Noninvertible anomalies and mapping-class-group transformation of anomalous partition functions

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Recently, it was realized that anomalies can be completely classified by topological orders, symmetry protected topological orders, and symmetry enriched topological orders in one higher dimension. The anomalies that people used to study are invertible anomalies that correspond to invertible topological orders and/or symmetry protected topological orders in one higher dimension. In this paper, we introduce a notion of noninvertible anomaly, which describes the boundary of generic topological order. It is characterized by two features. First, a theory with noninvertible anomaly has a multicomponent partition function. Second, under the mapping class group transformation of space-time, the vector of partition functions transform covariantly. In fact, the anomalous partition functions transform in the same way as the degenerate ground states of the corresponding topological order in one higher dimension. This general theory of noninvertible anomaly may have wide applications. As an example, we show that the irreducible gapless boundary of 2+1D double-semion topological order must have central charge $c = \bar{c} > \frac{25}{28}$.

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I. INTRODUCTION

A classical field theory described by an action may have a gauge symmetry if the action is gauge invariant. The corresponding theory is called a classical gauge theory. A gauge anomaly is an obstruction to quantize the classical gauge theory, since the path integral measure may not be gauge invariant [1,2]. Similarly, a classical action may have a diffeomorphism invariance. Then a gravitational anomaly is an obstruction to have a diffeomorphic invariant path integral [3]. So the standard point of view of anomaly corresponds to the obstruction to go from classical theory to quantum theory. This kind of gauge anomaly and gravitational anomaly are always *invertible*, i.e., can be canceled by another anomalous theory. The examples include 1+1D U(1)-gauged chiral fermion theory

$$S = \int dx\, dt\ \psi^\dagger(\partial_t + iA_t - \partial_x - iA_x)\psi, \tag{1}$$

which has both perturbative U(1) gauge anomaly and perturbative gravitational anomaly. We like to remark that anomaly defined as such is not a property of physical systems, but a property of a formalism trying to convert a classical theory to a quantum theory.

There is another invertible anomaly—'t Hooft anomaly, that can be defined within a quantum system with a global symmetry, and is a property of physical systems. It is not an obstruction to go from classical theory to quantum theory, but rather an obstruction to gauge a global symmetry within a quantum system [4]. It is quite amazing that the obstruction to quantize a classical gauge theory (gauge anomaly) is closely related to the obstruction to gauge a global symmetry within a quantum system.

Motivated by some early results [5,6], in recent years, we started to have a new understanding of anomaly as a physical property of quantum systems [7–9], rather than an obstruction to quantize a classical theory: (1) gravitational anomaly in a theory directly corresponds to topological order [10,11] in one higher dimension; (2) 't Hooft anomaly for global symmetry G in a theory directly corresponds to symmetry protected topological (SPT) order [12–14] with on-site symmetry G in one higher dimension, and (3) such an anomalous theory is realized as a boundary theory of the corresponding topological order and/or SPT order [see Fig. 1(b)].

So an anomaly is nothing but a topological order and/or a SPT order in one higher dimension, and the anomalies can be classified via the classification of topological orders and SPT orders in one higher dimension [7,15]. The boundary of topological order realizes all possible gravitational anomaly, and the boundary of SPT order realizes all possible 't Hooft anomaly and mixed gravity/'t Hooft anomaly. The boundary of symmetry enriched topological order will corresponds to new types of gravitational anomaly with symmetry. This point of view of anomaly plus Atiyah formulation of topological quantum field theory [16] allow us to develop a general theory of anomaly [7–9].

The anomaly from this new point of view is not the same as the previously defined anomaly before 2013, and is more general. This is because the previously defined anomalies are always invertible (i.e., can be canceled by another anomaly). Those anomalies are classified by invertible topological orders and/or SPT orders in one higher dimension, and are realized by the boundary of the invertible topological orders and/or the SPT orders. We know that generic topological orders are usually not invertible [8,17]. Hence, the anomalies realized

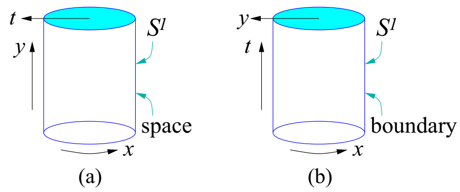

FIG. 1. (a) A particular time t evolution produces a particular ground state in the degenerate ground-state subspace on the space $S^1 \times S^1$. (b) A particular extension of a space-time $S^1 \times S^1$ as the boundary of a bulk $D^2 \times S^1$ produces a particular anomalous partition function in the vector space of partition functions on space-time $S^1 \times S^1$ (i.e., the boundary).

by the boundary of generic topological orders are usually noninvertible (i.e., cannot be canceled by any other anomaly). Those noninvertible topological orders will give rise to a new kind of gravitational anomalies on the boundary, which will be called noninvertible anomaly. For example, the chiral conformal field theory (CFT) on the boundary of a generic Chern-Simons theory is an example of noninvertible anomaly [18].

We like to mention that, in addition to the above general point of view proposed in Refs. [7–9] that include noninvertible anomalies, Refs. [19,20] also proposed a similar general point of view based mathematical category theory and cobordism theory. In particular, those general points of view suggest that the partition function becomes a vector in a vector space for a theory with noninvertible anomaly, as suggested in Refs. [16,21]. In fact, the vector space that contains partition functions of an anomalous theory can be identified with the degenerate ground-state subspace [10,11] [see Fig. 1(a)] of the topological order in one higher dimension that characterizes the anomaly. This is because if we regard a space direction as time direction, the space manifold can be viewed as a boundary of space-time, and the ground degenerate subspace becomes the vector space of partition functions (see Fig. 1) [8,22].

In this paper, we will study some simplest noninvertible anomalies—bosonic global gravitational anomalies in 1+1D which correspond to a 2+1D bosonic topological order. We will first give a general discussion, in particular the physical meaning of "multicomponent partition functions as a vector in a vector space." Then we will discuss some examples of 1+1D bosonic theories with noninvertible gravitational anomalies that correspond to (1) 2+1D bosonic Z_2 topological order [23,24] (i.e., the topological order described by the Z_2-gauge theory). (2) 2+1D bosonic double-semion (DS) topological order [25,26]. (3) 2+1D bosonic semion topological order (i.e., $\nu = 1/2$ quantum Hall state) [27]. (4) 2+1D bosonic Fibonacci topological order [25,26].

We will also discuss an application of invertible and noninvertible anomalies. There is a general belief that a gapless CFT has a partition function that is invariant under mapping-class-group (MCG) transformations of the space-time (the modular transformations for two-dimensional space-time), provided that the CFT can be put on a lattice. Being able to put a CFT on a lattice is nothing but the anomaly-free condition. This suggests that the MCG invariance of the partition function corresponds to the anomaly-free condition. So an anomalous CFT will have a partition function which is not MCG invariant, but *MCG covariant*.[1] Since the anomaly corresponds to a topological order in one higher dimension that is described by a higher category, the change of anomalous partition function can be described by the data of this higher category. In this paper, we will derive one such result.

Consider a CFT in d-dimensional closed space-time M^d, whose gravitational anomaly is described a $(d+1)$D topological order. The $(d+1)$D topological order has N-fold degenerate ground states on M^d. Let G_{M^d} be the MCG for M^d. Under a MCG transformation $g \in G_{M^d}$, the degenerate ground states transform according to a projective representation $R^{\text{top}}(g)$ of G_{M^d} [8,11,28,29]. Such a projective representation $R^{\text{top}}(g)$ is the data that characterize the $(d+1)$D topological order (and hence the anomaly). It was conjectured [11] that such data fully characterizes the topological order. From the correspondence described in Fig. 1, we find that a CFT with gravitational anomaly in d-dimensional space-time has several partition functions $Z(g_{\mu\nu}, i)$, $i = 1, \dots, \dim(R^{\text{top}})$, which transform as

$$Z(g \cdot g_{\mu\nu}, i) = R^{\text{top}}_{ij}(g) Z(g_{\mu\nu}, j), \tag{2}$$

where $g_{\mu\nu}$ is the metrics on the d-dimensional space-time M^d, which describes the shape of M^d, and $g \cdot g_{\mu\nu}$ is the MCG action on $g_{\mu\nu}$.

When $d = 2$ [Eq. (2)], becomes Eq. (24), which we will explain in detail. We like to remark that, in some dimensions (for examples, when $d = 2$ or when there is no perturbative gravitational anomaly), $R^{\text{top}}(g)$ can be a representation of MCG G_{M^d} [8]. In particular, in Eq. (24), the partition functions transform as a representation of MCG $SL(2, \mathbb{Z})$.

For an anomaly-free CFT, the corresponding $(d+1)$D topological order is trivial and $R^{\text{top}} = 1$ are 1-by-1 matrices. In the case, the above becomes the usual MCG invariant condition on the partition function:

$$Z(g \cdot g_{\mu\nu}) = Z(g_{\mu\nu}). \tag{3}$$

It is likely that the MCG invariant partition functions on M^d completely classify anomaly-free CFTs. Thus, it is also likely that the modular covariant partition functions (2) completely classify anomalous CFTs [i.e., the boundaries of $(d+1)$D topological order described by $R^{\text{top}}(g)$].

We would like to point out that Eq. (2) also covers the cases of gapped boundaries of $(d+1)$D topological order. In this case $Z(g_{\mu\nu}, i) = Z(i)$ becomes $g_{\mu\nu}$ independent. The $d = 2$ case is studied in detail in Refs. [30,31], where $Z(i)$ is denoted as W^{1i} and is called fusion matrix or wave function overlap. Thus Eq. (2) is a unified description for both gapped and gapless boundaries.

As another application, we point out that anomaly-free fermionic theories exactly correspond a subset of the bosonic theories with the noninvertible gravitational anomaly described by the bosonic Z_2 topological order with emergent

[1]The partition functions of a fermionic CFT on a $d + 1$D lattice are MCG covariant in a special way, so that the sum of the partition functions are still invariant under the fermionic MCG. For example, in 1+1D, the fermionic MCG is generated by T^2 and S.

fermion (i.e., the twisted Z_2 gauge theory) in one higher dimension. Thus we can construct anomaly-free fermionic theories, such as their partition functions, by constructing bosonic theories and their partition functions with this particular noninvertible gravitational anomaly.

The general theory developed here is for both purely chiral and nonchiral CFT's on the boundary. A related work [32] has given a comprehensive theory of purely chiral CFT's on the boundary of a 2+1D topological order, based on tensor category theory. When the boundary is purely chiral, our theory is just a subset of the full theory developed in Ref. [32]. Both theories provide a unified approach for gapped and gapless boundaries. Recently, some nonchiral CFT's on the boundary of 2+1D topological order were studied in an independent work [33]. In particular, multicomponent partition functions on a 1+1D gapless boundary of 2+1D double-Ising topological order were calculated. A connection between the modular transformation of boundary partition functions and the S and T matrices that characterize the modular tensor category for the 2+1D bulk topological order was noticed. Our paper generalizes those results and provides a more systematic discussion.

We also like to remark that, in the presence of symmetry, there are also several partition functions from the different symmetry twisting boundary conditions in d-dimensional space-time [34–36]. Those partition functions with different boundary conditions also transform covariantly under MCG transformations. If the anomaly is not invertible, there will be several partition functions for each twisted boundary condition. This generalization and its MCG transformations will be discussed in Ref. [37], for $d = 2$ case.

II. TOPOLOGICAL INVARIANT AND PROPERTIES OF BOUNDARY PARTITION FUNCTION

First, let us describe the topological path integral that can realize various topological orders. The boundaries of those topological orders realize invertible and noninvertible anomalous theories. This way, we can relate anomalies with topological invariants in one higher dimensions.

A. Topological partition function as topological invariant

A very general way to characterize a topologically ordered phase is via its partition function $Z(M^D)$ on closed space-time M^D with all possible topologies. A detailed discussion on how to define the partition function via tensor network is given in Ref. [8] and in Appendix C. From this careful definition, we see that the partition function also depends on the branched triangulation of the space-time (see Appendix C), as well as the tensor associated with each simplex. We collectively denote the triangulation, the branching structure, and the tensors as $\mathcal{T}$. Thus the partition function should be more precisely denoted as $Z_{\text{TN}}(M^D, \mathcal{T})$. In a very fine triangulation limit (i.e., the thermodynamic limit), we believe that the partition function depends on $\mathcal{T}$ via an effective metric tensor $g_{\mu\nu}$ of the space-time manifold, if the tensor network describes a "liquid" state, as opposed to a foliated state (a nonliquid state) [38–42]. Thus the partition function can be denoted as $Z_{\text{field}}(M^D, g_{\mu\nu})$ in the thermodynamic limit. $Z_{\text{field}}(M^D, g_{\mu\nu})$ correspond to the partition function of a field theory where the different lattice regularizations $\mathcal{T}$ are not important as long as they produce the same equivalent metric $g_{\mu\nu}$. Here, $g_{\mu\nu}$ and $g'_{\mu\nu}$ are regarded as equivalent if they differ by a diffeomorphim since $Z_{\text{field}}(M^D, g_{\mu\nu}) = Z_{\text{field}}(M^D, g'_{\mu\nu})$. Let $\mathcal{M}_{M^D}$ be the space formed by all metrics $g_{\mu\nu}$ of M^D (up to diffeomorphic equivalence), which is called the moduli space of M^D. Thus the partition function $Z_{\text{field}}(M^D, g_{\mu\nu})$ is a complex function on the moduli space $\mathcal{M}_{M^D} \xrightarrow{Z_{\text{field}}(M^D,-)} \mathbb{C}$.

However, $Z_{\text{TN}}(M^D, \mathcal{T})$ [or $Z_{\text{TN}}(M^D, g_{\mu\nu})$] is not a topological invariant since it contains a so-called volume term $e^{-\int_{M^D} \epsilon\, d^Dx}$ where ϵ is the energy density. The volume term can be factored out. And we can obtain a topological partition function $Z^{\text{top}}_{\text{TN}}(M^D)$ which is believed to be a topological invariant [8,22]:

$$Z_{\text{TN}}(M^D, \mathcal{T}) = e^{-\int_{M^D} \epsilon\, d^Dx} Z^{\text{top}}_{\text{TN}}(M^D, \mathcal{T}). \tag{4}$$

Appendix C 4 describes the way to fine-tune the tensors to make the volume term vanishes (i.e., $\epsilon = 0$). In this case, the path integral directly produces the topological partition function. Such a topological invariant may completely characterize the topological order.

Let us describe topological invariant, the topological partition function of the field theory, $Z^{\text{top}}_{\text{field}}(M^D, g_{\mu\nu})$ [i.e., $Z^{\text{top}}_{\text{TN}}(M^D, \mathcal{T})$] in more details. The "topological property" of $Z^{\text{top}}_{\text{field}}(M^D, g_{\mu\nu})$ may appear in two ways [8]. (1) $Z^{\text{top}}_{\text{field}}(M^D, g_{\mu\nu})$ is a local constant function on $\mathcal{M}_{M^D}$. In this case, the topological partition function only depends on $\pi_0(\mathcal{M}_{M^D})$: $\pi_0(\mathcal{M}_{M^D}) \xrightarrow{Z_{\text{field}}(M^D,-)} \mathbb{C}$. Such a complex function on $\pi_0(\mathcal{M}_{M^D})$ is a topological invariant, since $Z^{\text{top}}_{\text{field}}(M^D, g_{\mu\nu})$ does not depend on any smooth change of $g_{\mu\nu}$. In this case, the boundary has a global gravitational anomaly.

(2) The reduction from lattice partition function $Z^{\text{top}}_{\text{TN}}(M^D, \mathcal{T})$ to the field theory partition function $Z^{\text{top}}_{\text{field}}(M^D, g_{\mu\nu})$ may have a phase ambiguity. However, we can define the change of phase for $Z^{\text{top}}_{\text{field}}(M^D, g_{\mu\nu})$ as we go along a segment I in $\mathcal{M}_{M^D}$ without ambiguity:

$$\text{phase change} = e^{i\,2\pi \oint_I \alpha} = e^{i\,2\pi \oint_{M^D \times I} \Omega}, \tag{5}$$

where α is a 1-form on $\mathcal{M}_{M^D}$ and Ω is closed $D+1$ form constructed from the curvature tensor on $M^D \times I$. In this case, the boundary has a perturbative gravitational anomaly. For example, when $D = 3$, $\Omega = \frac{\Delta c}{24} p_1$, where p_1 is the first Pontryagin class on 4-manifold. $Z^{\text{top}}_{\text{field}}(M^3, g_{\mu\nu})$ is given by

$$Z^{\text{top}}_{\text{field}}(M^3, g_{\mu\nu}) = e^{i\,\frac{2\pi \Delta c}{24} \oint_{M^D} \omega_3}, \tag{6}$$

where the 3-form ω_3 satisfies $d\omega_3 = p_1$ and corresponds to the gravitational Chern-Simons term. The coefficient Δc is the chiral central charge of the boundary state. In this case, $Z^{\text{top}}_{\text{field}}(M^3, g_{\mu\nu})$ depends on the smooth change of $g_{\mu\nu}$ and is not a topological invariant in the usual sense.

