Analogy between equilibrium beach profiles and closed universes

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We reformulate the variational problem describing equilibrium beach profiles in the thermodynamic approach of Jenkins and Inman [J. Geophys. Res.: Oceans **111**, C02003 (2006)]. A first integral of the resulting Euler-Lagrange equation coincides formally with the Friedmann equation ruling closed universes in relativistic cosmology, leading to a useful analogy. Using the machinery of Friedmann-Lemaître-Robertson-Walker cosmology, qualitative properties and analytic solutions of beach profiles, which are the subject of a controversy, are elucidated.

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I. INTRODUCTION

Since the early work of Bruun [1], the profile of a beach, measured from the shore seaward and perpendicular to the shoreline, has been one of the most studied features of coastal morphology. It is important not only from the scientific point of view, but also because of its relevance to human activities [2] (early research was motivated by interest in military operations). A beach profile is dynamical and undergoes seasonal changes [3], therefore, research has focused on the simpler problem of *equilibrium* beach profiles, on which there is a significant literature [2,4]. Data show an undulating relief where the landward side of the topography increases for a while, while the seaward side decreases [5–7]. A beach profile is then modeled by matching two different curves, each of which satisfies an appropriate ordinary differential equation (e.g., [4]).

Research on the subject has moved from mere data fitting to developing theories of beach profiles under different conditions (e.g., breaking or nonbreaking waves). The most promising approach is probably that of Jenkins and Inman [4], which is based on thermodynamics. Near the shore, wave motion causes turbulence and energy dissipation and the main idea of Ref. [4] consists of maximizing the rate of energy dissipation of both breaking and nonbreaking waves. This extremization leads to an elegant variational principle formulation of the problem and to an associated Euler-Lagrange equation for the curves describing the equilibrium beach profiles. Since this equation is nonlinear, the search for its solutions is nontrivial. Analytic solutions were proposed in [4], but they are not easily reproducible and have recently been criticized in [8].

Instead of formulating the variational problem for a functional of the beach profile $h(x)$, Ref. [4] expresses it in terms of the inverse function $x(h)$. We reformulate the problem in terms of $h(x)$ and it is then easy to find a first integral of the Euler-Lagrange equation arising from a symmetry. The key point of this work is the realization that this first integral is formally equivalent to the Friedmann equation ruling the evolution of closed universes in relativistic cosmology, provided that the cosmic fluid that causes their space-time curvature is of a specific type. This fluid is indeed very reasonable from the physical point of view. The cosmological analogy turns out to be very useful because a wealth of information is now available about the equations of relativistic cosmology and their solutions. Research in cosmology has been much more intensive, and dates back to the 1920's (see, e.g., [9] for a historical perspective), which is longer than the time spanned by the research on beach profiles. We apply the standard exact solutions of the Einstein-Friedmann equations of cosmology [10–13], supplemented by recent mathematical results and methods for the Friedmann equation [14–16], to the analog beach problem. This use of the analogy leads us to clarifying several issues about beach profiles and to a comprehensive treatment of analytic solutions of the nonlinear differential equation ruling beach profiles in the thermodynamic approach of [4].

While it is understandable that the cosmological analogy was missed in the literature because of the enormous gap between the communities of cosmologists and ocean scientists, it is surprising that another, rather obvious, analogy between any beach profile ordinary differential equation (ODE) and the one-dimensional motion of a point particle was also missed. While this second analogy is much less useful than the first one, it nevertheless provides some insight on the qualitative nature of the solutions of the beach profile equation, and we discuss it briefly.

The structure of this paper is as follows: In Sec. II we reformulate the Jenkins-Inman variational problem and we rewrite the resulting first integral of (our version of) the Euler-Lagrange equation in a form analogous to the Friedmann equation. Section III discusses the mechanical analogy. Section IV develops the cosmological analogy, while Sec. V discusses in detail the analytic solutions of the beach profile equation and their deep water approximation. Section VI

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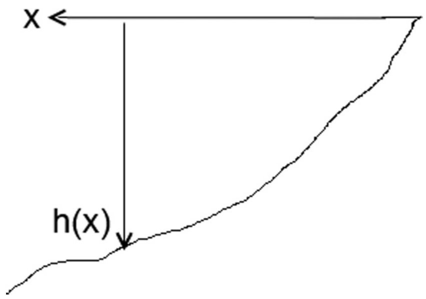

FIG. 1. The x axis points seaward horizontally from the shore and $h(x)$, measured downward, is the local water depth.

contains a summary and the conclusions. We follow the notation of Ref. [10]; the signature of the space-time metric is $-+++$, and we use units in which Newton's constant G and the speed of light c are unity.

