] (as recalled above), but instead employs the sandwiched Rényi relative entropy and its data processing inequality. The following proposition was also reported recently in Ref. [90]:

*Proposition 6.* Given states $\rho$ and $\sigma$, the following bound holds for all $\alpha>1$ and $\varepsilon\in(0,1)$:

$$D^{\varepsilon}_{\max}(\rho\|\sigma)\leqslant\widetilde{D}_{\alpha}(\rho\|\sigma)+\frac{1}{\alpha-1}\log_2\left(\frac{1}{\varepsilon^2}\right)+\log_2\left(\frac{1}{1-\varepsilon^2}\right). \quad \text{(K49)}$$

*Proof.* The first steps are exactly the same as (K17)–(K25). Now consider from the data processing inequality under the channel

$$\Delta(\omega):=\mathrm{Tr}[\Pi\omega]|0\rangle\langle 0|+\mathrm{Tr}[\hat{\Pi}\omega]|1\rangle\langle 1| \quad \text{(K50)}$$

that

$$\widetilde{D}_{\alpha}(\rho\|\sigma)\geqslant\widetilde{D}_{\alpha}(\Delta(\rho)\|\Delta(\sigma)) \quad \text{(K51)}$$

$$=\frac{1}{\alpha-1}\log_2((\mathrm{Tr}[\Pi\rho])^{\alpha}(\mathrm{Tr}[\Pi\sigma])^{1-\alpha}+(\mathrm{Tr}[\hat{\Pi}\rho])^{\alpha}(\mathrm{Tr}[\hat{\Pi}\sigma])^{1-\alpha}) \quad \text{(K52)}$$

$$\geqslant\frac{1}{\alpha-1}\log_2\left((\mathrm{Tr}[\Pi\rho])^{\alpha}(\mathrm{Tr}[\Pi\sigma])^{1-\alpha}\right) \quad \text{(K53)}$$

$$=\frac{1}{\alpha-1}\log_2\left(\mathrm{Tr}[\Pi\rho]\left(\frac{\mathrm{Tr}[\Pi\rho]}{\mathrm{Tr}[\Pi\sigma]}\right)^{\alpha-1}\right) \quad \text{(K54)}$$

$$=\frac{1}{\alpha-1}\log_2(\mathrm{Tr}[\Pi\rho])+\log_2\left(\frac{\mathrm{Tr}[\Pi\rho]}{\mathrm{Tr}[\Pi\sigma]}\right) \quad \text{(K55)}$$

$$\geqslant\frac{1}{\alpha-1}\log_2(\mathrm{Tr}[\Pi\rho])+\lambda. \quad \text{(K56)}$$

Now picking

$$\lambda=\widetilde{D}_{\alpha}(\rho\|\sigma)+\frac{1}{\alpha-1}\log_2\left(\frac{1}{\varepsilon^2}\right), \quad \text{(K57)}$$

we conclude that

$$\mathrm{Tr}[\Pi\rho] \leqslant \varepsilon^2. \tag{K58}$$

The rest of the proof then proceeds as in Eqs. (K34)–(K48), and we find that $D^{\varepsilon}_{\max}(\rho\|\sigma) \leqslant \lambda + \log_2(1/[1-\varepsilon^2])$. ■

## APPENDIX L: RESOURCE THEORY OF ASYMMETRIC DISTINGUISHABILITY BASED ON INFIDELITY

In this paper, we employed the normalized trace distance throughout as the measure for approximation in approximate box transformations. As emphasized in the main text, the primary reason for doing so is due to the interpretation of normalized trace distance as the error in a single-shot experiment, as discussed around (11). Another advantage is that the optimizations corresponding to the one-shot operational tasks of distillation and dilution are characterized by semidefinite programs in both the theory presented here and in Ref. [43].

One could alternatively employ the infidelity $1 - F(\rho, \widetilde{\rho})$ as the measure for approximation. The main disadvantage in doing so is that the interpretation in terms of error is not as strong as it is for normalized trace distance. Furthermore, in the resource theory of asymmetric distinguishability for quantum channels, it is not clear whether the optimizations for the operational tasks of distillation and dilution are characterized by semidefinite programs [43].

However, there are some advantages to using the infidelity, which we highlight briefly here while avoiding detailed proofs. For the rest of this Appendix, we employ the following shorthand:

$$\rho \approx_{\varepsilon_F} \widetilde{\rho} \Leftrightarrow 1 - F(\widetilde{\rho}, \rho) \leqslant \varepsilon_F, \tag{L1}$$

using the notation $\varepsilon_F$ to emphasize that the error is with respect to infidelity.