B. Invertible and noninvertible topological orders

Most topological orders are not invertible under the stacking operation. (Here, by definition, an invertible order [8,17] can be canceled by another order, i.e., the stacking of the two orders gives rise to the trivial order.) The invertible topological orders form a subset of topological orders. The topological invariant for invertible topological orders, $Z^{\text{top}}_{\text{TN}}(M^D, \mathcal{T})$, is a pure phase factor: $|Z^{\text{top}}_{\text{TN}}(M^D, \mathcal{T})| = 1$ [8,43–45].

The boundaries of invertible topological orders can have (standard) perturbative gravitational anomalies, which are the most studied in the literature. The boundaries of noninvertible topological order can have different gravitational anomalies. We will call the latter as noninvertible gravitational anomalies, and call the standard gravitational anomalies as invertible gravitational anomalies. In this paper, we will concentrate on the noninvertible anomalies.

To give an example of invertible anomalies, let us consider a E_8 bosonic quantum hall state described by the following K matrix [46,47]:

$$K_{E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \tag{7}$$

which has an invertible topological order, since $\det(K_{E_8}) = 1$. Its boundary is described by the $(E_8)_1$ CFT that has a perturbative gravitational anomaly, due to its nonzero chiral central charge $c = 8$. It is a chiral CFT whose partition function has a single character,

$$Z(\tau) = \chi^{E_8}(\tau) = \frac{\Theta_{K_{E_8}}(q)}{\eta^8(q)}, \quad q \equiv e^{2i\pi\tau} \tag{8}$$

where $\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n)$ is the Dedekind eta function, and Θ_K is the theta function for a lattice characterized by an integer symmetric matrix K:

$$\Theta_K(q) = \sum_{\boldsymbol{n}\in\mathbb{Z}^{\dim K}} q^{\boldsymbol{n}^\top K \boldsymbol{n}/2}, \tag{9}$$

and K_{E_8} is the E_8 root lattice, given by Eq. (7). The first a few terms in the expansion is

$$\chi^{E_8} = q^{-1/3}(1 + 248q + 4124q^2 + O(q^3)), \tag{10}$$

where the 248 generators of E_8 are counted in the second term in this single sector. χ^{E_8} transforms according to the one-dimensional representation of the modular group

$$\chi^{E_8}(-1/\tau) = \chi^{E_8}(\tau), \quad \chi^{E_8}(\tau + 1) = e^{-i\frac{2\pi}{3}}\chi^{E_8}(\tau). \tag{11}$$

The 1+1D perturbative gravitational anomaly characterized by the chiral central charge Δc constrains the boundary partition function in 1+1D:

$$\lim_{q\to 0} Z(q) = \text{integer} \times q^{-\frac{c}{24}}\bar{q}^{-\frac{\bar{c}}{24}}, \quad \Delta c = c - \bar{c}. \tag{12}$$

Thus, knowing the 1+1D boundary partition function, we can also determine its perturbative gravitational anomaly Δc. In this paper, we will try to go one step further. We will determine the global gravitational anomaly of 1+1-dimensional theories from its partition function.

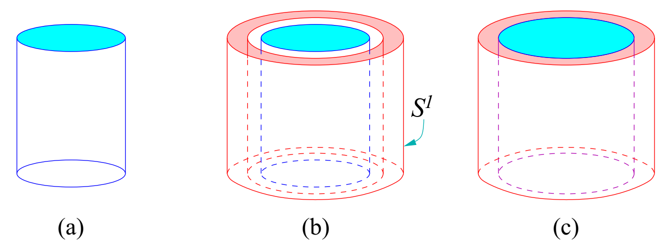

FIG. 2. (a) Space-time $D^2 \times S^1$ (solid cylinder). (b) $I \times S^1 \times S^1$ (cylinder) and $D^2 \times S^1$ (solid cylinder). (c) Gluing the cylinder with solid cylinder, along the $S^1 \times S^1 = T^2$ boundary, reproduces the space-time $D^2 \times S^1$. The tensor networks on the solid cylinder and the cylinder define the path integral. The tensors on the inner solid cylinder are the bulk tensors that describe a topological path integral. The tensors on the outer cylinder can be anything, which may describe a gapless CFT at long distance. Different choices of boundary tensor network on the outer cylinder give rise to different types of boundaries.

C. Properties of boundary partition function

To concentrate on global anomaly, we will assume that there is no perturbative anomaly. In this case, the global anomaly is characterized by the bulk topological invariant $Z^{\text{top}}_{\text{field}}(M^D, \mathcal{T})$, which can be realized by the topological path integral described in Appendix C 4 [8]. In this paper, we assume the bulk theory is always described by the topological path integral, whose partition function directly corresponds to the topological invariant $Z^{\text{top}}_{\text{field}}(M^D, \mathcal{T})$.

To link such a topological invariant (i.e., topological path integral), $Z^{\text{top}}_{\text{TN}}(M^D, \mathcal{T})$ to the partition function on the boundary B^d, $d = D - 1$, we note that the boundary partition function is given by [Fig. 2(a)] [8,22]

$$Z(B^d; M^D, \mathcal{T}) = Z^{\text{top}}_{\text{TN}}(M^D, \mathcal{T}), \quad B^d = \partial M^D. \tag{13}$$

The boundary is the so-called natural boundary described in Appendix C 3, but here we sum over the boundary degrees of freedom. We note that the bulk is gapped. Thus the low-energy properties of the boundary (below the bulk gap) are described by the above $Z(B^d, \mathcal{T}_B)$.

We may obtain a more general boundary by stacking a d-dimensional system described by a d-dimensional tensor network, $Z_{\text{TN}}(B^d, \mathcal{T}_B)$, to the boundary [see Fig. 2(b)]. The resulting boundary partition function has a form

$$Z(B^d, \mathcal{T}_B; M^D, \mathcal{T}) = Z_{\text{TN}}(B^d, \mathcal{T}_B) Z^{\text{top}}_{\text{TN}}(M^D, \mathcal{T}) \tag{14}$$

We may also allow the boundary and bulk degrees of freedom to interact with each other by gluing the boundary to the bulk as in Fig. 2(c). We see that the boundary partition function $Z(B^d, \mathcal{T}_B; M^D, \mathcal{T})$ is not purely given by a tensor network on the boundary B^d, which gives rise to a partition function $Z_{\text{TN}}(B^d, \mathcal{T}_B)$. It may contain a bulk topological term $Z^{\text{top}}_{\text{TN}}(M^D, \mathcal{T})$. The nonvanishing bulk topological term implies the boundary quantum system defined by $Z(B^d, \mathcal{T}_B; M^D, \mathcal{T})$ to be potentially anomalous. If the boundary partition function is

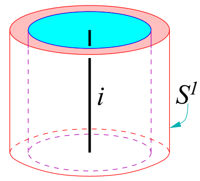

FIG. 3. The space-time $D^2 \times S^1$ with a world line of type-i topological excitation, wrapping in the S^1 direction. The path integral on the inner solid cylinder is a topological path integral with world line, as described in Appendix C 5.

given purely by a tensor network $Z_{\text{TN}}(B^d, \mathcal{T}_B)$ on the boundary [i.e., when $Z^{\text{top}}_{\text{TN}}(M^D, \mathcal{T}) = 1$], such a quantum system will be anomaly-free.

D. 1+1D anomalous theory on space-time torus T^2

In this section, we will concentrate on 1+1D anomalous theories. To define its partition function on a space-time torus T^2, we consider a 2+1D tensor network path integral (see Appendix C) on a simple extension of T^2 – $D^2 \times S^1$ [see Fig. 2(c)]

$$Z(T^2; D^2 \times S^1) = Z^{\text{top}}_{\text{TN}}(D^2 \times S^1), \quad \partial D^2 = S^1. \tag{15}$$

The tensors on the inner solid cylinder define a topological path integral described in Appendix C that realizes a topological order corresponding to the anomaly under consideration. The tensors on the outer cylinder [see Fig. 2(b)] can have more than one choice, representing different kinds of boundaries.

We can define a more general partition function for 1+1D anomalous theory by extending T^2 to more complicated M^3, $T^2 = \partial M^3$ Ref. [8]. Furthermore, in M^3, we can insert a world line of a topological excitation, and the partition function is generalized to (see Fig. 3)

$$Z\big(T^2; D_i^2 \times S^1\big) = Z^{\text{top}}_{\text{TN}}\big(D_i^2 \times S^1\big). \tag{16}$$

Note that the surface of the inner solid cylinder in Fig. 3 (after integrating out only the bulk degrees of freedom as in Appendix C 3) corresponds to a wave function, $|\psi_i\rangle$, that describes one of the degenerate ground states of the bulk topological order on the torus. If the path integral on the inner solid cylinder is a topological path integral, $|\psi_i\rangle$ automatically normalizes to 1: $\langle\psi_i|\psi_i\rangle = 1$ (as discussed in Ref. [31]). Thus, more precisely, the 1+1D partition function for an anomalous theory is given by

$$Z(T^2, |\psi_i\rangle) = Z^{\text{top}}_{\text{TN}}\big(D_i^2 \times S^1\big). \tag{17}$$

Here the degenerate ground-state wave functions $|\psi_i\rangle$ are labeled by the type-i of the topological excitations. For the trivial excitations labeled by $\mathbf{1}$, $Z(T^2, |\psi_{\mathbf{1}}\rangle)$ correspond to the partition function for the space-time in Fig. 2(c) without any world-line insertion.

The dependence on the ground-state wave function $|\psi_i\rangle$ of the topological order on the torus is the key character of anomalous partition function [8,16]. (1) If $|\psi_i\rangle$ is a product state, then $Z(T^2, |\psi_i\rangle)$ is a partition function of an anomaly-free theory. (2) If $|\psi_i\rangle$ is unique (i.e., the topological order has a nondegenerate ground state on the torus), then $Z(T^2, |\psi_i\rangle)$ is a partition function of a theory with invertible anomaly. (3) If $|\psi_i\rangle$ is not unique (i.e., the topological order has degenerate ground states on torus), then $Z(T^2, |\psi_i\rangle)$ is a partition function of a theory with noninvertible anomaly.

E. Modular transformations of the partition function for an anomalous theory

Let us fine tune the action of the 1+1D anomalous theory, so that it has a vanishing ground-state energy density. In this case, its partition function on T^2 will not depend on the size of the space-time, but only depend on the shape of the space-time. The shape of a torus T^2 can be described by a complex number τ. Thus we may write the 1+1D partition function as

$$Z(\tau, \bar{\tau}, |\psi_i\rangle) = Z^{\text{top}}_{\text{TN}}\big(D_i^2 \times S^1\big). \tag{18}$$

The complex parameters τ and $\tau' = \tau + 1$ describe the same shape of a torus, related by a coordinate transformation. Therefore we expect the partition function of an anomaly-free 1+1D theory to satisfy

$$Z(\tau, \bar{\tau}) = Z(\tau + 1, \bar{\tau} + 1). \tag{19}$$

On the other hand, for an anomalous 1+1D theory, it satisfies

$$Z\big(\tau, \bar{\tau}, T^{\text{top}}_{ij}|\psi_j\rangle\big) = Z(\tau + 1, \bar{\tau} + 1, |\psi_i\rangle), \tag{20}$$

since the coordinate transformation acts nontrivially on the bulk ground-state wave function $|\psi_i\rangle$ on torus. Here the unitary matrix T^{top}_{ij} describes such a nontrivial action, which is a modular transformation of the torus ground states of the 2+1D bulk topological order [11,28]. Similarly, τ and $\tau' = -1/\tau$ also describe the same shape after a coordinate transformation. Thus

$$Z\big(\tau, \bar{\tau}, S^{\text{top}}_{ij}|\psi_j\rangle\big) = Z(-1/\tau, -1/\bar{\tau}, |\psi_i\rangle), \tag{21}$$

where the unitary matrix S^{top}_{ij} describes another modular transformation of the ground states on a torus of the bulk topological order.

The partition function $Z(\tau, \bar{\tau}, |\psi_i\rangle)$ depends on $|\psi_i\rangle$ in a linear fashion

$$Z(\tau, \bar{\tau}, M_{ij}|\psi_j\rangle) = M_{ij} Z(\tau, \bar{\tau}, |\psi_j\rangle). \tag{22}$$

To see this, note that the path integral that sums over the degrees of freedom in the bulk and the outer surface of outer cylinder [see Fig. 2(b)] gives rise to a wave function $\langle\phi|$ that lives on the inner surface of the outer cylinder. The partition function $Z(\tau, \bar{\tau}, |\psi_i\rangle)$ is simply

$$\langle\phi|\psi_i\rangle = Z(\tau, \bar{\tau}, |\psi_i\rangle). \tag{23}$$

Thus $Z(\tau, \bar{\tau}, |\psi_i\rangle)$ is a linear function of $|\psi_i\rangle$. As a result, Eqs. (20) and (21) can be rewritten as

$$\begin{aligned} T^{\text{top}}_{ij} Z(\tau, \bar{\tau}; j) &= Z(\tau + 1, \bar{\tau} + 1; i), \\ S^{\text{top}}_{ij} Z(\tau, \bar{\tau}; j) &= Z(-1/\tau, -1/\bar{\tau}; i), \end{aligned} \tag{24}$$

where $Z(\tau, \bar{\tau}, i) \equiv Z(\tau, \bar{\tau}, |\psi_i\rangle)$. Equation (24) is a key result of this paper. It describes the modular transformation properties of the partition functions for an anomalous theory, i.e., a boundary theory of a 2+1D topological order characterized by T^{top} and S^{top}.

For a gapped anomalous theory, the partition functions do not depend on τ. Equation (24) becomes

$$Z(i) = T^{\text{top}}_{ij} Z(j), \quad Z(i) = S^{\text{top}}_{ij} Z(j). \tag{25}$$

We recover a condition for gapped boundary of a topological order obtained in Refs. [30,31], where $Z(i)$ was denoted as $W^{\mathbf{1}i}$. Note that for the gapped case, the partition functions $Z(i)$ are ground-state degeneracy of the system with a world line of the type-i topological excitation inserted in the bulk and are nonnegative integers.

The above is a general discussion of 1+1D anomalous theory, which can have a noninvertible anomaly. In particular, the boundary CFT may have separate right-moving part and left-moving part, and each part (i.e., the corresponding set of characters) transforms according to certain $S_{R,L}$ and $T_{R,L}$ matrices. Those boundary $S_{R,L}$ and $T_{R,L}$ matrices may be *different* from the S^{top} and T^{top} matrices for the bulk topological order. However, after we combine the right movers and left movers to construct multicomponent partition functions $Z(\tau, \bar{\tau}; i)$, we find that $Z(\tau, \bar{\tau}; i)$ transform according to the bulk S^{top} and T^{top} matrices. In the following, we will discuss some simple examples of 1+1D noninvertible anomaly.

We like to remark that equation of form Eq. (24) has appeared before, but in different physical context. For example, the characters (the conformal blocks) of a purely chiral CFT transform according to Eq. (24), which relates the purely chiral CFT to a modular tensor category since both as characterized by the S and T matrices. If we include the defect lines in a CFT, we may also obtain multiple partition functions labeled by the defect lines [48]. Those partition functions also transform as Eq. (24). The defect lines associated with symmetry twist are always invertible. Reference [48] also described noninvertible defect lines which is beyond the symmetry twist. Many multicomponent partition function discussed in this paper are directly related to the partition function with those noninvertible defect lines.

The new prospect offered in this paper is the connection between the boundary CFT and the bulk topological order. In other words, in this paper, we view Eq. (24) as a constraint on the boundary theory of a 2+1D topological order, characterized by S^{top} and T^{top}. This helps us to classify different possible boundaries of a given 2+1D topological order. The character point of view, the defect line point of view, and the topological order point of view for Eq. (24), although different, have some close relations. In this paper, we give the CFT characters and the partition functions induced by defect lines another physical interpretation, by viewing them as noninvertible anomaly (i.e., as boundary of 2+1D topological order).