II. EQUILIBRIUM BEACH PROFILES

Let x be the cross-shore distance (the x axis is horizontal and pointing seaward) and $h(x)$ be the local water depth, measured downward from a (constant) mean sea level (Fig. 1). The authors of [4] seek to maximize the entropy by extremizing the functional

$$I[x(h)] = \int_{h_1}^{h_2} (h(x))^{\frac{-3(n+1)}{4}} \sqrt{1+\left(\frac{dx}{dh}\right)^2}\, dh\,, \tag{2.1}$$

where $n > 0$ is an exponent appearing in the relation between the shear stress amplitude τ_0 and the water velocity $u_m(x)$ at the sea floor

$$\tau_0(x) = K_\tau \rho u_m^n(x)\,. \tag{2.2}$$

Here, ρ is the seawater density and the proportionality constant K_τ is independent of u_m [4].

Instead of studying the variational principle $\delta I = 0$ for $x(h)$, it is convenient to recast the problem in terms of the actual depth profile $h(x)$ as[1]

$$J[h(x)] = \int_{x_1}^{x_2} dx (h(x))^{\frac{-3(n+1)}{4}} \sqrt{1+\left(\frac{dh}{dx}\right)^2}. \tag{2.3}$$

The Lagrangian is

$$L(h,h') = (h(x))^{\frac{-3(n+1)}{4}} \sqrt{1+(h')^2}, \tag{2.4}$$

where $h' \equiv dh/dx$. Since $\partial L/\partial x = 0$, the Hamiltonian

$$\mathcal{H} = p_h h' - L(h,h') \tag{2.5}$$

is conserved, where

$$p_h \equiv \frac{\partial L}{\partial h'} = \frac{h^{\frac{-3(n+1)}{4}} h'}{\sqrt{1+(h')^2}} \tag{2.6}$$

is the momentum canonically conjugated to h. The conservation of

$$\mathcal{H} = -\frac{1}{h^{\frac{3(n+1)}{4}} \sqrt{1+h'^2}} \tag{2.7}$$

yields the first integral of motion

$$h^{\frac{3(n+1)}{2}}(1+h'^2) = C^2, \tag{2.8}$$

where C is an integration constant. It is clear that it must be $C \neq 0$, otherwise, the solution is $h(x) = 0$ everywhere. Imposing the boundary condition of zero depth at the origin, $h(0) = 0$, rules out any constant solutions (which would be unphysical anyway) and forces $h'(x)$ to diverge as $x \to 0$ in order to keep the left-hand side of Eq. (2.8) constant. The presence of this cusp prevents the applicability of the usual existence and uniqueness theorems for the initial value problem at $x = 0$ [19].[2] A physical consequence of this cusp is that the shallow water approximation used in [4] breaks down near the shore.

Equation (2.8) can be rearranged as

$$\left(\frac{h'}{h}\right)^2 = \frac{C^2}{h^{\frac{3n+7}{2}}} - \frac{1}{h^2}. \tag{2.9}$$

This equation is formally the same as the Friedmann equation ruling the evolution of certain spatially homogeneous and isotropic (Friedmann-Lemaître-Robertson-Walker, in short FLRW) universes in general relativity [10–13]. This fact gives rise to a very useful formal analogy between equilibrium beach profiles and closed universes in Einstein's theory of gravity. Given that the study of the cosmological equations has a long history [9], it is easy to infer mathematical solutions for the analog beach profile problem. Moreover, recent results on the mathematical properties of solutions of the Friedmann equation play a significant role. As we shall see, the analogy sheds some light on the mathematical solutions of Eq. (2.8) describing beach profiles, which are currently the subject of a controversy [8]. Note that the analogy with cosmology emerges only when the variational problem for the beach profiles is formulated in terms of $h(x)$ instead of $x(h)$. Before discussing it, however, it is useful to visit another analogy (missed in the literature thus far) between equilibrium beach profiles and point particle mechanics, which illustrates graphically certain qualitative properties of the solutions of Eq. (2.8).

