Composite non-Abelian strings with Grassmannian models on the worldsheet

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Most of the non-Abelian string vortices studied so far are characterized by two-dimensional $\mathbb{CP}(N)$ models with various degrees of supersymmetry on their worldsheet. We generalize this construction to "composite" non-Abelian strings supporting the Grassmann $\mathcal{G}(L, M)$ models (here $L + M = N$). The generalization is straightforward and provides, among other results, a simple and transparent way for counting the number of vacua in $\mathcal{N} = (2, 2)$ Grassmannian model.

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I. INTRODUCTION

The 2D $\mathbb{CP}(N-1)$ nonlinear sigma model has recently undergone much analysis, in particular appearing as worldsheet theories on the simplest non-Abelian string vortices [1–4] (see Refs. [5–8] for reviews) including its heterotic versions [9]. Non-Abelian BPS strings appear in four-dimensional theories with a U(N) gauge group and a certain scalar Higgs potential [1–4] ensuring that $\mathrm{U}(N)_{\mathrm{gauge}}$ and $\mathrm{SU}(N)_{\mathrm{flavor}}$ are spontaneously broken down to the diagonal SU(N) in the vacuum. Unlike the Abrikosov string, they carry orientational moduli due to the fact that, on the non-Abelian string solution, the above diagonal symmetry is further broken down to

$$\frac{\mathrm{SU}(N)}{\mathrm{SU}(N-1)\times \mathrm{U}(1)} = \mathbb{CP}(N-1) \qquad (1)$$

leading to the $\mathbb{CP}(N-1)$ model on the worldsheet.

In this paper, we will generalize the above construction for a non-Abelian multi-string with L units of flux, introducing a symmetry breaking pattern of the string solution

$$\frac{\mathrm{U}(N)}{\mathrm{U}(L)\times \mathrm{U}(M)} = \mathcal{G}(L, M), \quad L + M = N, \qquad (2)$$

which leads to the Grassmannian model on the worldsheet. In terms of gauge linear sigma models, this Grassmannian model can be described as a two-dimensional U(L) gauge theory. Since the original four-dimensional gauge theory is $\mathcal{N} = 2$ and the string is 1/2-BPS saturated, the two-dimensional theory on its worldsheet is $\mathcal{N} = (2, 2)$. We will, in parts, summarize and clarify some previously derived results about higher-winding non-Abelian strings first obtained in Refs. [4,10–13]: these works present various physical setups in which such Grassmannian sigma models arise.

In particular, the full worldsheet theory for the L multi-string as a U(L) gauge theory was first suggested in Ref. [4] using a brane construction. The dimension of its target space, in other words the number of zero modes promoted to two-dimensional fields of the composite string, was shown to be

$$\dim \mathcal{M}_{N,L} = 2LN = 2LM + 2L^2, \qquad (3)$$

where the first term in the final decomposition is the overall number of the orientational moduli, while $2L^2$ describes $2L$ relative positions and orientations of L components (i.e., the elementary strings with the unit flux). Attempts to reproduce these results in field theory do not lead to a transparent description of the worldsheet theory, see Ref. [14] and a review [6].

In this paper, we address a simplified problem. We suppress "relative orientations" and positions of the component strings assuming that axes of all L strings coincide. One last remaining positional zero mode remains, the position of the collective center of the stack of strings. This degree of freedom decouples from the zero mode dynamics of the gauge sector in sufficiently supersymmetric settings: certain deformations of the 4d theory (and consequently the 2d theory also) may couple the positional and internal fermionic zero modes (see Ref. [15] for a review), it is sufficient to assume (2, 2) worldsheet supersymmetry to ensure they will never couple. The dimension of this reduced moduli space is

$$\dim \mathcal{M}^{\mathrm{reduced}}_{N,L} = 2L(N-L) \equiv 2LM \qquad (4)$$

[see Eq. (2)] coincides with the dimension of the Grassmannian. In this setup, we construct explicit multi-string solution and derive U(L) gauge linear sigma model on the string worldsheet. We then detail its vacuum structure and check that it coincides with known exact results for such theories.

The paper is organized as follows. In Sec. II, we introduce four-dimensional $\mathcal{N} = 2$ SQCD and in Sec. III construct the solution for a composite string. In Sec. IV, we discuss the full worldsheet theory in the gauge description and study its classical vacua. Section V contains our conclusions.