III. A NONINVERTIBLE BOSONIC GLOBAL GRAVITATIONAL ANOMALY FROM 2+1D Z_2 TOPOLOGICAL ORDER

A 2+1D Z_2 topological order has four type of excitations, $\mathbf{1}$, e, m, f, where e, m are bosons and f is a fermion. e, m, f are topological excitations with π mutual statistics respect to each other. (Remember that a topological excitation is defined as the excitation that cannot be created by any local operators). Such a topological order can have many different boundaries, which all carry the same noninvertible gravitational anomaly. In this section, we will discuss some of those boundary theories [8].

A. Two gapped boundaries of the 2+1D Z_2 topological order

A gapped boundary of the 2+1D Z_2 topological order is induced by m particle condensation. This boundary has only one type of topological excitations e. The topological excitation e has a Z_2 fusion $e \otimes e = 1$, and is described by a symmetric fusion category $\mathcal{R}\text{ep}(Z_2)$ (which is the fusion category formed by the representations of Z_2 group). Such a boundary described by $\mathcal{R}\text{ep}(Z_2)$ has a nondegenerate ground state. Its partition function is given by $Z(\tau, \bar{\tau}, \mathbf{1}) = 1$ (where $\mathbf{1}$ means that there is no insertion of world line, i.e., $i = \mathbf{1}$ in Fig. 3).

The insertion of a world line of m-type topological excitations (see Fig. 3) produces another boundary, where e on the boundary S^1 acquires a π phase as it goes around the boundary. The partition function for such a boundary is still given by $Z(\tau, \bar{\tau}, m) = 1$.

If we insert a world line of e-type or a f-type, the resulting boundary will carry an un-paired e excitations. Such an un-paired e costs a finite energy ϵ_e. These boundaries will have partition functions $Z(\tau, \bar{\tau}, e) = Z(\tau, \bar{\tau}, f) = \#\mathrm{e}^{-\epsilon_e \beta}|_{\beta\to\infty} = 0$, when the size of space-time β approaches to infinity.

So the first gapped boundary of Z_2 topological order is described by four partition functions in the basis of topological excitations $(\mathbf{1}, e, m, f)$

$$\begin{aligned} Z(\tau, \bar{\tau}, \mathbf{1}) &= Z(\tau, \bar{\tau}, m) = 1, \\ Z(\tau, \bar{\tau}, e) &= Z(\tau, \bar{\tau}, f) = 0. \end{aligned} \tag{26}$$

They can be viewed as the partition function for an anomalous $c = 0$ CFT (i.e., a gapped theory). One can check that these four partition functions in the excitations basis satisfy Eq. (25) [30,31], since for Z_2 topological order, S^{top} and T^{top} are given by

$$\begin{aligned} T^{\text{top}}_{Z_2} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\ S^{\text{top}}_{Z_2} &= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \end{aligned} \tag{27}$$

Let us obtain another gapped boundary of the 2+1D Z_2 topological order, by lowering the energy of e to a negative value. This will drive a "Z_2 symmetry" breaking transition and obtain an e-condensed state, which have a twofold ground-state degeneracy on a ring. (If we condensed e particle on an open segment on the boundary, we will also get a twofold ground-state degeneracy.) This new boundary is described by the following four partition functions

$$\begin{aligned} Z(\tau, \bar{\tau}, \mathbf{1}) &= Z(\tau, \bar{\tau}, e) = 2, \\ Z(\tau, \bar{\tau}, m) &= Z(\tau, \bar{\tau}, f) = 0. \end{aligned} \tag{28}$$

They again satisfy Eq. (24).

Here $Z(\tau,\overline{\tau},\mathbf{1})=2$ means the Z_2 topological order on D^2 [i.e., the boundary state on S^1, see Fig. 2(c)] has a twofold degeneracy. This twofold degeneracy comes from the emergent mod-2 conservation of e particles on the boundary, and subsequently the spontaneous breaking of this emergent Z_2 symmetry. However, since the Z_2 symmetry is emergent, when the boundary S^1 has a finite density n_e of the e particles, the emergent mod-2 conservation may be explicitly broken by an amount $\mathrm{e}^{-1/n_e\xi}$ where ξ is a length scale. In this case, the twofold degeneracy is lifted by an amount $\mathrm{e}^{-1/n_e\xi}$. So the boundary described by Eq. (28) is unstable. After the lifting of the degeneracy, the boundary is actually described by

$$
\begin{aligned}
Z(\tau,\overline{\tau},\mathbf{1}) &= Z(\tau,\overline{\tau},e)=1,\\
Z(\tau,\overline{\tau},m) &= Z(\tau,\overline{\tau},f)=0,
\end{aligned}
\tag{29}
$$

which correspond to the boundary of the 2+1D Z_2 topological order induced by e condensation [while the boundary induced by m condensation is described by Eq. (26)].

B. A gapless boundary of the 2+1D Z_2 topological order

A gapless boundary of the 2+1D Z_2 topological order is given by a 1+1D gapless system described by a Majorana fermion field

$$
H=\int dx\,(\lambda_R i\,\partial_x\lambda_R-\lambda_L i\,\partial_x\lambda_L). \tag{30}
$$

We like to stress that such a 1+1D gapless system is actually a bosonic system where *the states in the many-body Hilbert are all bosonic (i.e., contain an even number of Majorana fermions)*. We refer such a 1+1D gapless system as the boson-restricted Majorana fermion theory. It is different from the usual Majorana fermion theory.

We can give the Majorana fermion a mass gap to obtain a gapped boundary:

$$
H=\int dx\,(\lambda_R\,i\,\partial_x\lambda_R-\lambda_L\,i\,\partial_x\lambda_L+i\,m\lambda_R\lambda_L). \tag{31}
$$

This gapped boundary corresponds to the gapped boundary described above. If we lower m to a negative value, we should drive the “Z_2 symmetry” breaking transition described above and obtain a twofold ground-state degeneracy on a ring. This is different from the standard Majorana fermion theory where the negative m also gives rise to nondegenerate ground state. So for our boson-restricted Majorana fermion theory, a positive m gives rise to nondegenerate ground state while a negative m gives rise to a twofold ground state degeneracy on a ring. If we only change the sign of m on an open segment, then both the standard Majorana fermion theory and our bosonic Majorana fermion theory will give rise to a twofold ground-state degeneracy.

So when $m=0$ the gapless bosonic Majorana fermion theory describes the critical point of the Z_2 symmetry breaking phase transition mentioned above. The gapless boson-restricted Majorana fermion theory describes a conformal field theory (CFT) with a noninvertible gravitational anomaly. In this paper, we like to understand this anomalous CFT in detail. In particular, we would like to compute its partition function and their properties under modular transformations.

To understand the critical CFT for the “Z_2 symmetry” breaking transition, let us introduce a 1d lattice Hamiltonian *on a ring* to describe the gapped boundary in Sec. III A

$$
H=-U\sum_i\sigma_i^z-J\sum_i\sigma_i^x\sigma_{i+1}^x,\quad U,J>0, \tag{32}
$$

where σ^l, $l=x,y,z$ are Pauli matrices. Here an up-spin $\sigma_i^z=1$ correspond to an empty site and an down-spin $\sigma_i^z=-1$ correspond to a site occupied with an e particle. Since number of the e particles is always even, thus *the Hilbert space $\mathcal{V}$ of our model is formed by states with even numbers of down spins* $\sigma_i^z=-1$. Note that our Hilbert space is *nonlocal*, i.e., it does not have a tensor product decomposition:

$$
\mathcal{V}\neq\otimes_i\mathcal{V}_i, \tag{33}
$$

where $\mathcal{V}_i$ is the two dimensional Hilbert space for site i. It is this property that make our model to have a noninvertible gravitational anomaly.

We like to mention that, we can view the 2+1D Z_2 topological order as a gauged Z_2 symmetric state with a trivial SPT order. The boundary of the 2+1D Z_2 symmetric state can be described by a transverse Ising model (32) with the standard Hilbert space (i.e., without the $\prod_i\sigma_i^z=1$ constraint). The boundary can be in a symmetric phase [described by Eq. (32) with $U\gg J$] or a Z_2 symmetry breaking phase [described by Eq. (32) with $U\ll J$]. We see that after gauging the Z_2 symmetry to obtain the Z_2 topological order in the bulk, the only change in the boundary theory is the addition of the constraint $\prod_i\sigma_i^z=1$ [49] that changes the many-body Hilbert space to make it nonlocal (i.e., make the boundary theory to have a noninvertible gravitational anomaly).

In our model (32) for the boundary, the J term is allowed since the e particles have only a mod-2 conservation. In the $U\gg J$ limit, the above lattice model describes the gapped phase in Sec. III A. As we change U to $U\ll J$, we will drive a “Z_2 symmetry” breaking phase transition. The critical point at $U=J$ is described by a CFT with noninvertible gravitational anomaly. Such a CFT is described by the boson-restricted Majorana fermion theory mentioned in Sec. III B. The Majorana fermion theory is obtained from Eq. (32) via the Jordan-Wigner transformation.

To obtain the partition function of the anomalous CFT, let us first consider the partition function of the transverse Ising model (32) at critical point $U=J$. There are four partition functions Z_{a_x,a_t} for the transverse Ising model, with different $\mathbb{Z}_2$ boundary conditions $a_x=\pm1$ and $a_t=\pm1$. The partition functions are given by the characters $\chi_{\mathbf{1}}(\tau)$, $\chi_\psi(\tau)$, $\chi_\sigma(\tau)$, of two Ising CFTs (see Appendix A 3), one for right movers and the other for the left movers. We find

$$
\begin{aligned}
Z_{1,1} &= |\chi_{\mathbf{1}}|^2+|\chi_\psi|^2+|\chi_\sigma|^2,\\
Z_{1,-1} &= |\chi_{\mathbf{1}}|^2+|\chi_\psi|^2-|\chi_\sigma|^2,\\
Z_{-1,1} &= \chi_{\mathbf{1}}\overline{\chi}_\psi+\chi_\psi\overline{\chi}_{\mathbf{1}}+|\chi_\mu|^2,\\
Z_{-1,-1} &= -\chi_{\mathbf{1}}\overline{\chi}_\psi-\chi_\psi\overline{\chi}_{\mathbf{1}}+|\chi_\mu|^2.
\end{aligned}
\tag{34}
$$

This means that the partition functions for the even and odd Z_2 sectors are given by

$$Z_{\text{even}} = \frac{Z_{1,1}+Z_{1,-1}}{2} = |\chi_{\mathbf{1}}|^2 + |\chi_\psi|^2,$$
$$Z_{\text{odd}} = \frac{Z_{1,1}-Z_{1,-1}}{2} = |\chi_\sigma|^2. \tag{35}$$

For the anomalous CFT on the boundary of 2+1D Z_2 topological order, its partition function is given by the partition function of the Ising model for the even Z_2 sector

$$Z(\tau,\bar\tau,\mathbf{1}) = |\chi_{\mathbf{1}}|^2 + |\chi_\psi|^2. \tag{36}$$

If we insert the e world line in the bulk (see Fig. 2), the corresponding partition function $Z(\tau,\bar\tau,e)$ is given by $Z_{\text{odd}}(\tau,\bar\tau)$:

$$Z(\tau,\bar\tau,e) = |\chi_\sigma|^2. \tag{37}$$

Similarly, we find

$$Z(\tau,\bar\tau,m) = |\chi_\mu|^2 \tag{38}$$

and

$$Z(\tau,\bar\tau,f) = \chi_{\mathbf{1}}\bar\chi_\psi + \chi_\psi\bar\chi_{\mathbf{1}}. \tag{39}$$

We find that the above partition functions $Z(\tau,\bar\tau,i)$, $i=\mathbf{1},e,m,f$, indeed satisfy Eq. (24). Those partition functions describe a 1+1D gapless theory with a noninvertible gravitational anomaly, which can appear as a boundary of the 2+1D Z_2 topological order.

IV. A NONINVERTIBLE BOSONIC GLOBAL GRAVITATIONAL ANOMALY FROM 2+1D DS TOPOLOGICAL ORDER

Now let us consider the boundary of the 2+1D DS topological order. Since the DS topological order can be viewed as a gauged 2+1D Z_2 symmetric state with the nontrivial Z_2 SPT order. Let us first review a boundary theory of the 2+1D Z_2 SPT state on a 1d ring with even number of sites, with the following Hamiltonian [50]:

$$H = -U\sum_i \sigma_i^z\sigma_{i+1}^z - J\sum_i\left(\sigma_i^x + \sigma_{i-1}^z\sigma_i^x\sigma_{i+1}^z\right),\quad U,J>0. \tag{40}$$

The above Hamiltonian has a non-on-site Z_2 symmetry generated by

$$U = \prod_i \sigma_i^x \prod_i CZ_{i,i+1}, \tag{41}$$

where CZ_{ij} acts on two spins as

$$CZ_{ij} = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\downarrow\rangle\langle\downarrow\downarrow|.$$
$$= \frac{1+\sigma_i^z+\sigma_j^z-\sigma_i^z\sigma_j^z}{2} \tag{42}$$

From Appendix B, we see that the above Hamiltonian in Eq. (40) is Z_2 symmetric. However, the Z_2 symmetry has a 't Hooft anomaly.

To have a theory that is defined on rings with both even and odd sites, we should consider different (but equivalent) non-on-site Z_2 symmetry:

$$U = \prod_i \sigma_i^x \prod_i s_{i,i+1}, \tag{43}$$

where s_{ij} acts on two spins as

$$s_{ij} = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|.$$
$$= \tfrac{1}{2}\left(1-\sigma_i^z+\sigma_j^z+\sigma_i^z\sigma_j^z\right). \tag{44}$$

The Z_2 transformation has a simple picture: it flips all the spins and include a $(-)^{N_{\uparrow\to\downarrow}}$ phase, where $N_{\uparrow\to\downarrow}$ is the number of $\uparrow\to\downarrow$ domain wall. From Appendix B, we see that the following Hamiltonian is invariant under the new Z_2 transformation:

$$H = -U\sum_i \sigma_i^z\sigma_{i+1}^z - J\sum_i\left(\sigma_i^x - \sigma_{i-1}^z\sigma_i^x\sigma_{i+1}^z\right),\quad U,J>0. \tag{45}$$

The boundary of 2+1D Z_2 SPT state described by Eq. (45) has a symmetry breaking phase when $U \gg J$. The boundary can also be gapless described by a $c=\bar c=1$ CFT when $U=0$. Equation (45) has no symmetric gapped phase, since the Z_2 symmetry is not on-site (i.e., has a 't Hooft anomaly) [13].

When $U=0$, the model (45) can be mapped to the XY model on 1d lattice [50]:

$$H_{\text{XY}} = -J\sum_i\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y\right). \tag{46}$$

In this case, the anomalous 1+1D theory is gapless and is described by a $u1_4\times\overline{u1}_4$ CFT (see Appendix A 1). It has a partition function

$$Z_{\text{XY}}(q,\bar q) \equiv Z_{Z_2\text{-SPT}}(q,\bar q) = \sum_{i=0}^{3}\left|\chi_i^{u1_4}(q)\right|^2 \tag{47}$$

with primary fields of dimension $(h_i,\bar h_i) = (\frac{i^2}{8},\frac{i^2}{8})$. The above partition function can be rewritten as

$$Z_{Z_2\text{-SPT}}(q,\bar q) = \frac{1}{|\eta(q)|^2}\sum_{(a,b)\in\Gamma} q^{\frac12 a^2}\bar q^{\frac12 b^2}, \tag{48}$$

where (a,b) form a lattice Γ (see Fig. 4):

$$(a,b) = \tfrac12(l+2m, l-2m),\quad l,m\in\mathbb{Z}. \tag{49}$$

So all the U(1) vertex operators in our XY model can be labeled by (l,m) which have scaling dimension

$$(h,\bar h) = \left(\frac12 a^2, \frac12 b^2\right) = \left(\frac{(l+2m)^2}{8}, \frac{(l-2m)^2}{8}\right). \tag{50}$$

This is the labeling scheme used in Ref. [50]. It was found that the U(1) vertex operator labeled by (l,m) carry the Z_2 charge $l+m$ mod 2.