III. MECHANICAL ANALOGY

Let us rewrite the ordinary differential equation (2.8) as

$$\frac{h'^2}{2} + V(h) = E, \tag{3.1}$$

where

$$V(h) = -\frac{C^2}{2h^{3(n+1)/2}} \tag{3.2}$$

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[1]For $n = -\frac{7}{3}$, the Lagrangian reduces to $L = h\sqrt{1+(dh/dx)^2}$ and gives rise to the classic catenary problem [17,18], but negative values of n are excluded in [4].

[2]Curiously, this situation resembles the fact that the longitudinal profile of a glacier as described by the Vialov equation of glaciology necessarily has a cusp at its terminus. This is because the Vialov ODE exhibits a feature similar to Eq. (2.8) [20–24].

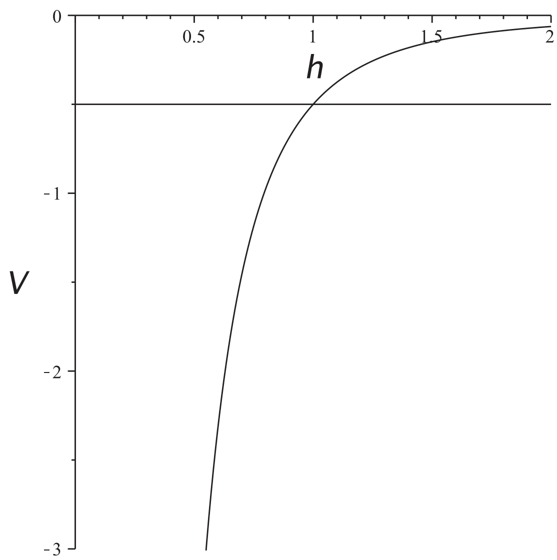

FIG. 2. In the region $h > 0$, there is always a unique intersection between the horizontal line $E = -\frac{1}{2}$ and the potential energy $V(h)$, therefore, the motion is always confined between the origin and the turning point.

and $E = -\frac{1}{2}$. In the form (3.1), Eq. (2.8) can be interpreted formally as describing as the position of a particle of unit mass and kinetic energy $(h')^2/2$ in one-dimensional motion along the h axis, subject to the potential energy $V(h)$, as time x goes by. Since this fictitious particle is subject only to the conservative force $-dV/dh$, its total mechanical energy is conserved and has the constant value $E = -\frac{1}{2}$. Equation (3.1) is a first integral of Newton's second law $d^2h/dx^2 = -dV/dh$ expressing energy conservation. Following the Weierstrass approach [17,25–27], one obtains a qualitative understanding and a graphical representation of the possible motions [i.e., of the possible solutions of Eq. (2.8)] from the graph of the potential $V(h)$ and its intersections with the horizontal line $E = -\frac{1}{2}$ (see Fig. 2).

The function $V(h)$ has a vertical asymptote at $h = 0$, the h axis as a horizontal asymptote, and only the region $h \geqslant 0$ is physical. Since $E \geqslant V$, the possible motions [i.e., the solutions $h(x)$ of Eq. (2.8)] are always confined to the interval $0 \leqslant h \leqslant h_*$, where the turning point h_* is the horizontal coordinate of the unique intersection between the line $E = -\frac{1}{2}$ and $V(h)$. This turning point is

$$(h_*, V_*) = \left(\left(\frac{C^2}{2|E|} \right)^{\frac{2}{3(n+1)}}, E \right) \tag{3.3}$$

and is always present for all negative energies E, in particular for $E = -\frac{1}{2}$. It is unique. A solution $h(x)$ of Eq. (2.8) describes only a segment of the beach profile [5–7] (in Ref. [4], two ellipsoidal cycloids are matched at a point to form a realistic profile).