II. FROM FOUR TO TWO DIMENSIONS. NON-ABELIAN STRINGS

For what follows we need to briefly review the construction of the "minimal" non-Abelian strings with the goal of generalizing it to the "composite" strings.

We start off in four-dimensional $\mathcal{N}=2$ U(N) SQCD, with $N_f=N$ flavors. The field content reduces to two gauge fields, A_μ and A_μ^a (one Abelian and the other not) as well as N flavors of squarks in the fundamental representation of the gauge group, Φ_A^k where k and A are respectively the color and flavor indices,[1]

$$\mathcal{L}=\frac{1}{4g^2}\operatorname{Tr}\left(\left(F_{\mu\nu}^a\right)^2+(F_{\mu\nu})^2\right)+|D\Phi|^2 + \frac{g^2}{2}\operatorname{Tr}(\Phi^\dagger T^a\Phi)^2+\frac{g^2}{8}(\operatorname{Tr}(\Phi^\dagger\Phi)-N\xi)^2. \quad (5)$$

The bosonic action above is a simplified version of the actual bosonic action of $\mathcal{N}=2$ SQCD. The vector supermultiplet also contains scalar complex superpartners of gauge fields, A_μ and A_μ^a, while squark fields are described by two sets of scalars, Φ_A^k and $\tilde{\Phi}_k^A$. We dropped both adjoint matter and squark $\tilde{\Phi}_k^A$ for simplicity as well as associated F terms in Eq. (5) because these fields have no VEVs and play no role in the string solution, see review [7].

The gauge symmetry becomes spontaneously broken (Higgsed) by the introduction of a Fayet-Iliopoulos term ξ. The scalar equations of motion show that the field Φ gains a diagonal VEV, enforcing a color-flavor locked phase

$$\left(\Phi_A^k\right)|_{\text{vac}}=\sqrt{\xi}\{\mathbb{1}\}_A^k. \quad (6)$$

This means the ground state is invariant under locked color-flavor transformations $\mathrm{U}(N)_{\text{diag}}$. Already, this generates distinct topological sectors due to the following nontrivial homotopy structure

$$\pi_1\left(\frac{\mathrm{U}(N)\times\mathrm{U}(N)}{\mathrm{U}(N)}\right)\sim\pi_1(\mathrm{U}(N))=\mathbb{Z}. \quad (7)$$

The integer that labels the equivalence classes of this homotopy is the overall winding number of a vortex.

Without breaking center symmetry, we can only prepare vortices in which all flavors have the same winding number; for $k\in\mathbb{Z}$, at infinity the scalar fields must tend to

$$\left(\Phi_A^k\right)\big|_{\text{large circle}}=\sqrt{\xi}\begin{pmatrix} e^{ik\theta} & 0 & 0\\ 0 & \dots & 0\\ 0 & 0 & e^{ik\theta}\end{pmatrix}, \quad (8)$$

which is a configuration with winding number kN. This seems nonminimal, we are not exploring the full range of integer winding number values. Besides, the physics of the object above reduces immediately to that of the Abrikosov-Nielsen-Olesen vortex, i.e., the Abelian variety. To obtain a proper non-Abelian vortex, we require to break $\mathbb{Z}_N$ center symmetry.

[1] We will for now forget about the fermionic matter content, which is present but fully determined by (5) through supersymmetry.

In doing so, we cause an additional homotopy structure to arise, since

$$\pi_1\left(\frac{\mathrm{U}(1)\times\mathrm{SU}(N)}{\mathbb{Z}_N}\right)=\mathbb{Z}_N. \quad (9)$$

Each of these N solutions are easy to find: one can wind separately any individual element on the diagonal in Eq. (6) as we go around a large circle in the perpendicular plane, introducing one unit of magnetic flux. The U(1) and SU(N) gauge fields are rotated correspondingly. Charging the last color flavor, for instance, we get that the boundary condition at infinity for the scalar field is

$$\left(\Phi_A^k\right)\big|_{\text{large circle}}=\sqrt{\xi}\begin{pmatrix}1&0&0&0\\0&1&\dots&0\\0&\dots&1&0\\0&0&0&e^{i\theta}\end{pmatrix}. \quad (10)$$