We see that each character $|\chi_i^{u1_4}|^2$ contains U(1) vertex operators with different Z_2 charges. Thus it is more convenient to rewrite the partition function in terms of the $u1_{16}$ characters

$$Z_{Z_2\text{-SPT}}(q) = \sum_{i=0}^{7}\left|\chi_{2i}^{u1_{16}}\right|^2 + \sum_{i=0}^{7}\bar\chi_{2i+4}^{u1_{16}}\chi_{2i}^{u1_{16}}. \tag{51}$$

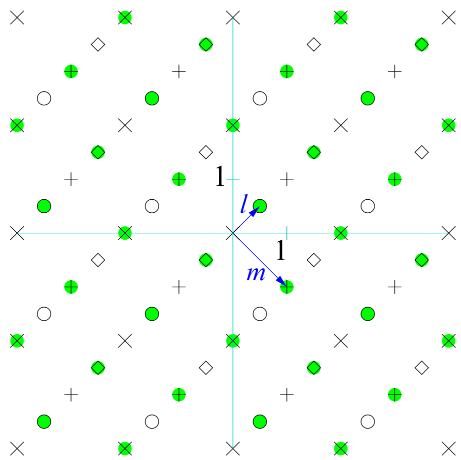

FIG. 4. The lattice Γ formed by points (a, b). Each point corresponds to a U(1) vertex operator with scaling dimension $(h, \bar{h}) = (\frac{1}{2}a^2, \frac{1}{2}b^2)$. The "$\times$" points give rise to $|\chi_0^{u1_4}|^2$. The "$\circ$" points give rise to $|\chi_1^{u1_4}|^2$. The "$+$" points give rise to $|\chi_2^{u1_4}|^2$. The "$\diamond$" points give rise to $|\chi_3^{u1_4}|^2$. We also mark the directions of the l and m labels. The shaded points carry the Z_2 charge $l + m = 1$ mod 2, and the unshaded points carry the Z_2 charge $l + m = 0$ mod 2.

The U(1) vertex operators in $|\chi_{2i}^{u1_{16}}|^2$ carry the Z_2 charge i mod 2. The U(1) vertex operators in $\bar{\chi}_{2i+4}^{u1_{16}} \chi_{2i}^{u1_{16}}$ carry the Z_2-charge $i + 1$ mod 2.

In the presence of the Z_2 symmetry, we can define 4-partition functions for different Z_2-symmetry twists in the space and time directions $(a_x, a_t) = (\pm 1, \pm 1)$. $Z_{Z_2\text{-SPT}}$ is the partition function with no symmetry twist $(a_x, a_t) = (1, 1)$:

$$Z_{1,1} = \sum_{i=0}^{7} \left|\chi_{2i}^{u1_{16}}\right|^2 + \sum_{i=0}^{7} \chi_{2i+8}^{u1_{16}} \bar{\chi}_{2i}^{u1_{16}}. \tag{52}$$

with a Z_2-symmetry twist in time direction the terms with Z_2 charge 1 acquiring a $-$ sign:

$$Z_{1,-1} = \sum_{i=0}^{7} (-)^i \left|\chi_{2i}^{u1_{16}}\right|^2 - \sum_{i=0}^{7} (-)^i \chi_{2i+8}^{u1_{16}} \bar{\chi}_{2i}^{u1_{16}}. \tag{53}$$

After an S transformation of $u1_{16}$ (see Appendix A 1), we get

$$Z_{-1,1} = \sum_{i=0}^{7} \chi_{2i+1}^{u1_{16}} \bar{\chi}_{2i+5}^{u1_{16}} + \sum_{i=0}^{7} \chi_{2i+1}^{u1_{16}} \bar{\chi}_{2i+13}^{u1_{16}}. \tag{54}$$

From $Z_{-1,1}$ we find

$$Z_{-1,-1} = \sum_{i=0}^{7} (-)^i \chi_{2i+1}^{u1_{16}} \bar{\chi}_{2i+5}^{u1_{16}} - \sum_{i=0}^{7} (-)^i \chi_{2i+1}^{u1_{16}} \bar{\chi}_{2i+13}^{u1_{16}}. \tag{55}$$

by adding a $-$ sign to the terms with Z_2 charge 1.

Now we gauge the Z_2 on-site symmetry in the 2+1D SPT state to obtain the 2+1D DS topological order. The 2+1D DS topological order has a gapped boundary which contains topological excitation s that satisfies a Z_2 fusion role $s \otimes s = 1$. The 1d particles with Z_2 fusion rule are described by one of the two fusion categories. The first one is $\mathcal{R}\text{ep}(Z_2)$ mentioned in the last section. The second one is a different fusion category, which we refer as the semion fusion category [25,26]. Such a gapped boundary can be described by Eq. (45) in $U \gg J$ limit (i.e., in the Z_2 symmetry breaking phase), where the Z_2 domain walls correspond to the boundary particle s. The fusion of those domain walls is described by the semion fusion category, provided that the fusion processes preserve the nonon-site Z_2 symmetry (43),

However, there is one problem with the above picture: in the Z_2 symmetry breaking phase, all the domain wall configurations have twofold degeneracy induced by the Z_2 transformation (43). To resolve this, we need to modify the many-body Hilbert space on a ring by imposing the constraint

$$\prod_i \sigma_i^x \prod_i s_{i,i+1} = 1, \tag{56}$$

i.e., we include only even Z_2-charge states in our many-body Hilbert space. The model Eq. (45), together with the Z_2-even Hilbert space, describes the boundary of the 2+1D DS topological order. Such a 1+1D theory has a noninvertible gravitational anomaly described by 2+1D DS topological order.

Now we see that using the partition functions Z_{a_x,a_t} of the model (45) with different Z_2-symmetry twists, we can construct the four partition functions for the gapless boundary of 2+1D DS topological order. For example, the partition function of the model (45) in the even Z_2 charge sector, $\frac{Z_{1,1}+Z_{1,-1}}{2}$, corresponds to the partition function for the boundary of the DS topological order without inserting any anyon world line, $Z(\tau, \bar{\tau}, \mathbf{1})$. Note that the DS topological order has four types of excitations: trivial excitation $\mathbf{1}$, semion s, conjugate semion s^*, and topological boson b. Thus the boundary has four partition functions $Z(\tau, \bar{\tau}, \mathbf{1})$, $Z(\tau, \bar{\tau}, s)$, $Z(\tau, \bar{\tau}, s^*)$, and $Z(\tau, \bar{\tau}, b)$, which are given by

$$\begin{aligned} Z(\mathbf{1}) &= \frac{Z_{1,1} + Z_{1,-1}}{2}, \quad Z(s) = \frac{Z_{-1,1} + Z_{-1,-1}}{2}, \\ Z(b) &= \frac{Z_{1,1} - Z_{1,-1}}{2}, \quad Z(s^*) = \frac{Z_{-1,1} - Z_{-1,-1}}{2}. \end{aligned} \tag{57}$$

The 2+1D DS topological order is characterized by (in the basis of $\mathbf{1}, s, s^*, b$)

$$\begin{aligned} T_{\text{DS}}^{\text{top}} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ S_{\text{DS}}^{\text{top}} &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \end{aligned} \tag{58}$$

Using the above $S_{\text{DS}}^{\text{top}}$ and $T_{\text{DS}}^{\text{top}}$ and the modular transformations of $u1_{16}$ in Appendix A 1, we can explicitly check that the four boundary partition functions (57) satisfy the modular covariance Eq. (24).

V. NONINVERTIBLE GRAVITATIONAL ANOMALY AND "NONLOCALITY" OF HILBERT SPACE

In Ref. [7], it was stressed that the 't Hooft anomaly of a global symmetry in a theory is not an obstruction for the theory to have an ultraviolate (UV) completion (i.e., to have a lattice realization). Such an anomalous theory can still be realized on a lattice, however, the global symmetry has to be realized as an *nonon-site* symmetry in the lattice model.

In this section, we would like to propose that a noninvertible gravitational anomaly in a theory is not an obstruction for the theory to have a UV completion. The anomalous theory can still be realized on a lattice, if there is no perturbative gravitational anomaly. However, the Hilbert space of UV theory $\mathcal{V}$ is not given by the lattice Hilbert space $\mathcal{V}_{\text{latt}}$: $\mathcal{V} \neq \mathcal{V}_{\text{latt}}$.

The lattice Hilbert space $\mathcal{V}_{\text{latt}}$ has a tensor product decomposition

$$\mathcal{V}_{\text{latt}} = \otimes_i \mathcal{V}_i, \tag{59}$$

where $\mathcal{V}_i$ is the Hilbert space for each lattice site. We call the Hilbert space with such a tensor product decomposition as a *local* Hilbert space. A system with such a local Hilbert space is free of gravitational anomaly, by definition.

In contrast to an anomaly-free theory, the UV completion of a theory with noninvertible gravitational anomaly does not have a local Hilbert space (i.e., with the above tensor product decomposition). In other words, a noninvertible gravitational anomaly is not an obstruction to have a UV completion, but for the UV completion to have a local Hilbert space. This understanding of noninvertible gravitational anomaly is supported by the example discussed in the last section.

In last section, we pointed out the boundary of 2+1D Z_2 topological order (which has a noninvertible gravitational anomaly) has a UV completion described by a lattice model (32), with a constraint on the Hilbert space $\prod_i \sigma_i^z = 1$. It is the constraint $\prod_i \sigma_i^z = 1$ that makes the Hilbert space nonlocal.

Let us describe the above result using a more physical reasoning. One boundary of 2+1D Z_2 topological order has a single type of topological excitations e, which is mod 2 conserved. The Hilbert space always has an even number of e particles. On the other hand, when there is no e-particle excitations, the boundary ground state is not degenerate. Here we like to point out that *the even-particle constraint (i.e., Z_2 fusion) plus the nondegeneracy of the ground state is a sign of 1+1D noninvertible gravitational anomaly*.

The example in the last section supports such a claim. The even-particle constraint is imposed by $\prod_i \sigma_i^z = 1$. The nondegenerate ground state is given by $\otimes_i|\sigma^z = 1\rangle$. Such a theory describes a boundary of 2+1D Z_2 topological order and has a noninvertible anomaly. The Ising model may also be in the symmetry breaking phase. Due to the constraint $\otimes_i|\sigma^z = 1\rangle$, the symmetry breaking phase also has a unique ground state $\otimes_i|\sigma^x = 1\rangle + \otimes_i|\sigma^x = -1\rangle$. In such a symmetry breaking phase, there are always an even number of domain walls, that correspond to an even number of topological excitations.

On the other hand, an even-particle constraint plus twofold degenerate ground states will lead to an anomaly-free theory. We can consider an Ising model in symmetry breaking phase and without the constraint on Hilbert space. Such a phase has twofold degenerate ground states and the number of domain walls (which correspond to the e particles) is always even. Thus even-particle constraint plus twofold degenerate ground states can be realized by a lattice model with local Hilbert space and is thus anomaly-free.

There is also a mathematical way to understand the above claim. The e particles with mod 2 conservation in 1+1D can be described by a fusion category with a Z_2 fusion ring. There are only two different fusion categories with a Z_2 fusion ring, both have 1+1D noninvertible anomaly. One fusion category describes the boundary of 2+1D Z_2 topological order, and the other describes the boundary of DS topological order. Both noninvertible anomalies can be described by the following Ising model (in a gapped phase)

$$H = -\sum_i \sigma_i^z \sigma_{i+1}^z \tag{60}$$

but with different constraints on Hilbert space. The anomaly corresponds to the Z_2 topological order has a constraint $\prod_i \sigma_i^x = 1$, and the anomaly corresponds to the DS topological order has a constraint $\prod_i \sigma_i^x \prod_i s_{i,i+1} = 1$ [see Eq. (43)].

VI. SYSTEMATICAL SEARCH OF GAPPED AND GAPLESS BOUNDARIES OF A 2+1D TOPOLOGICAL ORDER

A. Boundaries of 2+1D topological order

In this section, we want to systematically find gapped and gapless boundary theories of a 2+1D topological order by solving Eq. (24) from the data S^{top} and T^{top} of the bulk topological order. This is a generalization of finding possible 1+1D critical theories via finding modular invariant partition functions. Note that, regardless of whether the boundary is gapped or gapless, it always has the same anomaly characterized by the bulk topological order.

To solve Eq. (24), we may start with a CFT with partition functions $Z_{\text{bdy}}(\tau, \overline{\tau}, I)$, which transform as

$$\begin{aligned} T_{IJ} Z_{\text{bdy}}(\tau, \overline{\tau}, J) &= Z_{\text{bdy}}(\tau + 1, \overline{\tau} + 1, I), \\ S_{IJ} Z_{\text{bdy}}(\tau, \overline{\tau}, J) &= Z_{\text{bdy}}(-1/\tau, -1/\overline{\tau}, I). \end{aligned} \tag{61}$$

We then construct $Z(\tau, \overline{\tau}, i)$ as a linear summation of $Z(\tau, \bar{\tau}, I)$'s,

$$Z(\tau, \overline{\tau}, i) = M_{iI} Z_{\text{bdy}}(\tau, \overline{\tau}, I). \tag{62}$$

Now Eq. (24) becomes

$$\begin{aligned} M_{iI} Z_{\text{bdy}}(\tau + 1, \overline{\tau} + 1, I) &= M_{iI} T_{IJ} Z_{\text{bdy}}(\tau, \overline{\tau}, J) \\ &= (T^{\text{top}})_{ij} M_{jJ} Z_{\text{bdy}}(\tau, \overline{\tau}, J). \end{aligned} \tag{63}$$

We see that M_{iI} must satisfy

$$M_{iI} = (T^{\text{top}})_{ij} T^*_{IJ} M_{jJ}, \quad M_{iI} = (S^{\text{top}})_{ij} S^*_{IJ} M_{jJ}. \tag{64}$$

We also note that, for a fixed i, $Z(\tau, \overline{\tau}, i)$ can be zero, indicating the always presence of gapped excitations on the boundary. $Z(\tau, \overline{\tau}, i)$ can also be a τ-independent positive integer. It means that the ground states are gapped and have a degeneracy given by $Z(\tau, \overline{\tau}, i)$. Otherwise, $Z(\tau, \overline{\tau}, i)$ has an expansion

$$\begin{aligned} Z(\tau, \overline{\tau}, i) &= q^{-\frac{c}{24}} \overline{q}^{-\frac{\bar{c}}{24}} \sum_{n,\bar{n}=0}^{\infty} D_{n,\bar{n}}(i) q^{n+h_i} \overline{q}^{\bar{n}+\bar{h}_i}, \\ q &= e^{i\,2\pi\tau}, \quad D_{n,\bar{n}}(i) = \text{nonnegative integer}, \end{aligned} \tag{65}$$

where $(h_i, \overline{h}_i)$ are the scaling dimensions for the type-i topological excitation. Such an expansion describes the many-body spectrum of the gapless boundary of the disk D_i^2, with a type-i topological excitation at the center of the disk. Here the subscript i in D_i^2 indicates the type-i excitation on the disk. Let us assume the boundary $S^1 = \partial D_i^2$ has a length L. Then $D_{n,\bar{n}}(i)$ is number of many-body states on D_i^2 with energy $(n + h_i + \bar{n} + \bar{h}_i)\frac{2\pi}{L}$, and momentum $(n + h_i - \bar{n} - \bar{h}_i)\frac{2\pi}{L}$. Here we have assumed that velocity of the gapless excitations is $v = 1$. Thus $D_{n,\bar{n}}(i)$ are nonnegative integers.

Also $D_{0,0}(i)$ is the ground-state degeneracy on the boundary of the disk D_i^2. Since the boundary can be gapless, the ground-state degeneracy needs to be defined carefully. Here, we view two energy levels with an energy difference of order $2\pi/L$ as nondegenerate. We view two energy levels with an energy difference smaller than $(2\pi/L)^\alpha, \alpha > 1$, as degenerate. It is in this sense we define the ground-state degeneracy $D_{0,0}(i)$ for a gapless system in $L \to \infty$ limit. We believe that the ground state degeneracy on disk D^2 is always 1. Therefore, we like to impose a nondegeneracy condition on the boundary $D_{0,0}(\mathbf{1}) = 1$. $Z_{\text{bdy}}(\tau, \bar{\tau}, I)$ satisfies a similar quantization condition.

From Eq. (62), we see that M_{iI} is the multiplicity of the number of energy levels in the many-body spectrum of the boundary theory. Therefore, for a fixed i, if $M_{iI} \neq 0$, then

$$M_{iI} \text{ are quantized to make } D_{n,\bar{n}}(i) \text{ to be nonnegative integer and } D_{0,\bar{0}}(\mathbf{1}) = 1. \tag{66}$$

In practice, to find M_{iI}, we may compute the eigenvectors of $T^{\text{top}} \otimes T^* + S^{\text{top}} \otimes S^*$ with eigenvalue 2, that satisfy the above quantization condition.