Since $V(h) \to -\infty$ as $h \to 0^+$, a particle approaching $h = 0$ from the right must have diverging kinetic energy to keep the total energy E finite (and equal to $-\frac{1}{2}$). This means that it is always $h' \to +\infty$ as $h \to 0^+$ [which we had already concluded by inspection of Eq. (2.8)]. The origin of this divergence can be traced to the fact that Eq. (2.8) was derived in [4] under the approximation of a mild slope of the profile

$$\frac{\tan\beta}{kh} = \frac{h'}{kh} \ll 1, \tag{3.4}$$

where k is the wave vector and $\tan\beta = h'$ is the bottom slope. It is shown in Ref. [4] that $k \simeq \frac{\omega}{\sqrt{gh}}$ (where ω is the angular frequency of the breaking wave and g is the acceleration of gravity), which yields

$$\frac{h'}{kh} \simeq \frac{h'}{\sqrt{h}} \to \infty \quad \text{as} \quad x, h \to 0^+, \tag{3.5}$$

violating the mild slope approximation near the shore. There is nothing else to gain from the analogy with the one-dimensional motion of a point particle, and we now turn to the richer analogy with cosmology.

IV. COSMOLOGICAL ANALOGY

Here, we recall the essentials of FLRW cosmology and develop the analogy with equilibrium beach profiles.

In relativistic cosmology, the geometry of a spatially homogeneous and isotropic universe is necessarily given by the four-dimensional FLRW line element

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right], \tag{4.1}$$

written here in comoving polar coordinates (t, r, θ, φ). The scale factor $a(t)$ describes how two points at fixed comoving coordinate distance r_0 (for example, two typical galaxies without proper motions) separate as the universe expands. At time t, the physical distance between these two points is $l = a(t)r_0$ and it increases if $a(t)$ increases to describe an expanding universe. The function $a(t)$ embodies the expansion history of the universe. The constant K in Eq. (4.1) is normalized to the only three possible values $K = 1, 0, -1$ describing, respectively, a closed universe (closed three-dimensional spatial sections $t =$ const), Euclidean spatial sections, or hyperbolic 3-spaces [10–13]. This classification includes all the possible FLRW geometries and all the dynamics is encoded in the evolution of the scale factor $a(t)$ as a function of the comoving time t.

It is common in cosmology to describe the matter content of the universe, which generates the space-time curvature, as a perfect fluid of energy density $\rho(t)$ and isotropic pressure $P(t)$ related by some equation of state. The functions $a(t)$, $\rho(t)$, and $P(t)$ satisfy the Einstein-Friedmann equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{K}{a^2}, \tag{4.2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P), \tag{4.3}$$

$$\dot{\rho} + 3H(P + \rho) = 0, \tag{4.4}$$

where an overdot denotes differentiation with respect to t and $H(t) \equiv \dot{a}/a$ is the Hubble parameter [10–13]. Only two of these three equations are independent; given any two, the third one can be derived from them. For convenience, and without loss of generality, we take the Friedmann equation (4.2) and the energy conservation equation (4.4) as primary, and the acceleration equation (4.3) as derived.

Equation (4.2) with $K = +1$ is formally the same as Eq. (2.9) ruling equilibrium beach profiles if we exchange the variables $(x, h(x)) \longrightarrow (t, a(t))$. The analogy holds if a suitable cosmological fluid fills the analog universe. By comparing Eqs. (4.2) and (2.9), we see that it must be

$$\rho(t) = \frac{\rho_0}{(a(t))^{\frac{3n+7}{2}}}, \tag{4.5}$$

where ρ_0 is a positive integration constant determined by the initial conditions. This relation is familiar in cosmology, where it is common to assume that the cosmic fluid satisfies the barotropic equation of state

$$P = w\rho \tag{4.6}$$

for a suitable constant w ("equation-of-state parameter").[3] Then, Eq. (4.4) is integrated to give

$$\rho(a) = \frac{\rho_0}{a^{3(w+1)}}. \tag{4.7}$$

By comparing Eqs. (4.5) and (4.7), one concludes that the analogy between beach profiles and cosmology is valid if the universe is filled with a perfect fluid with $P = w\rho$ and equation-of-state parameter

$$w = \frac{3n+1}{6}. \tag{4.8}$$

Since it must be $n > 0$ in the model of Ref. [4], it is $w > \frac{1}{6}$. Well-known cases discussed in cosmology textbooks are a radiation fluid $w = \frac{1}{3}$ (corresponding to $n = \frac{1}{3}$) and a stiff fluid $w = 1$ (corresponding to $n = \frac{5}{3}$), which is realized by a free scalar field acting as an effective fluid [10–13].