The resulting string has tension

$$T=2\pi\xi, \quad (11)$$

to be compared with the tension

$$T_{\text{ANO}}=2\pi N\xi \quad (12)$$

of the Abrikosov-Nielsen-Olesen string [16] in which all of the N flavors contribute magnetic flux to the vortex. The topological indices of this configuration is then not merely winding number, but the winding numbers of each flavor, which are now distinguishable by the action of $\mathbb{Z}_N$. For the elementary non-Abelian string produced above, its indices are

$$(0,0,\dots,1). \quad (13)$$

While this example is simple, it seems generalizable; one may ask what happens if, instead of rotating a single diagonal element in (6), we wind, say, two elements, leaving $N-2$ unwound. In the general case, we can wind L elements in (6) combining the action of the U(1) generator with the action of L generators from the Cartan subalgebra of SU(N),

$$\left(\Phi_A^k\right)\big|_{\text{large circle}}=\sqrt{\xi}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&e^{i\theta}&0\\0&0&0&0&e^{i\theta}\end{pmatrix}. \quad (14)$$

In the example above, Eq. (21), we have $L=2$, $M=3$, and $N=5$. Moreover, θ is the polar angle in the orthogonal plane. This configuration has topological indices

$$(0,0,0,1,1), \quad (15)$$

which makes it physically distinct from the previous example of a non-Abelian string. It appears as though it is the result of the fusion of two elementary non-Abelian strings.

Needless to say that in this construction the symmetry of N elements of the U(N) Cartan subalgebra under permutations is broken. This will lead us to a non-Abelian string with the $\mathcal{G}(L,M)$ Grassmann model on the worldsheet. We expect its tension will be

$$T_L=2\pi L\xi. \quad (16)$$

The symmetry under

$$L\leftrightarrow M \quad (17)$$

in the tension is realized in a curious way, namely,

$$T_L + T_M = 0 \bmod (T_{\mathrm{ANO}}). \tag{18}$$

We can remark here that we have explicitly chosen the lower L components to bear magnetic flux, but this choice is arbitrary: any L of the N components can be turned on, there are exactly $\binom{N}{L}$ such choices, corresponding to different solutions. These combinatorics are clear from the topological indices. The number of distinct strings (21) reduces to the combinatorial coefficient,

$$\nu_{L,M} = \binom{N}{L} = \frac{N!}{L!\,M!}, \tag{19}$$

whose symmetry under (17) is evident. We will return to this question later.

One may wonder what kind of object arises from giving different, nonzero fluxes to different colors, such as in the following example, if at infinity the scalar field becomes

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{2i\theta} \end{pmatrix}. \tag{20}$$

This is a much more complicated scenario which will be treated in a follow up study.

This extra selection sector becomes lifted if we allow residual $\mathrm{U}(N)_{\mathrm{diag.}}$ transformations to act on the string. Introducing a unitary matrix U, we endow the string solution with residual degrees of freedom in the following way:

$$\left(\Phi_A^k\right)\Big|_{\text{large circle}} = \sqrt{\xi}\, U \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\theta} \end{pmatrix} U^\dagger. \tag{21}$$

In doing so we can change the flavor of individual magnetic fluxes, although not their number. Total magnetic flux, the topological quantity arising from color-flavor locking, is still a good topological index. As a result, we have introduced extra degrees of freedom on the string: these will turn out to live inside a Grassmannian sigma model once worldsheet fluctuations are considered for all moduli of the string. Let us see how this proceeds.

III. BUILDING THE "COMPOSITE" STRING VORTEX

The ansatz for the string solution in a regular gauge has the form

$$\Phi_A^k = U \left(\begin{array}{ccc|ccc} \Phi_M(r) & & & & & \\ & \dots & & & 0 & \\ & & \Phi_M(r) & & & \\ \hline & & & e^{i\theta}\Phi_L(r) & & \\ & 0 & & & \dots & \\ & & & & & e^{i\theta}\Phi_L(r) \end{array}\right) U^\dagger, \tag{22}$$

$$A_\ell^a T^a = \frac{1}{N} U \left(\begin{array}{ccc|ccc} L & & & & & \\ & \dots & & & 0 & \\ & & L & & & \\ \hline & & & -M & & \\ & 0 & & & \dots & \\ & & & & & -M \end{array}\right) \times U^\dagger\, \partial_\ell \theta(-1 + f_N(r)), \tag{23}$$

$$A_\ell = \frac{L}{N} \partial_\ell \theta (1 - f(r)), \quad \ell = 1, 2, \tag{24}$$

where ℓ denotes spatial coordinates in the perpendicular plane, $\theta = \arctan(x_2/x_1)$ and r is the distance from the string axis in the perpendicular plane. For the time being we ignore the fermion fields: the object we create is BPS protected. These block-diagonal matrices above are split into a top-left $M \times M$ block and a bottom-right $L \times L$ block, then the non-Abelian gauge potential is indeed traceless with this choice of conventions.