B. Z_2 topological order

To find a CFT that describes a boundary of 2+1D Z_2 topological order, we need to solve Eq. (24) with S^{top} and T^{top} given by Eq. (27) that characterize the 2+1D Z_2 topological order. Let us first try to find gapped boundaries by choosing $Z_{\text{bdy}}(\tau, \bar{\tau}) = 1$, the partition function of a trivial gapped 1+1D state. Now Eq. (64) reduces to

$$Z(i) = \left(T_{Z_2}^{\text{top}}\right)_{ij} Z(j), \quad Z(i) = \left(S_{Z_2}^{\text{top}}\right)_{ij} Z(j). \tag{67}$$

So we need to find common eigenvectors of $S_{Z_2}^{\text{top}}$ and $T_{Z_2}^{\text{top}}$, both with eigenvalue 1. We also require the solutions to satisfy the quantization condition (66), i.e., the components of the solutions are all nonnegative integers. The condition $D_{0,0}(\mathbf{1}) = 1$ becomes $Z(\mathbf{1}) = 1$. This agrees with the fact that the ground state of 2+1D Z_2 topological order on a disk D^2 is nondegenerate if there is no accidental degeneracy. This can be achieved by finding eigenvectors of $S_{Z_2}^{\text{top}} + T_{Z_2}^{\text{top}}$ that satisfy Eq. (66).

We find that $S_{Z_2}^{\text{top}} + T_{Z_2}^{\text{top}}$ has two eigenvectors with eigenvalue 2, given by

$$(Z_{m\text{-cond}}(i)) = (1, 0, 1, 0), \quad (Z_{e\text{-cond}}(i)) = (1, 1, 0, 0), \tag{68}$$

where $i = (\mathbf{1}, e, m, f)$. They are the only two nonnegative integral eigenvectors with $Z(\mathbf{1}) = 1$. Thus the 2+1D Z_2 topological order has only two types of gapped boundaries, an e condensed boundary described by $Z_{e\text{-cond}}(i)$ and an m condensed boundary described by $Z_{m\text{-cond}}(i)$ [30].

If we choose $Z_{\text{bdy}}(\tau, \bar{\tau}, I)$ to be the partition functions (the characters) of Is $\otimes$ $\overline{\text{Is}}$ CFT (see Appendix A 3), then S and T will be 9×9 matrices:

$$S_{\text{Is}\otimes\overline{\text{Is}}} = S_{\text{Is}}^* \otimes S_{\text{Is}}, \quad T_{\text{Is}\otimes\overline{\text{Is}}} = T_{\text{Is}}^* \otimes T_{\text{Is}}, \tag{69}$$

where $S_{\text{Is}}, T_{\text{Is}}$ are given in Eq. (A11). We find eigenvalue 2 for $T_{Z_2}^{\text{top}} \otimes T_{\text{Is}\otimes\overline{\text{Is}}}^* + S_{Z_2}^{\text{top}} \otimes S_{\text{Is}\otimes\overline{\text{Is}}}^*$ to be threefold degenerate. We obtain the following three solutions of Eq. (24)

$$\begin{pmatrix} Z(\tau, \bar{\tau}, \mathbf{1}) \\ Z(\tau, \bar{\tau}, e) \\ Z(\tau, \bar{\tau}, m) \\ Z(\tau, \bar{\tau}, f) \end{pmatrix} = \begin{pmatrix} |\chi_{\mathbf{1}}(\tau)|^2 + |\chi_\psi(\tau)|^2 + |\chi_\sigma(\tau)|^2 \\ |\chi_{\mathbf{1}}(\tau)|^2 + |\chi_\psi(\tau)|^2 + |\chi_\sigma(\tau)|^2 \\ 0 \\ 0 \end{pmatrix}, \tag{70}$$

$$\begin{pmatrix} Z(\tau, \bar{\tau}, \mathbf{1}) \\ Z(\tau, \bar{\tau}, e) \\ Z(\tau, \bar{\tau}, m) \\ Z(\tau, \bar{\tau}, f) \end{pmatrix} = \begin{pmatrix} |\chi_{\mathbf{1}}(\tau)|^2 + |\chi_\psi(\tau)|^2 + |\chi_\sigma(\tau)|^2 \\ 0 \\ |\chi_{\mathbf{1}}(\tau)|^2 + |\chi_\psi(\tau)|^2 + |\chi_\sigma(\tau)|^2 \\ 0 \end{pmatrix}, \tag{71}$$

$$\begin{pmatrix} Z(\tau, \bar{\tau}, \mathbf{1}) \\ Z(\tau, \bar{\tau}, e) \\ Z(\tau, \bar{\tau}, m) \\ Z(\tau, \bar{\tau}, f) \end{pmatrix} = \begin{pmatrix} |\chi_{\mathbf{1}}(\tau)|^2 + |\chi_\psi(\tau)|^2 \\ |\chi_\sigma(\tau)|^2 \\ |\chi_\mu(\tau)|^2 \\ \chi_{\mathbf{1}}(\tau)\bar{\chi}_\psi(\tau) + \chi_\psi(\tau)\bar{\chi}_{\mathbf{1}}(\tau) \end{pmatrix}, \tag{72}$$

that satisfy the quantization condition (66).

The first two solutions correspond to the two gapped boundaries of the 2+1D Z_2 topological order induced by e and m condensation respectively, and then stacking with a transverse Ising model at critical point. So the first two solutions are regarded as gapped boundaries. Here we would like to introduce the notion of reducible boundary. If the partition functions $Z(\tau, \bar{\tau}, i)$ of a boundary can be factorized into a form

$$Z(\tau, \bar{\tau}, i) = Z_{\text{inv}}(\tau, \bar{\tau}) Z'(\tau, \bar{\tau}, i), \tag{73}$$

then we say the boundary is reducible. We will call the boundary described by $Z'(\tau, \bar{\tau}, i)$ as the reduced boundary. Here, $Z(\tau, \bar{\tau}, i)$ and $Z'(\tau, \bar{\tau}, i)$ are partition functions satisfying (24) and Eq. (65), and $Z_{\text{inv}}(\tau, \bar{\tau})$ is a modular invariant partition function satisfying Eq. (65). Noticing that $|\chi_{\mathbf{1}}(\tau)|^2 + |\chi_\psi(\tau)|^2 + |\chi_\sigma(\tau)|^2$ is modular invariant, so the first two boundaries are reducible and their reduced boundary are gapped boundaries described by Eq. (68).

The third solution (72) corresponds to an irreducible gapless boundary. Let us consider the stability of such $c = \bar{c} = \frac{1}{2}$ gapless boundary. To begin with, we review the stability of the critical point of transverse Ising model described by

$$Z_{\text{Is}}(\tau, \bar{\tau}) = |\chi_{\mathbf{1}}(\tau)|^2 + |\chi_\psi(\tau)|^2 + |\chi_\sigma(\tau)|^2. \tag{74}$$

From the above partition function, we see that there are two relevant operators: $\bar{\psi}\psi$ with scaling dimension $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$, and σ with scaling dimension $(h, \bar{h}) = (\frac{1}{16}, \frac{1}{16})$. Among the two, σ is odd under the Z_2 symmetry of the transverse Ising model.

In analogy, to examine the stability of the gapless boundary (72), we examine the partition function $Z(\tau, \bar{\tau}, \mathbf{1})$. We do not consider other partition functions, since the partition function $Z(\tau, \bar{\tau}, \mathbf{1})$ describes the physical boundary of Fig. 2 without the insertion of the world line. From $Z(\tau, \bar{\tau}, \mathbf{1})$, we see that gapless boundary (72) has only one relevant operator $\bar{\psi}\psi$ with scaling dimension $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$. So the gapless boundary (72) can be the phase transition point between two gapped boundaries. In fact, according to the discussion in Sec. III B, the third gapless boundary is the critical transition point between the gapped e condensed boundary and the m condensed boundary.

We can also use the characters χ_h^{m4} of the (4,5) minimal model (or tricritical Ising model [51], see Appendix A 3) with central charge $c=\bar{c}=\frac{7}{10}$, to construct the boundary partition functions $Z_{\rm bdy}(\tau,\bar\tau,I)$ that have the Z_2 noninvertible anomaly [i.e., satisfy Eq. (24)]. We obtain

$$\begin{pmatrix} Z(\mathbf{1})\\ Z(e)\\ Z(m)\\ Z(f)\end{pmatrix}=\begin{pmatrix} |\chi_0^{m4}|^2+|\chi_{\frac{1}{10}}^{m4}|^2+|\chi_{\frac{3}{5}}^{m4}|^2+|\chi_{\frac{3}{2}}^{m4}|^2\\ |\chi_{\frac{7}{16}}^{m4}|^2+|\chi_{\frac{3}{80}}^{m4}|^2\\ |\chi_{\frac{7}{16}}^{m4}|^2+|\chi_{\frac{3}{80}}^{m4}|^2\\ \chi_0^{m4}\bar\chi_{\frac{3}{2}}^{m4}+\chi_{\frac{1}{10}}^{m4}\bar\chi_{\frac{3}{5}}^{m4}+\chi_{\frac{3}{5}}^{m4}\bar\chi_{\frac{1}{10}}^{m4}+\chi_{\frac{3}{2}}^{m4}\bar\chi_0^{m4}\end{pmatrix}. \tag{75}$$

If we choose $Z_{\rm bdy}(\tau,\bar\tau,I)$ to be built from the characters of $u(1)_M\otimes\overline{u(1)}_M$ CFT, we obtain the following simple solution of gapless boundary

$$\begin{pmatrix} Z(\tau,\bar\tau;\mathbf{1})\\ Z(\tau,\bar\tau;e)\\ Z(\tau,\bar\tau;m)\\ Z(\tau,\bar\tau;f)\end{pmatrix}=\begin{pmatrix} |\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\\ |\chi_1^{u1_4}|^2+|\chi_3^{u1_4}|^2\\ \chi_1^{u1_4}\bar\chi_3^{u1_4}+\chi_3^{u1_4}\bar\chi_1^{u1_4}\\ \chi_0^{u1_4}\bar\chi_2^{u1_4}+\chi_2^{u1_4}\bar\chi_0^{u1_4}\end{pmatrix}. \tag{76}$$

Note that $Z(\tau,\bar\tau,e)$ and $Z(\tau,\bar\tau,m)$ are no longer identical, but differ by a charge conjugation, whose action induce on the characters is $C:\chi_i^{u1_M}\bar\chi_j^{u1_M}\to\chi_i^{u1_M}\bar\chi_{M-j}^{u1_M}$.

C. Double-semion topological order

To find gapped boundaries of 2+1D DS topological order, we need to solve

$$Z(i)=\left(T_{\rm DS}^{\rm top}\right)_{ij}Z(j),\quad Z(i)=\left(S_{\rm DS}^{\rm top}\right)_{ij}Z(j), \tag{77}$$

where $T_{\rm DS}^{\rm top}$ and $S_{\rm DS}^{\rm top}$ are given by Eq. (58). We find that $S_{\rm DS}^{\rm top}+T_{\rm DS}^{\rm top}$ has only one eigenvector with eigenvalue 2, given by

$$(Z_b(i))=(1,0,0,1), \tag{78}$$

where $i=(\mathbf{1},s,s^*,b)$. Thus the 2+1D DS topological order has only one type of gapped boundary, a b condensed boundary [30].

Next, we consider possible gapless boundaries of DS topological order described by $\text{Is}\otimes\overline{\text{Is}}$ CFT, by solving Eq. (64) for solutions satisfying Eq. (66). We find only one eigenvector for $T_{\rm DS}^{\rm top}\otimes T^*_{\text{Is}\otimes\overline{\text{Is}}}+S_{\rm DS}^{\rm top}\otimes S^*_{\text{Is}\otimes\overline{\text{Is}}}$ with eigenvalue 2. We obtain the following unique solution of Eq. (24):

$$\begin{pmatrix} Z(\tau,\bar\tau,\mathbf{1})\\ Z(\tau,\bar\tau,s)\\ Z(\tau,\bar\tau,s^*)\\ Z(\tau,\bar\tau,b)\end{pmatrix}=\begin{pmatrix} |\chi_{\mathbf{1}}(\tau)|^2+|\chi_\psi(\tau)|^2+|\chi_\sigma(\tau)|^2\\ 0\\ 0\\ |\chi_{\mathbf{1}}(\tau)|^2+|\chi_\psi(\tau)|^2+|\chi_\sigma(\tau)|^2\end{pmatrix}. \tag{79}$$

Such a solution corresponds to the gapped boundary of 2+1D DS topological order, and then stacking with a transverse Ising model at critical point. So this solution is regarded as a gapped boundary. There is no irreducible gapless boundary described by $\text{Is}\otimes\overline{\text{Is}}$.

Actually, we can obtain an even stronger result 2+1D DS topological order has no irreducible gapless boundary with central charge $c=\bar c\leqslant\frac{25}{28}$. This result is obtained by realizing that the DS anomalous partition functions, for irreducible gapless boundary, has a nonzero component $Z(\tau,\bar\tau,s)$. Otherwise, the gapless boundary can be viewed as a gapped boundary stacked with an anomaly-free 1+1D CFT. The condition (24) for the $T^{\rm top}$ transformation requires that the excitations in the partition function has topological spin $h-\bar h=\frac14$ mod 1. This constrains the central charge of the anomalous CFT. If the unitary CFT has a central charge $c=\bar c<1$, then the boundary CFT must be given by a chiral-antichiral minimal model $C^{\rm ft}_{p,p+1}\times\overline{C}^{\rm ft}_{p,p+1}$. The topological spin for the operators in such CFT is given by $s_{r,s,r',s'}=h_{r,s}-h_{r',s'}$ [see Eq. (A10)]. We find that, for $p<7$, $s_{r,s,r',s'}$ cannot be $\frac14$ mod 1. Thus the condition Eq. (24) cannot be satisfied for $T^{\rm top}$ transformation.

Last, we consider possible gapless boundary theories of DS topological order described by $u1_M\otimes\overline{u1}_M$ CFT, by solving Eq. (64) for solutions satisfying Eq. (66). This includes many cases, one for each choice of M. So we need to consider each case separately.

For $M=16$, we have found an irreducible gapless boundary described by $u1_{16}\otimes\overline{u1}_{16}$ CFT:

$$\begin{pmatrix} Z(\mathbf{1})\\ Z(s)\\ Z(s^*)\\ Z(b)\end{pmatrix}=\begin{pmatrix} \sum_{i=0}^3|\chi_{4i}^{u1_{16}}|^2+\sum_{i=0}^3\chi_{4i+10}^{u1_{16}}\bar\chi_{4i+2}^{u1_{16}}\\ \sum_{i=0}^3\chi_{4i+1}^{u1_{16}}\bar\chi_{4i+5}^{u1_{16}}+\sum_{i=0}^3\chi_{4i+3}^{u1_{16}}\bar\chi_{4i+15}^{u1_{16}}\\ \sum_{i=0}^3\chi_{4i+3}^{u1_{16}}\bar\chi_{4i+7}^{u1_{16}}+\sum_{i=0}^3\chi_{4i+1}^{u1_{16}}\bar\chi_{4i+13}^{u1_{16}}\\ \sum_{i=0}^3|\chi_{4i+2}^{u1_{16}}|^2+\sum_{i=0}^3\chi_{4i+8}^{u1_{16}}\bar\chi_{4i}^{u1_{16}}\end{pmatrix}. \tag{80}$$

From the partition function $Z(\tau,\bar\tau,\mathbf{1})$, we see that there is an relevant operator with scaling dimension $(h,\bar h)=(\frac{4^2}{2\times16},\frac{4^2}{2\times16})=(\frac12,\frac12)$. So the gapless boundary (80) is unstable. It describes the transition point between two gapped phases in Eq. (45). One gapped phase for $U>0$ and other gapped phase for $U<0$. The gapless critical point is described by $U=0$.

For $M=4$, we find that there is no irreducible gapless boundary described by $u1_4\otimes\overline{u1}_4$ CFT.