Since $w > \frac{1}{6}$, the acceleration equation (4.3) implies that the analog universe always decelerates, i.e., $\ddot{a} < 0$ [only if $P < -\rho/3$ does the universe accelerate, as is clear by inspecting the right-hand side of the acceleration equation (4.3)].

V. SOLUTIONS OF THE BEACH PROFILE EQUATION VIA FRIEDMANN ANALOG

Let us analyze the solutions of Eq. (2.9), which are the subject of an ongoing controversy [8], in the light of the analog Friedmann equation. It is convenient to begin with the simplest case [we refer the reader to standard textbooks (e.g., [17,18]) for the classic catenary problem obtained for the unphysical value $n = -\frac{7}{3}$].

A. The case $n = \frac{1}{3}$

In the special case $n = \frac{1}{3}$, corresponding to $w = \frac{1}{3}$ in the analog universe dominated by a gas of photons, Eq. (4.5) gives the typical blackbody scaling of the energy density $\rho(a) = \rho_0/a^4$ and the scale factor [10–13]

$$a(t) = \sqrt{C'}\sqrt{1 - \left(1 - \frac{t}{\sqrt{C'}}\right)^2}, \tag{5.1}$$

where C' is a positive integration constant. This solution describes a closed universe that begins at a big bang singularity $a = 0$ at $t = 0$, expands to a maximum size $\sqrt{C'}$, and collapses to a big crunch singularity at $t = 2\sqrt{C'}$. The corresponding equilibrium beach profile is

$$h(x) = h_0\sqrt{1 - \left(1 - \frac{x}{h_0}\right)^2}, \tag{5.2}$$

with h_0 a constant length. The graph of $h(x)$ in the interval $x \in (0, 2h_0)$ is a cycloid (a semicircle), i.e., the trajectory of a point located on the rim of a circle of radius h_0 that rolls without slipping on the x axis.

B. Value $n = -1$ (linearly expanding universe)

Other special cases give simple exact solutions well known in cosmology, but they correspond to negative values of n, which are unphysical in the thermodynamic model of [4]. We report them here nevertheless.

If $w = -\frac{1}{3}$, corresponding to $n = -1$, the acceleration equation (4.3) gives the linear solution. In terms of the analog beach profile, it is

$$h(x) = h_0 x + h_1. \tag{5.3}$$

Linear beach profiles are considered in [8] and, in the shallow water approximation, they are reported in [28].

It is easy to see that a linear solution is the only possible power-law solution of Eq. (2.8) (here we refer to *exact* solutions: approximate solutions can be power law, as we will see later). In fact, assuming $h(x) = Ax^\alpha$ with A and α constants, substitution into Eq. (2.8) yields immediately $(n, \alpha) = (-1, 1)$ and $1 + A = \pm C$.

C. Value $n = -\frac{1}{3}$ (cosmic dust)

Another special case corresponds to a cosmic-dust fluid $w = 0$, obtained for $n = -\frac{1}{3}$. In this case the explicit solution in parametric form is [10–13]

$$h(\eta) = \frac{C}{2}(1 - \cos\eta), \tag{5.4}$$

$$x(\eta) = \frac{C}{2}(\eta - \sin\eta). \tag{5.5}$$

Expanding for $\eta \ll 1$ yields

$$h(\eta) \simeq \frac{C}{4}\eta^2, \tag{5.6}$$

$$x(\eta) \simeq \frac{C}{12}\eta^3. \tag{5.7}$$

Then, $h/x \approx 3/\eta \gg 1$, which shows the meaning of the approximation $\eta \ll 1$: it corresponds to deep water. By eliminating the parameter η, one obtains

$$h(x) \simeq \left(\frac{9C}{4}\right)^{1/3} x^{2/3}. \tag{5.8}$$

[3] The assumption that w is constant is often relaxed [12,13], but this complication is not necessary, nor useful, here.

This profile was obtained in Ref. [28] and claimed to be a good fit to field data.