First, it is worthwhile to note that the one-shot distillable distinguishability is unchanged:

$$D^{\varepsilon_F}_d(\rho, \sigma) = D^{\varepsilon_F}_{\min}(\rho\|\sigma), \tag{L2}$$

while the one-shot distinguishability cost becomes

$$D^{\varepsilon_F}_c(\rho, \sigma) = D^{\varepsilon_F}_{\max}(\rho\|\sigma), \tag{L3}$$

where

$$D^{\varepsilon_F}_{\max}(\rho\|\sigma) := \inf_{\widetilde{\rho}\,:\,1-F(\widetilde{\rho},\rho)\leqslant\varepsilon} D_{\max}(\widetilde{\rho}\|\sigma). \tag{L4}$$

In the above, the superscript $\varepsilon_F$ serves to distinguish $D^{\varepsilon_F}_{\max}(\rho\|\sigma)$ from the smooth max-relative entropy in Eq. (49). Then we have the following expansions:

$$D^{\varepsilon_F}_d((\rho^{\otimes n}, \sigma^{\otimes n})) = nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon_F) + O(\ln n), \tag{L5}$$

$$D^{\varepsilon_F}_c((\rho^{\otimes n}, \sigma^{\otimes n})) = nD(\rho\|\sigma) - \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon_F) + O(\ln n), \tag{L6}$$

with the key difference being that the second-order characterization of the $\varepsilon_F$-approximate distinguishability cost is now tight. The inequality in Eq. (51) becomes as follows:

$$D^{\varepsilon_F}_{\min}(\rho\|\sigma) \leqslant D^{\varepsilon_F'}_{\max}(\rho\|\sigma) - \log_2(1 - [\sqrt{\varepsilon_F} + \sqrt{\varepsilon_F'}]^2), \tag{L7}$$

for $\varepsilon_F, \varepsilon_F' \geqslant 0$ and $\sqrt{\varepsilon_F} + \sqrt{\varepsilon_F'} < 1$, which follows by employing the triangle inequality for the sine distance [91–94]. The inequality in Eq. (I1) becomes as follows:

$$D^{\varepsilon_F}_{\max}(\rho\|\sigma) \leqslant D^{1-\varepsilon_F}_{\min}(\rho\|\sigma) + \log_2|\mathrm{spec}(\sigma)| + \log_2\left(\frac{1}{1-\varepsilon_F}\right), \tag{L8}$$

which is a key reason why we obtain (L6). The inequality in Eq. (K1) becomes

$$D^{\varepsilon_F}_{\max}(\rho\|\sigma) \geqslant \widetilde{D}_\alpha(\rho\|\sigma) + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{1-\varepsilon_F}\right), \tag{L9}$$

for $\alpha \in [1/2, 1)$, while (K49) becomes

$$D^{\varepsilon_F}_{\max}(\rho\|\sigma) \leqslant \widetilde{D}_\alpha(\rho\|\sigma) + \log_2\left(\frac{1}{1-\varepsilon_F}\right) + \frac{1}{\alpha-1}\log_2\left(\frac{1}{\varepsilon_F}\right), \tag{L10}$$

for $\alpha \in (1, \infty)$. The converse bound in proposition 1 becomes

$$-\frac{1}{n}\log_2(1-\varepsilon_F) \geqslant \left(\frac{1-\alpha}{\alpha}\right)(R\,\widetilde{D}_\alpha(\tau\|\omega) - \widetilde{D}_\beta(\rho\|\sigma)) \tag{L11}$$

$$= \left(\frac{\beta-1}{\beta}\right)(R\,\widetilde{D}_\alpha(\tau\|\omega) - \widetilde{D}_\beta(\rho\|\sigma)), \tag{L12}$$

holding for an arbitrary $(n, m, \varepsilon_F)$ box transformation protocol [i.e., so that $\mathcal{N}^{(n)}(\rho^{\otimes n}) \approx_{\varepsilon_F} \tau^{\otimes m}$ and $\mathcal{N}^{(n)}(\sigma^{\otimes n}) = \omega^{\otimes m}$], $\alpha \in (1/2, 1)$, and $\beta = \alpha/(2\alpha - 1)$. For distinguishability distillation with $\tau = |0\rangle\langle 0|$ and $\omega = \pi$, so that $\widetilde{D}_\alpha(\tau\|\omega) = 1$, this bound reduces to

$$-\frac{1}{n}\log_2(1-\varepsilon_F) \geqslant \left(\frac{\beta-1}{\beta}\right)(R - \widetilde{D}_\beta(\rho\|\sigma)), \tag{L13}$$

holding for all $\beta > 1$, which is the optimal strong converse exponent, as shown in Ref. [53]. For distinguishability dilution with $\rho = |0\rangle\langle 0|$ and $\sigma = \pi$, so that $\widetilde{D}_\beta(\rho\|\sigma) = 1$, and by multiplying (L11) by $n/m$ and setting the rate $S = n/m$, the bound becomes

$$-\frac{1}{m}\log_2(1-\varepsilon_F) \geqslant \left(\frac{1-\alpha}{\alpha}\right)(\widetilde{D}_\alpha(\tau\|\omega) - S), \tag{L14}$$

holding for all $\alpha \in [1/2, 1)$. It is an interesting open question to determine the optimal strong converse exponent for distinguishability dilution.

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