For $M=2$, we find that there is only one irreducible gapless boundary described by $u1_2\otimes\overline{u1}_2$ CFT:

$$\begin{pmatrix} Z(\tau,\bar\tau,\mathbf{1})\\ Z(\tau,\bar\tau,s)\\ Z(\tau,\bar\tau,s^*)\\ Z(\tau,\bar\tau,b)\end{pmatrix}=\begin{pmatrix} |\chi_0^{u1_2}|^2\\ \chi_1^{u1_2}\bar\chi_0^{u1_2}\\ \chi_0^{u1_2}\bar\chi_1^{u1_2}\\ |\chi_1^{u1_2}|^2\end{pmatrix}. \tag{81}$$

There is no other irreducible gapless boundary described by $u1_2\otimes\overline{u1}_2$ CFT. But there is a reducible gapless boundary described by

$$\begin{pmatrix} Z(\tau,\bar\tau,\mathbf{1})\\ Z(\tau,\bar\tau,s)\\ Z(\tau,\bar\tau,s^*)\\ Z(\tau,\bar\tau,b)\end{pmatrix}=\begin{pmatrix} |\chi_0^{u1_2}|^2+|\chi_1^{u1_2}|^2\\ 0\\ 0\\ |\chi_0^{u1_2}|^2+|\chi_1^{u1_2}|^2\end{pmatrix}, \tag{82}$$

which is a stacking of a gapped boundary described by Eq. (78) and the CFT for spin-1/2 Heisenberg chain.

From the partition function $Z(\tau,\bar{\tau},\mathbf{1})$, we find that the irreducible boundary (81) has no relevant operator. It has only several marginal operators, such as $\bar{J}J$, $\mathrm{e}^{\pm i\sqrt{2}\phi}\mathrm{e}^{\pm i\sqrt{2}\bar{\phi}}$ with scaling dimension $(h,\bar{h})=(1,1)$. Here, J is the U(1) current operator and $\mathrm{e}^{\pm i\sqrt{2}\phi}$ are U(1)-charged operators. Those operators can be marginally relevant. If there is only one marginally relevant operator $g\hat{O}$ in the Hamiltonian, the renomalization group (RG) flow of the coupling constant g is given by

$$\frac{dg}{d\beta}=\alpha g^2. \tag{83}$$

We see that regardless the sign of α, there is a finite region of g where g flows to zero. In this case, the CFT can be stable. When there are many marginally relevant operators $g_i\hat{O}_i$, RG flow of the coupling constants g_i is given by [52]

$$\frac{dg_i}{d\beta}=\alpha_{ijk}g_jg_k. \tag{84}$$

In Appendix D, we discuss the above RG equation in more details and show that generic coupling constants g_i always flow to infinite. Thus the CFT is unstable, and the 2+1D DS topological order always has a gapped boundary without fine tuning.

We would like to remark that from this gapless boundary of DS topological order, and apply the relations (57), we can find another gapless boundary theory of $\mathbb{Z}_2$ SPT, whose partition function is given by

$$Z_{Z_2\text{-SPT}}(q,\bar{q})=\sum_{i=0}^{1}\left|\chi_i^{u1_2}(q)\right|^2, \tag{85}$$

which is different from Eq. (47). The partition function (85) can also be rewritten as

$$Z_{Z_2\text{-SPT}}(q,\bar{q})=|\eta(q)|^{-1}\sum_{i=0}^{1}q^{\frac{1}{2}a^2}\bar{q}^{\frac{1}{2}b^2}, \tag{86}$$

where (a,b) form a lattice Γ^{u1_2},

$$(a,b)=\frac{1}{\sqrt{2}}(l+m,l-m),\quad l,m\in\mathbb{Z}. \tag{87}$$

The $\mathbb{Z}_2$ charges of the vertex operators in $|\chi_i^{u1_2}|^2$ is $i \mod 2$, or $(l+m) \mod 2$ on the lattice.

From the Z_2-even partition function $|\chi_0^{u1_2}(q)|^2$, we find that the gapless boundary (85) has no Z_2-even relevant operator. However, as mentioned above, there may be many marginally relevant operators, and it is yet to be verified if the gapless boundary (85) of 2+1D Z_2-SPT order is perturbatively stable or not. In some previous studies, a gapless boundary (47) for the same 2+1D Z_2-SPT order is found to be perturbatively unstable against Z_2 symmetric perturbations [50,53], via relevant perturbations (with total scaling dimension less than 2). In this paper, we found a gapless boundary of Z_2-SPT state (85), which is more stable against Z_2 symmetric perturbations, in the sense that the instability only come from potentially marginally relevant operators (with total scaling dimension equal to 2).

D. Semion topological order

There is a close relative of 2+1D DS topological order—2+1D semion topological order, which has only two types of excitations: trivial excitation $\mathbf{1}$ and semion s. The 2+1D semion topological order can be realized by $\nu=1/2$ bosonic Laughlin state.

Let us describe the data that characterize the 2+1D semion topological order. The topological spins and the quantum dimensions of $\mathbf{1}$ and s are $(s_{\mathbf{1}},s_s)=(0,\frac{1}{4})$ and $(d_{\mathbf{1}},d_s)=(1,1)$. The topological $S^{\text{top}}_{\text{sem}}$ and $T^{\text{top}}_{\text{sem}}$ matrices are

$$T^{\text{top}}_{\text{sem}}=\mathrm{e}^{-i\frac{2\pi}{24}}\begin{pmatrix}1&0\\0&\mathrm{e}^{i\frac{2\pi}{4}}\end{pmatrix},$$
$$S^{\text{top}}_{\text{sem}}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}. \tag{88}$$

To obtain the possible boundaries of 2+1D semion topological order, we just need to solve Eq. (24). We find a simple boundary described by the following partition function [in terms of $u1_2$ characters (A1)]

$$\begin{pmatrix}Z(\tau,\bar{\tau},\mathbf{1})\\Z(\tau,\bar{\tau},s)\end{pmatrix}=\begin{pmatrix}\chi_0^{u1_2}(\tau)\\\chi_1^{u1_2}(\tau)\end{pmatrix}. \tag{89}$$

The 1+1D theory described by the above partition functions has both perturbative and global gravitational anomaly.

E. Fibonacci topological order

Another simple 2+1D topological order is the Fibonacci topological order. It is characterized by the following topological data. The central charge is $\frac{14}{5} \mod 8$. There are two types of excitations $\mathbf{1}$ and γ. Their topological spins and the quantum dimensions are $(s_{\mathbf{1}},s_\gamma)=(0,\frac{2}{5})$ and $(d_{\mathbf{1}},d_s)=(1,\phi)$, where $\phi=\frac{\sqrt{5}+1}{2}$, the golden ratio. The topological $S^{\text{top}}_{\text{Fib}}$ and $T^{\text{top}}_{\text{Fib}}$ matrices are

$$T^{\text{top}}_{\text{Fib}}=\mathrm{e}^{-i\frac{2\pi}{24}\frac{14}{5}}\begin{pmatrix}1&0\\0&\mathrm{e}^{i2\pi\frac{2}{5}}\end{pmatrix},$$
$$S^{\text{top}}_{\text{Fib}}=\frac{1}{\sqrt{\phi+2}}\begin{pmatrix}1&\phi\\\phi&-1\end{pmatrix}. \tag{90}$$

Solving Eq. (24), we can find several gapless boundary of the Fibonacci topological order: (1) $(G_2)_1$ CFT with central charge $(c,\bar{c})=(\frac{14}{5},0)$, with the partition functions

$$\begin{pmatrix}Z(\tau,\mathbf{1})\\Z(\tau,\gamma)\end{pmatrix}=\begin{pmatrix}\chi_0^{G2_1}(\tau)\\\chi_1^{G2_1}(\tau)\end{pmatrix}$$
$$=\mathrm{e}^{-i\frac{2\pi}{24}\frac{14}{5}}\begin{pmatrix}1+14q+42q^2+O(q^3)\\q^{\frac{2}{5}}(7+34q+119q^2+O(q^3))\end{pmatrix}, \tag{91}$$

where $\chi_i^{G2_1}(\tau)$ are the characters of level-1 G_2 current algebra, see Appendix A 4. The first multiplicity equaling 7 appearing in $Z(\tau,\gamma)$ implies that when there is a Fibonacci anyon in the bulk, the boundary has sevenfold degeneracy. The degeneracy cannot be split unless the anyon is moved to the boundary.

(2) $su(2)_3\times u(1)_M$ CFT has a central charge $c=\frac{9}{5}+1=\frac{14}{5}$ and $\bar{c}=0$. The $su2_3$ CFT has four chiral characters $\chi_j^{su2_3}$, labeled by the spin $j=0,\frac{1}{2},1,\frac{3}{2}$ (see Appendix A 2) with S

and T matrices

$$T_{su2_3} = \mathrm{e}^{-i\frac{2\pi}{24}\frac{9}{5}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{i\,2\pi\frac{3}{20}} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{i\,2\pi\frac{2}{5}} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{i\,2\pi\frac{3}{4}} \end{pmatrix},$$

$$S_{su2_3} = \frac{1}{\sqrt{2(\phi+2)}} \begin{pmatrix} 1 & \phi & \phi & 1 \\ \phi & 1 & -1 & -\phi \\ \phi & -1 & -1 & \phi \\ 1 & -\phi & \phi & -1 \end{pmatrix}. \tag{92}$$

When $M=2$, we find a solution of Eq. (24):

$$\begin{pmatrix} Z(\tau,\mathbf{1}) \\ Z(\tau,\gamma) \end{pmatrix} = \begin{pmatrix} \chi_0^{u1_2}\chi_0^{su2_3} + \chi_1^{u1_2}\chi_{\frac{3}{2}}^{su2_3} \\ \chi_1^{u1_2}\chi_{\frac{1}{2}}^{su2_3} + \chi_0^{u1_2}\chi_1^{su2_3} \end{pmatrix}. \tag{93}$$

In fact, we find the expansion of the $Z(\tau,i)$ in Eq. (93) in terms of modular parameter $q=\mathrm{e}^{i\,2\pi\tau}$ to be the same as that of Eq. (91).

(3) The same result also arises in $su(2)_{28}$ with $c=\frac{14}{5}$,

$$\begin{aligned} Z(\tau,\mathbf{1}) &= \chi_0^{su2_{28}} + \chi_5^{su2_{28}} + \chi_9^{su2_{28}} + \chi_{14}^{su2_{28}}, \\ Z(\tau,\gamma) &= \chi_3^{su2_{28}} + \chi_6^{su2_{28}} + \chi_8^{su2_{28}} + \chi_{11}^{su2_{28}}, \end{aligned} \tag{94}$$

and see Appendix A 2 for explicit forms of characters.

(4) $(E_8)_1 \times \overline{(F_4)_1}$ CFT, with central charge $(c,\bar c) = (8, \frac{26}{5})$, and $c-\bar c = \frac{14}{5}$. $(F_4)_1$ CFT has the S and T matrices:

$$T_{(F_4)_1} = \mathrm{e}^{-i\frac{2\pi}{24}\frac{26}{5}} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{i\,2\pi\frac{3}{5}} \end{pmatrix}, \quad S_{(F_4)_1} = S^{\mathrm{top}}_{\mathrm{Fib}}. \tag{95}$$

Therefore $(E_8)_1 \times \overline{(F_4)_1}$ is also a gapless boundary of the Fibonacci topological order.

$$\begin{pmatrix} Z(\tau,\bar\tau,\mathbf{1}) \\ Z(\tau,\bar\tau,\gamma) \end{pmatrix} = \begin{pmatrix} \chi^{(E_8)_1}(\tau)\overline{\chi}_0^{(F_4)_1}(\bar\tau) \\ \chi^{(E_8)_1}(\tau)\overline{\chi}_1^{(F_4)_1}(\bar\tau) \end{pmatrix}. \tag{96}$$

VII. DETECT ANOMALIES FROM 1+1D PARTITION FUNCTIONS

So far, we have discussed how to use anomaly to constrain the structure of 1+1D partition function. In this section, we are going to consider a different problem: given a partition function, how to determine its anomaly? We have mentioned that the 1+1D perturbative gravitational anomaly can be partially detected via $q\to 0$ limit of partition function [see (12)]. So here we will concentrate on global gravitational anomalies.

Let us consider partition functions constructed using the characters of Ising CFT:

$$Z_M(\tau,\bar\tau) = \sum_{i,j=1,\psi,\sigma} \overline{\chi}_i(\bar\tau) M_{ij} \chi_j(\tau). \tag{97}$$

Under modular transformation Z_M transforms as

$$\begin{aligned} Z_M(\tau+1,\bar\tau+1) &= Z_{M_T}(\tau,\bar\tau), \quad & M_T &= T^\dagger_{\mathrm{Is}} M T_{\mathrm{Is}}, \\ Z_M(-1/\tau,-1/\bar\tau) &= Z_{M_S}(\tau,\bar\tau), \quad & M_S &= S^\dagger_{\mathrm{Is}} M S_{\mathrm{Is}}, \end{aligned} \tag{98}$$

where S_{Is} and T_{Is} are given by Eq. (A11).

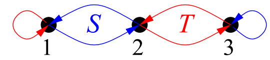

FIG. 5. The modular transformations on the partition functions $Z_{M_n}, n=\mathbf{1},2,3$, for a gapless boundary of a 2+1D Z_2 topological order. For example, the two red lines to the right represent the following T transformations: $M_2 \to M_3 : Z_{M_2}(\tau+1,\bar\tau+1) = Z_{M_3}(\tau,\bar\tau)$ and $M_3 \to M_2 : Z_{M_3}(\tau+1,\bar\tau+1) = Z_{M_2}(\tau,\bar\tau)$. The blue lines represent the S transformations. The pattern of the transformations characterizes an 1+1D noninvertible gravitational anomaly described by 2+1D Z_2 topological order.

Let us consider a particular partition function

$$Z(\tau,\bar\tau,\mathbf{1}) = Z_{M_\mathbf{1}}(\tau,\bar\tau), \quad M_\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{99}$$

which is not modular invariant. Starting from $M_\mathbf{1}$, the modular transformations (98) generate two other partition functions described by

$$M_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{100}$$

The actions of modular transformations on Z_{M_1}, Z_{M_2}, and Z_{M_3} are described by Fig. 5. Such orbits of modular transformations can be used to characterize the anomaly in the partition function. At this point, it is not fully clear if such a characterization is complete or not, i.e., it is not clear if different anomalies always have different orbits. However, the orbits in Fig. 5 are consistent with the 1+1D anomaly described by 2+1D Z_2 topological order. This is because the $S^{\mathrm{top}}_{Z_2}$ and $T^{\mathrm{top}}_{Z_2}$ transformations of the 2+1D Z_2 topological order (27), when acting on

$$|\mathbf{1}\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{101}$$

will generate

$$\begin{aligned} |2\rangle &\equiv \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}(|\mathbf{1}\rangle + |e\rangle + |m\rangle + |f\rangle), \\ |3\rangle &\equiv \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{2}(|\mathbf{1}\rangle + |e\rangle + |m\rangle - |f\rangle). \end{aligned} \tag{102}$$

The actions of $S^{\mathrm{top}}_{Z_2}$ and $T^{\mathrm{top}}_{Z_2}$ on $|\mathbf{1}\rangle$, $|2\rangle$, and $|3\rangle$ will generate the same orbits as in Fig. 5.

We may also consider a partition function constructed using $u1_{16}$ characters:

$$Z(\tau,\bar\tau,\mathbf{1}) = \sum_{i=0}^{3} \left|\chi_{4i}^{u1_{16}}\right|^2 + \sum_{i=0}^{3} \chi_{4i+10}^{u1_{16}} \overline{\chi}_{4i+2}^{u1_{16}}. \tag{103}$$

Starting from $Z(\tau,\bar\tau,\mathbf{1}) = Z_1(\tau,\bar\tau)$, using modular transformations S and T in Eq. (A2), we can generate five

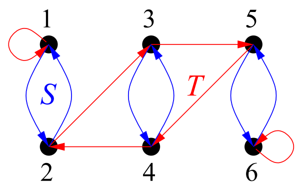

FIG. 6. The modular transformations on the partition functions Z_n, $n = 1, 2, \ldots, 6$, for a gapless boundary of 2+1D DS topological order. For example, the red lines in the middle represent the following T transformations: $Z_4 \to Z_2 : Z_4(\tau + 1, \bar{\tau} + 1) = Z_2(\tau, \bar{\tau})$ and $Z_5 \to Z_4 : Z_5(\tau + 1, \bar{\tau} + 1) = Z_4(\tau, \bar{\tau})$. The blue lines represent the S transformations. The pattern of the transformations characterizes an 1+1D noninvertible gravitational anomaly described by 2+1D DS topological order.

additional partition functions $Z_n(\tau, \bar{\tau})$, $n = 2, 3, 4, 5, 6$. Under the modular transformations S and T, the partition functions $Z_n(\tau, \bar{\tau})$, $n = 1, \ldots, 6$ change into each other. The orbits are described by Fig. 6. Such orbits are consistent with the 1+1D anomaly described by 2+1D DS topological order.