D. General case $w = \text{const}$

In the general case $w = \text{const}$, a solution of the cosmological equations (4.2)–(4.4) can be found in parametric form and up to a quadrature by performing a change of variable [29]. Let us adopt the conformal time η defined by $dt = a\, d\eta$. Then, the Einstein-Friedmann equations give

$$\eta = \pm \int \frac{da}{a\sqrt{\frac{8\pi}{3}\rho a^2 - K}}. \tag{5.9}$$

When $w = \text{const}$, the substitution of Eq. (4.7) yields

$$\eta = \pm \int \frac{da}{a\sqrt{\frac{8\pi}{3} a^{-(3w+1)} - K}}. \tag{5.10}$$

By introducing the rescaled variable

$$z \equiv \left(\frac{8\pi C_1}{3}\right)^{\frac{-1}{3w+1}} a \tag{5.11}$$

and using, for $K = +1$,

$$\int \frac{dz}{z\sqrt{z^m - 1}} = \frac{2}{m}\text{arcsec}(z^{m/2}), \tag{5.12}$$

one integrates Eq. (5.10) and inverts the result, obtaining the parametric solution with conformal time as the parameter [14,29,30]

$$a(\eta) = a_0[\cos(c\eta + d)]^{1/c}, \tag{5.13}$$

$$t(\eta) = a_0 \int_0^\eta d\eta' [\cos(c\eta' + d)]^{1/c}, \tag{5.14}$$

where

$$c = \frac{3w+1}{2} \tag{5.15}$$

and a_0 is a constant. The big bang boundary condition $a = 0$ at $t = 0$ (corresponding to $\eta = 0$) is satisfied if $d = -\pi/2$, which yields

$$a(\eta) = a_0\left[\sin\left(\frac{(3w+1)}{2}\eta\right)\right]^{\frac{2}{3w+1}}, \tag{5.16}$$

$$t(\eta) = a_0 \int_0^\eta d\eta' \left[\sin\left(\frac{(3w+1)}{2}\eta'\right)\right]^{\frac{2}{3w+1}}. \tag{5.17}$$

On the beach profile side, the analog of the conformal time parameter is defined by $d\eta = dx/h(x)$. Small increments of the dimensionless parameter η are small increments of the distance from the shoreline measured in units of the local water depth. In finite terms, Eq. (5.10) has the analog

$$\eta = \pm \int \frac{dh}{h\sqrt{\frac{8\pi}{3} h^{\frac{-(3n+7)}{2}} - 1}}, \tag{5.18}$$

which integrates to

$$h(\eta) = h_0\left[\sin\left(\frac{3(n+1)}{4}\eta\right)\right]^{\frac{4}{3(n+1)}}, \tag{5.19}$$

$$x(\eta) = h_0 \int_0^\eta d\eta' \left[\sin\left(\frac{3(n+1)}{4}\eta'\right)\right]^{\frac{4}{3(n+1)}}, \tag{5.20}$$

where $x(\eta)$ is reduced to a quadrature and c is given by Eq. (5.15).

In the special case $n = \frac{1}{3}$ considered in the previous subsection it is $c = 1$, the integration of Eq. (5.20) is trivial, and the parameter η can be eliminated obtaining the explicit solution $h(x)$ given by Eq. (5.1).

An alternative way to solve for the cosmic dynamics consists of reasoning on the acceleration equation and noting that, in conformal time η, the latter reduces to a Riccati equation [30]. Assuming that $w = \text{const}$, the acceleration equation (4.3) becomes

$$\frac{\ddot{a}}{a} + c\frac{\dot{a}^2}{a^2} + \frac{cK}{a^2} = 0. \tag{5.21}$$

For $K = +1$ and using conformal time, this equation is rewritten as

$$\frac{1}{a}\frac{d^2a}{d\eta^2} + \frac{(c-1)}{a^2}\left(\frac{da}{d\eta}\right)^2 + c = 0. \tag{5.22}$$

This standard Riccati equation [31,32] is solved by using the new variable

$$u \equiv \frac{1}{a}\frac{da}{d\eta} \tag{5.23}$$

and then setting

$$u \equiv \frac{1}{cv}\frac{dv}{d\eta}, \tag{5.24}$$

which reduces the Riccati equation (5.22) to the harmonic oscillator equation $v'' + c^2 v = 0$, with sine and cosine solutions. Going back to the original variable $a(\eta)$ reproduces the solutions (5.19) and (5.20) [30].

It is, of course, interesting to know when the solution can be expressed explicitly in terms of elementary functions, as in the case $n = \frac{1}{3}$ discussed above. This question is answered in Ref. [14] with the help of the Chebysev theorem of integration [33,34]. Manipulation of the Friedmann equation (4.2) yields [14]

$$t = \int da \frac{a^{\frac{3w+1}{2}}}{\sqrt{\frac{8\pi\rho_0}{3} - a^{3w+1}}} \tag{5.25}$$

or, introducing [14]

$$b_0 \equiv \frac{8\pi\rho_0}{3}, \quad u \equiv a^{\frac{3(w+1)}{2}}, \tag{5.26}$$

it is

$$t = \frac{2}{3(w+1)}\int \frac{du}{\sqrt{b_0 - u^\gamma}}, \tag{5.27}$$

where

$$\gamma = \frac{2(3w+1)}{3(w+1)} \tag{5.28}$$

for $w \neq -1$. According to Chebysev's theorem, the integral is elementary only if $1/\gamma$ or $\frac{2-\gamma}{2\gamma}$ is an integer [14]. Setting $1/\gamma = N = 0, \pm 1, \pm 2 \pm 3, \ldots$ yields $w = \frac{3-2N}{3(2N-1)}$ and

$$n = \frac{7-6N}{3(2N-1)}. \tag{5.29}$$

The requirement of Ref. [4] that $n > 0$ corresponds to $\frac{1}{2} < N < \frac{7}{6}$, which leaves only $N = 1$, corresponding to $n = w = \frac{1}{3}$. The other possibility $\frac{2-\gamma}{2\gamma} = N$ corresponds to $w = \frac{1-N}{3N}$ and to $n = (2-3N)/(3N)$. The requirement $n > 0$ is then equivalent to $0 < N < \frac{2}{3}$, which is not satisfied by any integer.

E. Deep water approximation

We can now derive a deep water approximation for the general solutions (5.19) and (5.20). Expanding these equations for $\eta \ll 1$ yields

$$h(\eta) \simeq h_0 \left(\frac{3(n+1)}{4} \right)^{\frac{4}{3(n+1)}} \eta^{\frac{4}{3(n+1)}}, \tag{5.30}$$

$$x(\eta) \simeq h_0 \left(\frac{3(n+1)}{4} \right)^{\frac{4}{3(n+1)}} \frac{3(n+1)}{7+3n} \eta^{\frac{7+3n}{3(n+1)}}. \tag{5.31}$$

We have

$$\frac{h}{x} \simeq \frac{(7+3n)}{3(n+1)} \frac{1}{\eta} \gg 1 \tag{5.32}$$

independent of the value of n. Therefore, $\eta \ll 1$ corresponds to the deep water approximation. Eliminating the parameter η, we obtain the approximate power-law solution

$$h(x) \simeq h_0^{\frac{3(n+1)}{7+3n}} \left(\frac{7+3n}{4} \right)^{\frac{4}{7+3n}} x^{\frac{4}{7+3n}} \tag{5.33}$$

(power-law beach profiles have been proposed since the early studies of this subject [1,28,35]). The power is equal to $\frac{2}{3}$ (the value advocated in Ref. [28]) if $n = -\frac{1}{3}$, as already seen in a special case. The different exponent $\frac{2}{5}$ advocated in [35] is achieved for $n = 1$. For all other values of n, the exponent is instead $4/(7+3n)$.

F. Roulettes

The qualitative study and the search for analytic solutions of the Einstein-Friedmann equations (4.2)–(4.4) are reviewed in [30,36,37]), while [14–16] report new efforts in this direction. A mathematical property of the Friedmann equation (4.2) demonstrated in [15] is that the graphs of all solutions of this equation are roulettes. A roulette is the locus of a point that lies on, or inside, a curve that rolls without slipping on a straight line.[4] Indeed, all the solutions of the beach profile equation (2.9) proposed in [4] have graphs that are elliptical cycloids, i.e., the curves described by a point on an ellipse as the latter rolls on the x axis. In the special case in which the ellipse reduces to a circle, one obtains an ordinary cycloid [a semicircle like the one given by Eq. (5.2)]. Chen *et al.* [15] study explicitly the Friedmann equation for a closed ($K = +1$) universe to derive the equation of the solution in polar coordinates (r, ϑ). We do not repeat their analysis, reporting only the results. In general, $r(\vartheta)$ is not explicit and is only obtained up to a quadrature, but there are integrable cases corresponding to particular fluids with energy density

$$\rho(a) = \frac{\alpha}{a^2} + \beta a^{\delta}, \tag{5.34}$$

where α, β, and δ are arbitrary constants [15] (although it must be $\alpha \geqslant 0$ and $\beta \geqslant 0$ to avoid negative densities). Our case is reproduced for $\alpha = 0$, $\beta = \rho_0$, and $\delta = -(3n+7)/2$. The solution, constructed as a roulette, is [15]

$$\frac{1}{r^{\frac{3n+7}{3n+1}}} = \cos\left(\frac{3n+7}{3n+1} \vartheta \right). \tag{5.35}$$