VIII. SUMMARY

In this paper, we study noninvertible gravitational anomalies that correspond to noninvertible topological orders in one higher dimension. A theory with a noninvertible anomaly can have many partition functions, which are linear combinations of N partition functions. For 1+1D noninvertible anomaly, N is the number of types of the topological excitations in the corresponding 2+1D topological order. The anomalous 1+1D partition functions $Z(\tau, \bar{\tau}, i)$, $i = 1, \ldots, N$, are not invariant under the modular transformation, but transform in a nontrivial way described by the modular matrices S^{top}_{ij} and T^{top}_{ij} that characterize the corresponding 2+1D topological order. Similarly, an anomalous theory on an arbitrary close space-time manifold M^d also has many partition functions $Z(M^d, i)$, which transforms according to a representation R_{M_d} of the mapping-class-group G_{M^d} of M^d. The G_{M^d} representation R_{M_d} describes how the ground states of the corresponding $(d+1)$D topological order transform on a spatial manifold M^d. As an application of our theory of noninvertible anomaly, we show that for 2+1D DS topological order, its irreducible gapless boundary must have central charge $c = \bar{c} > \frac{25}{28}$.

At the beginning of the paper, we mentioned that 't Hooft anomaly is an obstruction to gauge a global symmetry. However, if we include theories with noninvertible anomaly, then even global symmetry with 't Hooft anomaly can be gauged, which will result in a theory with a noninvertible anomaly. This is because a theory with 't Hooft anomaly can be realized as a boundary of SPT state in one dimension higher, where the global symmetry is realized as an on-site-symmetry. We can gauge the global on-site-symmetry in bulk and turn the SPT state into a topologically ordered state. The boundary of the resulting topologically ordered state is the theory obtained by gauging the anomalous global symmetry. This connection between 't Hooft anomaly and noninvertible gravitational anomaly allows us to use the theory of gravitational anomaly developed in this paper to systematically understand 't Hooft anomaly and its effect on low energy properties. Those issues will be studied in Ref. [37].

We like to remark that in this paper, we only studied the modular covariance of torus partition functions of 1+1D anomalous systems. However, in order for the partition functions to be realizable on a boundary of a 2+1D topological order, just requiring the modular covariance of torus partition functions is not enough. We should also require the partition functions on high genus surfaces to transform properly under the corresponding MCG transformations. The resulting partition functions should be realizable on a boundary of a 2+1D topological order.

ACKNOWLEDGMENTS

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APPENDIX A: CHARACTERS OF CHIRAL CFTS

1. $u1_M$ CFT

$u1_M$ current algebra is generated by the current $\partial_z \varphi(z)$ and $e^{i\sqrt{M}\varphi}$. The primary fields of the current algebra are $e^{i\frac{m}{\sqrt{M}}\varphi}, 0 \leqslant m \leqslant M-1$. The character $\chi^{u1_M}_m$ of $u1_M$ CFT is given by

$$\chi^{u1_M}_m(\tau) = \eta^{-1}(q) \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(\frac{m}{R}+nR)^2}, \tag{A1}$$

where $0 \leqslant m < M$ and $R^2 = M$. Under modular transformation S and T, the characters transform as follows:

$$\chi^{u1_M}_i\left(-\frac{1}{\tau}\right) = S_{ij}\chi^{u1_M}_j(\tau), \quad S_{ij} = \frac{e^{-i2\pi\frac{ij}{M}}}{\sqrt{M}},$$
$$\chi^{u1_M}_i(\tau+1) = T_{ij}\chi^{u1_M}_j(\tau), \quad T_{ij} = e^{-i\frac{2\pi}{24}} e^{i2\pi\frac{i^2}{2M}}\delta_{ij}. \tag{A2}$$

In the case of semion model, the left-moving part has two sectors, the vacuum and semion sector. They are primary fields of $u1_2$ current algebra:

$$\chi^{u1_2}_0 = \eta(q)^{-1}\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(2n)2^{-1}(2n)} = \eta(q)^{-1}\sum_{n\in\mathbb{Z}} q^{n^2},$$
$$\chi^{u1_2}_1 = \eta(q)^{-1}\sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})(n+\frac{1}{2})}. \tag{A3}$$

2. $su2_k$ CFT

The CFT of the level-k SU(2) current algebra, $su2_k$, has characters $\chi^{su2_k}_j(\tau)$:

$$\chi^{su2_k}_j(q) = \frac{q^{(2j+1)^2/4(k+2)}}{[\eta(q)]^3} \times \sum_{n\in\mathbb{Z}} [2j+1+2n(k+2)] q^{n[2j+1+(k+2)n]}, \tag{A4}$$

where $j \in \{0, \frac{1}{2}, \dots, \frac{k}{2}\}$. Their modular transformations are

$$\chi_j^{su2_k}(-1/\tau) = \sum_{l\in\mathcal{P}} S_{jl}\chi_l^{su2_k}(\tau),$$
$$S_{jl} = \sqrt{\frac{2}{k+2}}\sin\left[\frac{\pi(2j+1)(2l+1)}{k+2}\right], \quad \text{(A5)}$$
$$\chi_j^{su2_k}(\tau+1) = \mathrm{e}^{-i\,\frac{2\pi}{24}\frac{3k}{(k+2)}}\mathrm{e}^{i\,2\pi\,\frac{j(j+1)}{k+2}}\chi_j^{su2_k}(\tau).$$

3. The minimal model CFT

The chiral CFTs with central charge $c < 1$ are called the minimal models. They are labeled by two integers p, p' with $p, p' > 2$ and an equivalence $(p, p') \sim (p', p)$. We demote those CFTs as $C^{\text{ft}}_{p,p'}$. The central charge and the dimensions of primary fields are given by

$$c = 1 - \frac{6(p-p')^2}{pp'},$$
$$h_{r,s} = \frac{(rp'-sp)^2-(p-p')^2}{4pp'}, \quad \text{(A6)}$$
$$1 \leqslant r \leqslant p-1, \quad 1 \leqslant s \leqslant p'-1,$$

which satisfy

$$h_{r,s} = h_{p-r,p'-s} = h_{p+r,p'+s} \quad \text{(A7)}$$

The CFTs are unitary if and only if $|p'-p| = 1$. In this case, the character for the primary field (r, s) is given by

$$\chi_{r,s}(q) = \frac{q^{h_{r,s}}}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{n[(np+r)(p+1)-ps]}(1-q^{(2np+r)s}),$$
$$\eta(q) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n), \quad q = \mathrm{e}^{2i\pi\tau}, \quad \text{(A8)}$$

where $\chi_{r,s}(q) = \chi_{p-r,p'-s}(q)$. The S matrix is

$$S_{rs;\rho\sigma} = \sqrt{\frac{8}{pp'}}(-1)^{(1+s\rho+r\sigma)}\sin\left(\pi\frac{p'}{p}r\rho\right)\sin\left(\pi\frac{p}{p'}s\sigma\right). \quad \text{(A9)}$$

For unitary minimal models $(p, p') = (p, p+1)$, we have

$$c = 1 - \frac{6}{p(p+1)},$$
$$h_{r,s} = \frac{(r+rp-sp)^2-1}{4p(p+1)}, \quad 1 \leqslant r \leqslant p-1,\ 1 \leqslant s \leqslant p, \quad \text{(A10)}$$

For $c = 1/2$ Ising CFT, $p = 3$, $p' = 4$. $(r, s) = (1, 1)$ and (2,3) correspond to the identity primary field $\mathbf{1}$. $(r, s) = (1, 2)$ and (2,2) correspond to primary field σ with $h_\sigma = \frac{1}{16}$. $(r, s) = (1, 3)$ corresponds to primary field ψ with $h_\psi = \frac{1}{2}$. In the basis of $\{\chi_{\mathbf{1}}, \chi_\psi, \chi_\sigma\}$, the modular transformation is given by

$$S_{\text{Is}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix},$$
$$T_{\text{Is}} = \mathrm{e}^{-i\,\frac{\pi}{24}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mathrm{e}^{i\,\frac{2\pi}{16}} \end{pmatrix}. \quad \text{(A11)}$$

4. Exceptional current algebra CFT

Both $(G_2)_1$ and $(F_4)_1$ characters have the form as follows [54]:

$$\chi_0 = \left[\frac{\lambda(1-\lambda)}{16}\right]^{\frac{1-x}{6}} {}_2F_1\left(\frac{1}{2}-\frac{1}{6}x, \frac{1}{2}-\frac{1}{2}x; 1-\frac{1}{3}x; \lambda\right),$$
$$\chi_1 = N\left[\frac{\lambda(1-\lambda)}{16}\right]^{\frac{1+x}{6}} {}_2F_1\left(\frac{1}{2}+\frac{1}{6}x, \frac{1}{2}+\frac{1}{2}x; 1+\frac{1}{3}x; \lambda\right), \quad \text{(A12)}$$

where $\lambda(\tau) = (\frac{\theta_2(\tau)}{\theta_3(\tau)})^4$, in terms of theta functions,

$$\theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}, \quad \theta_4(\tau) = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2}{2}}. \quad \text{(A13)}$$

Under modular transformation $T : \lambda \to \lambda(\lambda-1)$ and $S : \lambda \to 1-\lambda$,

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \quad \text{(A14)}$$
$$(q)_n = \begin{cases} 1 & n = 0 \\ \frac{(q+n-1)!}{(q-1)!} & n > 0 \end{cases} \quad \text{(A15)}$$

is the hypergeometric function defined for $|z| < 1$, and $x = 1 + \frac{c}{2}$. The parameters for some examples are

$$(G_2)_1 : N = 7, \quad x = \tfrac{12}{5},$$
$$(F_4)_1 : N = 26, \quad x = \tfrac{18}{5},$$
$$(E_8)_1 : N = 2, \quad x = 4. \quad \text{(A16)}$$

APPENDIX B: NON-ON-SITE Z_2 SYMMETRY TRANSFORMATIONS

The first nonon-site Z_2 symmetry transformation (41) transforms σ_i^x in the following way [see (42)]:

$$\left(\prod_j \sigma_j^x \prod_j CZ_{j,j+1}\right)\sigma_i^x\left(\prod_j \sigma_j^x \prod_j CZ_{j,j+1}\right)$$
$$= \frac{1+\sigma_{i-1}^z+\sigma_i^z-\sigma_{i-1}^z\sigma_i^z}{2}\frac{1+\sigma_i^z+\sigma_{i+1}^z-\sigma_i^z\sigma_{i+1}^z}{2}\sigma_i^x\frac{1+\sigma_{i-1}^z+\sigma_i^z-\sigma_{i-1}^z\sigma_i^z}{2}\frac{1+\sigma_i^z+\sigma_{i+1}^z-\sigma_i^z\sigma_{i+1}^z}{2}$$
$$= \frac{1+\sigma_{i-1}^z+\sigma_i^z-\sigma_{i-1}^z\sigma_i^z}{2}\frac{1+\sigma_i^z+\sigma_{i+1}^z-\sigma_i^z\sigma_{i+1}^z}{2}\frac{1+\sigma_{i-1}^z-\sigma_i^z+\sigma_{i-1}^z\sigma_i^z}{2}\frac{1-\sigma_i^z+\sigma_{i+1}^z+\sigma_i^z\sigma_{i+1}^z}{2}\sigma_i^x$$

$$= \left(\frac{1+\sigma^z_{i-1}+\sigma^z_i-\sigma^z_{i-1}\sigma^z_i}{2}\frac{1+\sigma^z_{i-1}-\sigma^z_i+\sigma^z_{i-1}\sigma^z_i}{2}\right)\left(\frac{1+\sigma^z_i+\sigma^z_{i+1}-\sigma^z_i\sigma^z_{i+1}}{2}\frac{1-\sigma^z_i+\sigma^z_{i+1}+\sigma^z_i\sigma^z_{i+1}}{2}\right)\sigma^x_i$$
$$= \frac{(1+\sigma^z_{i-1})-(1-\sigma^z_{i-1})}{2}\frac{(1+\sigma^z_{i+1})-(1-\sigma^z_{i+1})}{2}\sigma^x_i = \sigma^z_{i-1}\sigma^x_i\sigma^z_{i+1}. \tag{B1}$$

The second non-on-site Z_2 symmetry transformation (43) transforms σ^x_i in a similar way [see Eq. (44)]:

$$\left(\prod_j \sigma^x_j \prod_j s_{j,j+1}\right)\sigma^x_i\left(\prod_j \sigma^x_j \prod_j s_{j,j+1}\right)$$
$$= \frac{1-\sigma^z_{i-1}+\sigma^z_i+\sigma^z_{i-1}\sigma^z_i}{2}\frac{1-\sigma^z_i+\sigma^z_{i+1}+\sigma^z_i\sigma^z_{i+1}}{2}\sigma^x_i\frac{1-\sigma^z_{i-1}+\sigma^z_i+\sigma^z_{i-1}\sigma^z_i}{2}\frac{1-\sigma^z_i+\sigma^z_{i+1}+\sigma^z_i\sigma^z_{i+1}}{2}$$
$$= \frac{1-\sigma^z_{i-1}+\sigma^z_i+\sigma^z_{i-1}\sigma^z_i}{2}\frac{1-\sigma^z_i+\sigma^z_{i+1}+\sigma^z_i\sigma^z_{i+1}}{2}\frac{1-\sigma^z_{i-1}-\sigma^z_i-\sigma^z_{i-1}\sigma^z_i}{2}\frac{1+\sigma^z_i+\sigma^z_{i+1}-\sigma^z_i\sigma^z_{i+1}}{2}\sigma^x_i$$
$$= \left(\frac{1-\sigma^z_{i-1}+\sigma^z_i+\sigma^z_{i-1}\sigma^z_i}{2}\frac{1-\sigma^z_{i-1}-\sigma^z_i-\sigma^z_{i-1}\sigma^z_i}{2}\right)\left(\frac{1-\sigma^z_i+\sigma^z_{i+1}+\sigma^z_i\sigma^z_{i+1}}{2}\frac{1+\sigma^z_i+\sigma^z_{i+1}-\sigma^z_i\sigma^z_{i+1}}{2}\right)\sigma^x_i$$
$$= \frac{(1-\sigma^z_{i-1})-(1+\sigma^z_{i-1})}{2}\frac{(1+\sigma^z_{i+1})-(1-\sigma^z_{i+1})}{2}\sigma^x_i = -\sigma^z_{i-1}\sigma^x_i\sigma^z_{i+1}. \tag{B2}$$

APPENDIX C: TOPOLOGICAL PATH INTEGRAL ON A SPACE-TIME WITH WORLD LINES

1. Space-time lattice and branching structure

To find the conditions on the domain-wall data, we need to use extensively the space-time path integral. Let us first describe how to define a space-time path integral. We first triangulate the three-dimensional space-time to obtain a simplicial complex $\mathcal{M}^3$ (see Fig. 7). Here we assume that all simplicial complexes are of bounded geometry in the sense that the number of edges that connect to one vertex is bounded by a fixed value. Also, the number of triangles that connect to one edge is bounded by a fixed value, etc.

In order to define a generic lattice theory on the space-time complex $\mathcal{M}^3$, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure [14,55]. A branching structure is a choice of the orientation of each edge in the n-dimensional complex so that there is no oriented loop on any triangle (see Fig. 8).

The branching structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming edges, and the second vertex is the vertex with only one incoming edge, etc. So the simplex in Fig. 8(a) has the following vertex ordering: "0"<"1"<"2"<"3."

The branching structure also gives the simplex (and its sub simplexes) an orientation denoted by $s_{ij\cdots k} = 1, *$. Figure 8 illustrates two 3-simplices with opposite orientations $s_{0123} = 1$ and $s_{0123} = *$. The red arrows indicate the orientations of the 2-simplices which are the subsimplices of the 3-simplices. The black arrows on the edges indicate the orientations of the 1-simplices.

The degrees of freedom of our lattice model live on the vertices (denoted by v_i where i labels the vertices), on the edges (denoted by e_{ij} where $\langle ij\rangle$ labels the edges), and on other high dimensional simplicies of the space-time complex (see Fig. 7).

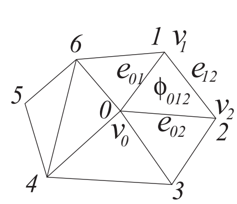


FIG. 7. A two-dimensional complex. The vertices (0-simplices) are labeled by i. The edges (1-simplices) are labeled by $\langle ij\rangle$. The faces (2-simplices) are labeled by $\langle ijk\rangle$. The degrees of freedoms may live on the vertices (labeled by v_i), on the edges (labeled by e_{ij}) and on the faces (labeled by ϕ_{ijk}).

2. Discrete path integral

In this paper, we only consider a type of 2+1D path integral that can be constructed from a tensor set T of two real and one complex tensors: $T = (w_{v_0}, d^{v_0v_1}_{e_{01}}, C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}_{v_0v_1v_2v_3;\phi_{013}\phi_{123}})$. The complex tensor $C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}_{v_0v_1v_2v_3;\phi_{013}\phi_{123}}$ can be associated with a tetrahedron, which has a branching structure (see Fig. 9). A branching structure is a choice of an orientation of each edge in the complex so that there is no oriented loop on any triangle

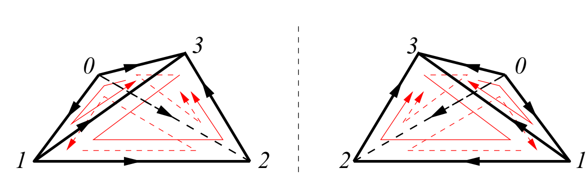


FIG. 8. Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

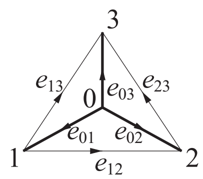


FIG. 9. The tensor $C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}_{v_0v_1v_2v_3;\phi_{013}\phi_{123}}$ is associated with a tetrahedron, which has a branching structure. If the vertex-0 is above the triangle-123, then the tetrahedron will have an orientation $s_{0123} = *$. If the vertex-0 is below the triangle-123, the tetrahedron will have an orientation $s_{0123} = 1$. The branching structure gives the vertices a local order: the ith vertex has i incoming edges.

(see Fig. 9). Here the v_0 index is associated with the vertex-0, the e_{01} index is associated with the edge-01, and the ϕ_{012} index is associated with the triangle-012. They represent the degrees of freedom on the vertices, edges, and triangles.

Using the tensors, we can define the path integral on any 3-complex that has no boundary:

$$Z(\mathcal{M}^3) = \sum_{v_0,\cdots;e_{01},\cdots;\phi_{012},\cdots} \prod_{\text{vertex}} w_{v_0} \prod_{\text{edge}} d^{v_0v_1}_{e_{01}} \times \prod_{\text{tetra}} [C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}_{v_0v_1v_2v_3;\phi_{013}\phi_{123}}]^{s_{0123}}, \tag{C1}$$

where $\sum_{v_0,\cdots;e_{01},\cdots;\phi_{012},\cdots}$ sums over all the vertex indices, the edge indices, and the face indices, $s_{0123} = 1$ or $*$ depending on the orientation of tetrahedron (see Fig. 9). We believe such type of path integral can realize any 2+1D topological order.

3. Path integral on space-time with natural boundary

On the complex $\mathcal{M}^3$ with boundary: $\mathcal{B}^2 = \partial\mathcal{M}^3$, the partition function is defined differently:

$$Z(\mathcal{M}^3) = \sum_{\{v_i;e_{ij};\phi_{ijk}\}} \prod_{\text{vertex}\notin\mathcal{B}^2} w_{v_0} \prod_{\text{edge}\notin\mathcal{B}^2} d^{v_0v_1}_{e_{01}} \times \prod_{\text{tetra}} [C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}_{v_0v_1v_2v_3;\phi_{013}\phi_{123}}]^{s_{0123}}, \tag{C2}$$

where $\sum_{v_i;e_{ij};\phi_{ijk}}$ only sums over the vertex indices, the edge indices, and the face indices that are not on the boundary. The resulting $Z(\mathcal{M}^3)$ is actually a complex function of v_i's, e_{ij}'s, and ϕ_{ijk}'s on the boundary $\mathcal{B}^2$: $Z(\mathcal{M}^3;\{v_i;e_{ij};\phi_{ijk}\})$. Such a function is a vector in the vector space $\mathcal{V}_{\mathcal{B}^2}$. (The vector space $\mathcal{V}_{\mathcal{B}^2}$ is the space of all complex function of the boundary indices on the boundary complex $\mathcal{B}^2$: $\Psi(\{v_i;e_{ij};\phi_{ijk}\}$.) We will denote such a vector as $|\Psi(\mathcal{M}^3)\rangle$. The boundary is attached with the tensors w_{v_i} and $d^{v_0v_1}_{e_{01}}$. The boundary (C2) defined above is called a natural boundary of the path integral.

We also note that only the vertices and the edges in the bulk (i.e., not on the boundaries) are summed over in $|\Psi(\mathcal{M}^3)\rangle$. When we glue two boundaries together, those tensors w_{v_i} and $d^{v_iv_j}_{e_{ij}}$ are added back. For example, let $\mathcal{M}^3$ and $\mathcal{N}^3$ to have the same boundary (with opposite orientations) $\partial\mathcal{M}^3 = -\partial\mathcal{N}^3 = \mathcal{B}^2$, which give rise to wave functions on the boundary $|\Psi(\mathcal{M}^3)\rangle$ and $\langle\Psi(\mathcal{N}^3)|$ after the path integral in the bulk. Gluing two boundaries together corresponds to the inner product $\langle\Psi(\mathcal{N}^3)|\Psi(\mathcal{M}^3)\rangle$. So the tensors w_{v_i} and

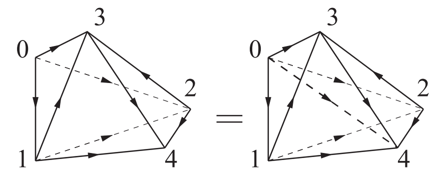


FIG. 10. A retriangulation of a 3D complex.

$d^{v_iv_j}_{e_{ij}}$ defines the inner product in the boundary Hilbert space $\mathcal{V}_{\mathcal{B}^2}$. Therefore we require w_{v_i} and $d^{v_iv_j}_{e_{ij}}$ to satisfy the following unitary condition:

$$w_{v_i} > 0, \quad d^{v_iv_j}_{e_{ij}} > 0. \tag{C3}$$

4. Topological path integral

We notice that the above path integral is defined for any space-time lattice. The partition function $Z(\mathcal{M}^3)$ depends on the choices of the space-time lattice. For example, $Z(\mathcal{M}^3)$ depends on the number of the cells in space-time, which give rise to the leading volume dependent term, in the large space-time limit (i.e., the thermodynamic limit)

$$Z(\mathcal{M}^3) = \mathrm{e}^{-\epsilon V} Z^{\text{top}}(\mathcal{M}^3), \tag{C4}$$

where V is the space-time volume, ϵ is the energy density of the ground state, and $Z^{\text{top}}(\mathcal{M}^3)$ is the volume independent partition function. It was conjectured that the volume independent partition function $Z^{\text{top}}(\mathcal{M}^3)$ in the thermodynamic limit, as a function of closed space-time $\mathcal{M}^3$, is a topological invariant that can fully characterize topological order [8,22]. So it is very desirable to fine tune the path integral to make the energy density $\epsilon = 0$. This can be achieved by fine-tuning the tensors w_{v_i} and $d^{v_iv_j}_{e_{ij}}$. However, we can do better. We can choose the tensor (w_{v_0}, $d^{v_0v_1}_{e_{01}}$, $C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}_{v_0v_1v_2v_3;\phi_{013}\phi_{123}}$) to be the fixed-point tensor-set under the renormalization group flow of the tensor network [12,56]. In this case, not only the volume factor $\mathrm{e}^{-\epsilon V}$ disappears, the volume independent partition function $Z^{\text{top}}(\mathcal{M}^3)$ is also retriangulation invariant, for any size of space-time lattice. In this case, we refer the path integral as a topological path integral, and denote the resulting partition function as $Z^{\text{top}}(\mathcal{M}^3)$. Z^{top} is also referred as the volume independent the partition function, which is a very important concept, since only the volume independent partition functions correspond to topological invariants. In particular, it was conjectured that such kind of topological path integrals describes all the topological order with gappable boundaries. For details, see Refs. [8,22].

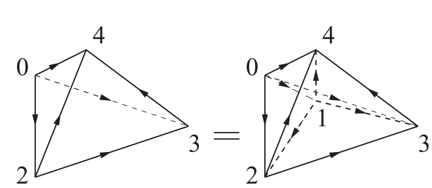


FIG. 11. A retriangulation of another 3D complex.

The invariance of partition function Z under the retriangulation in Figs. 10 and 11 requires that

$$\begin{aligned}\sum_{\phi_{123}} & C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}_{v_0v_1v_2v_3;\phi_{013}\phi_{123}} C^{e_{12}e_{13}e_{14}e_{23}e_{24}e_{34};\phi_{123}\phi_{134}}_{v_1v_2v_3v_4;\phi_{124}\phi_{234}} \\ &= \sum_{e_{04}} d^{v_0v_4}_{e_{04}} \sum_{\phi_{014}\phi_{024}\phi_{034}} C^{e_{01}e_{02}e_{04}e_{12}e_{14}e_{24};\phi_{012}\phi_{024}}_{v_0v_1v_2v_4;\phi_{014}\phi_{124}} \\ &\quad \times C^{*e_{01}e_{03}e_{04}e_{13}e_{14}e_{34};\phi_{013}\phi_{034}}_{v_0v_1v_3v_4;\phi_{014}\phi_{134}} C^{e_{02}e_{03}e_{04}e_{23}e_{24}e_{34};\phi_{023}\phi_{034}}_{v_0v_2v_3v_4;\phi_{024}\phi_{234}}, \end{aligned} \tag{C5}$$

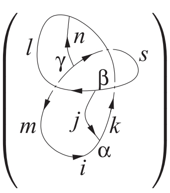
$$\begin{aligned} & C^{e_{02}e_{03}e_{04}e_{23}e_{24}e_{34};\phi_{023}\phi_{034}}_{v_0v_2v_3v_4;\phi_{024}\phi_{234}} \\ &= \sum_{e_{01}e_{12}e_{13}e_{14},v_1} w_{v_1} d^{v_0v_1}_{e_{01}} d^{v_1v_2}_{e_{12}} d^{v_1v_3}_{e_{13}} d^{v_1v_4}_{e_{14}} \sum_{\phi_{012}\phi_{013}\phi_{014}\phi_{123}\phi_{124}\phi_{134}} \\ &\quad \times C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}_{v_0v_1v_2v_3;\phi_{013}\phi_{123}} C^{*e_{01}e_{02}e_{04}e_{12}e_{14}e_{24};\phi_{012}\phi_{024}}_{v_0v_1v_2v_4;\phi_{014}\phi_{124}} \\ &\quad \times C^{e_{01}e_{03}e_{04}e_{13}e_{14}e_{34};\phi_{013}\phi_{034}}_{v_0v_1v_3v_4;\phi_{014}\phi_{134}} C^{e_{12}e_{13}e_{14}e_{23}e_{24}e_{34};\phi_{123}\phi_{134}}_{v_1v_2v_3v_4;\phi_{124}\phi_{234}}. \end{aligned} \tag{C6}$$

We would like to mention that there are other similar conditions for different choices of the branching structures. The branching structure of a tetrahedron affects the labeling of the vertices. For more details, see Ref. [57].

5. Topological path integral with world lines

In this paper, we also need to use the space-time path integral with world lines of topological excitations. We denote the resulting partition function as

$$Z\left(\quad \right), \tag{C7}$$



where $i, j, k, \ldots \in \{1, 2, \cdots, N\}$ label the type of topological excitations, and α, β, γ label the different fusion channels (i.e., different choices of actions at the junction of three world lines). The world lines are defined via a different choice of tensors for simplexes that touch the world lines. In this paper, we will choose the tensors very carefully, so that the path integral with world lines is also retriangulation invariant (even for the retriangulations that involve the world lines). The different choices of re-triangulation-invariant world lines are labeled by the different types of topological excitations. In this paper, we will only consider those topological path integrals with retriangulation invariance.

APPENDIX D: RENORMALIZATION GROUP FLOW OF MARGINAL PERTURBATIONS OF SU(2)$_1$ CFT

There are in total nine terms of marginal perturbations in SU(2)$_1$ CFT, composed of left and right currents. Let us first consider the following three couplings:

$$S_{\text{int}} = \sum_{i=1}^{3} \int g^i O_i, \quad O_i = J_i \bar{J}_i. \tag{D1}$$

The renormalization group (RG) equations have the form

$$\dot{g}_i = \alpha_{ijk} g_j g_k, \tag{D2}$$

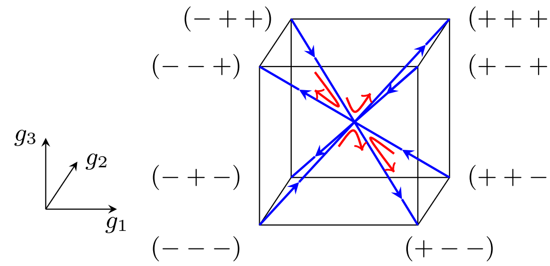


FIG. 12. The RG flow of $\delta S = \sum_{i=1,2,3} \int d^2x g_i J_i \bar{J}_i$. There are only fixed lines, shown as the blue lines. There are no stable sheets or regions, as indicated by the orange flow arrows. The corners are labeled by (s_1, s_2, s_3).

where α_{ijk} is proportional to the operator product expansion

$$\alpha_{ijk} = \langle O_i O_j O_k \rangle = \langle J_i J_j J_k \rangle \langle \bar{J}_i \bar{J}_j \bar{J}_k \rangle = (\epsilon_{ijk})^2. \tag{D3}$$

It follows that

$$\dot{g}_1 = g_2 g_3, \quad \dot{g}_2 = g_3 g_1, \quad \dot{g}_3 = g_1 g_2. \tag{D4}$$

The solution of the beta function has four fixed lines. To solve them, take the form $g_i(t) = \lambda_i f(t)$, and one finds $\frac{\lambda_1\lambda_2}{\lambda_3} = \frac{\lambda_2\lambda_3}{\lambda_1} = \frac{\lambda_3\lambda_1}{\lambda_2}$. Therefore $\lambda_i = s_i \alpha$, where $\alpha \geqslant 0$, $s_i = \pm 1$ to be determined. The RG equations become

$$\dot{f}(t) = s\alpha f^2(t), \tag{D5}$$

where $s = s_1 s_2 s_3$. The solution is

$$f(t) = \frac{f(0)}{1 - s\alpha f(0)t}. \tag{D6}$$

This leads to the RG solution of fixed lines

$$g_i(t) = \frac{g_i(0)}{1 - s|g_i(0)|t}, \quad |g_1(0)| = |g_2(0)| = |g_3(0)|. \tag{D7}$$

We find that (1) when $s > 0$, the following four fixed lines flow towards infinity:

$$\begin{aligned} g_1(t) = g_2(t) = g_3(t) > 0, \\ g_1(t) = -g_2(t) = -g_3(t) > 0, \\ -g_1(t) = g_2(t) = -g_3(t) > 0, \\ -g_1(t) = -g_2(t) = g_3(t) > 0. \end{aligned} \tag{D8}$$

(2) When $s < 0$, the following four fixed lines flow towards $g_1 = g_2 = g_3 = 0$:

$$\begin{aligned} g_1(t) = g_2(t) = g_3(t) < 0, \\ g_1(t) = -g_2(t) = -g_3(t) < 0, \\ -g_1(t) = g_2(t) = -g_3(t) < 0, \\ -g_1(t) = -g_2(t) = g_3(t) < 0. \end{aligned} \tag{D9}$$

This allows us to show that there are no stable regions or sheets in the (g_1, g_2, g_3) parameter space, as illustrated in Fig. 12.

Through the above example, we see a general pattern. If there is only one marginally relevant coupling, i.e., if we are on a fixed line, then there is a finite region, such that all the couplings in that region flow to zero. This finite region represents the region of stable gapless phase. If there are two marginally relevant couplings, i.e., if we are on a plane spanned by two fixed lines, then there is no finite region where the couplings flow to zero. When there are more marginally relevant couplings, the system is getting even more unstable. So we believe that, for our case with nine marginally relevant couplings, the corresponding CFT is unstable.

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