Particularly simple solutions correspond to $w = 0$ ($n = -\frac{1}{3}$, discussed separately in [15]) and $w = \frac{1}{9}$ ($n = -\frac{1}{9}$) which are excluded in the model of Ref. [4]. As already mentioned, all the solutions proposed in [4] are roulettes, but they are not reproduced by Eq. (5.35) (see also Ref. [8]).

VI. SUMMARY AND CONCLUSIONS

Using an analogy with relativistic cosmology and, to a much lesser extent, a different one with one-dimensional point-particle motion, we have derived and studied the nonlinear ODE (2.8) ruling beach profiles in the Jenkins-Inman thermodynamic approach to the problem of equilibrium beach profiles [4]. Contrary to Ref. [4], we first reformulate the variational principle in terms of the beach profile $h(x)$, instead of its inverse $x(h)$,[5] which uncovers two analogies.

The first is an analogy with the mechanics of a point particle in one-dimensional motion, which provides a graphic way of deducing basic qualitative properties of the solutions. The second, and much richer, analogy is with relativistic cosmology, as described by Einstein's theory of general relativity. It is rather surprising that there is a formal analogy between the Friedmann equation describing closed universes and the beach profile equation. Since there are *two* independent equations ruling the evolution of these universes, one extra condition must be imposed, i.e., the cosmic perfect fluid sourcing the analog universe must have a specific equation of state. *A priori*, this extra condition would be expected to generate a completely exotic fluid with an unphysical equation of state, which would make the analogy far less interesting. A similar analogy for the transversal (i.e., cross-sectional) profile of glaciated valleys holds [38]: in that case, the cosmic fluid is very exotic, with a nonlinear equation of state, albeit of a type considered by cosmologists studying dark energy [16,39]. In the beach profile analogy, however, the cosmic fluid required is physically very reasonable: its equation of state is barotropic, linear, and constant. This type of cosmic fluid is very common in the cosmology textbooks [10–13] and includes, as a special case,

[4]In a more general definition, the curve rolls without slipping along another curve, but this is an unnecessary complication here.

[5]Our equation (2.8) is not contained in Ref. [4], although the resulting beach profiles should be the same as those obtainable by these authors' equations once their function $x(h)$ is inverted.

a radiation fluid (i.e., an expanding blackbody distribution of incoherent photons with random phases, polarizations, and directions of propagation) describing the radiation era of the early universe [10–13].

Since there is a wealth of literature on analytic solutions of the Friedmann equation, one can use the analogy beach profiles–closed universes to discover the solutions of the beach profile equation (2.8), which are currently the subject of a controversy [8]. The solutions can be given in parametric form $(h(\eta), x(\eta))$ with $x(\eta)$ expressed up to a quadrature. The Jenkins-Inman formalism contains another parameter n which is related to shear stress and water velocity at the sea floor and is also related to the equation-of-state parameter of the cosmic fluid in the analogous universe. Special values of this parameter n corresponding to integrability of the first-order ODE (2.8) have been identified, and some simple exact solutions provided. Furthermore, recent results [15] demonstrate that all the solutions of the Friedmann equation and, therefore, all those of the beach profile equation (2.8), are roulettes. The solutions (proposed in polar coordinates) in Ref. [4] are indeed roulettes, but their form is not reproduced by the integrability cases listed in [15], lending support to the critique of [8]. At the end of the day, however, much is learned about analytic solutions for beach profiles in the thermodynamic approach thanks to the cosmological analogy (and, to a much lesser extent, to the mechanical analogy) developed here. Three-dimensional beach profiles not contemplated in [4] would be analogous to anisotropic universes (Bianchi models) in relativistic cosmology [40], and will be studied in the future.

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