We have introduced four scalar profiles Φ_L, Φ_M and f_N, f to be determined later through the equations of motion. Also we have introduced an arbitrary, constant unitary matrix $U \in \mathrm{SU}(N)$. The scalar functions obey the following boundary conditions required by regularity of the solution at 0:

$$\Phi_L(0) = 0, \quad f_N(0) = f(0) = 1, \tag{25}$$

$$\Phi_L(\infty) = \Phi_M(\infty) = \sqrt{\xi}, \quad f_N(0) = f(0) = 0. \tag{26}$$

In the regular gauge, it is clear that indeed L colors (flavors) experience winding, a set of scalars have a topological phase factor that depends on the angular coordinate, and the corresponding gauge field produces magnetic flux. However, it is more convenient for the remainder of the discussion to move to the singular gauge. At the cost of making the gauge fields ill-defined at the origin, we can absorb the phases of the L wound scalar fields and make them functions of the radial distance only, without inducing a winding or topological phase on the remainder M other scalars. Our ansatz becomes

$$\Phi_A^k = U \left(\begin{array}{ccc|ccc} \Phi_M(r) & & & & & \\ & \dots & & & 0 & \\ & & \Phi_M(r) & & & \\ \hline & & & \Phi_L(r) & & \\ & 0 & & & \dots & \\ & & & & & \Phi_L(r) \end{array}\right) U^\dagger, \tag{27}$$

$$A_{\ell=1,2}^a T^a = \frac{1}{N} U \left(\begin{array}{ccc|ccc} L & & & & & \\ & \dots & & & 0 & \\ & & L & & & \\ \hline & & & -M & & \\ & 0 & & & \dots & \\ & & & & & -M \end{array}\right) \times U^\dagger\, \partial_\ell \theta f_N(r), \tag{28}$$

$$A_\ell = -\frac{L}{N} \partial_\ell \theta f(r), \tag{29}$$

with unchanged boundary conditions. To ease the notation, in what follows we will denote the radial profile for the unwound and wound scalars as

$$\phi(r) \equiv \Phi_M(r), \quad \phi_w(r) \equiv \Phi_L(r). \tag{30}$$

Thanks to the fact that for the purpose of the classical solution our model we limit ourselves to the bosonic reduction of an $\mathcal{N}=2$ supersymmetric theory, the Lagrangian (5) is at the Bogomoln'yi point and hence half of supersymmetry is preserved on the solution. This allows to write first-order BPS equations of motion for the fields, constraining the profile functions. Namely,

$$\frac{d\phi(r)}{dr} = \frac{1}{r}\frac{L}{N}(f - f_N)\phi(r), \tag{31}$$

$$\frac{d\phi_w(r)}{dr} = \frac{1}{Nr}(Lf(r) - Mf_N(r))\phi_w(r), \tag{32}$$

$$\frac{L}{Nr}\frac{df(r)}{dr} = \frac{g^2}{4}\left(L\phi^2(r) + M\phi_w^2(r) - N\xi\right), \tag{33}$$

$$\frac{1}{r}\frac{df_N(r)}{dr} = \frac{g^2}{2}\left(\phi_w^2 - \phi^2\right). \tag{34}$$

A string satisfying these equations is BPS protected and can be viewed as a composite of L "elementary" strings. Indeed, compare their tension $T_L = 2\pi L\xi$ with that of the simplest strings [7] given in Eq. (11).

In the above ansatz, we introduced an arbitrary unitary matrix U, parametrizing an infinite family of solutions for the composite string. On quantum level, there is of course no spontaneous SU(N) symmetry breaking in two dimensions and much in the same way as in CP($N-1$) model the moduli space is lifted leaving us with discrete vacua. We will recover the $\binom{N}{L}$ vacuum structure at the classical level introducing twisted masses (see Sec. IV).

What is important for us now is that not every generic matrix U affects the solution of the type (21). Namely, any element of the form

$$U = \begin{pmatrix} \mathrm{U_M} & 0 \\ 0 & \mathrm{U}_L \end{pmatrix} \tag{35}$$

(where U_M and U_L are unitary matrices of the dimension $M \times M$ and $L \times L$, respectively) keeps it intact. Thus the space of distinguishable values the matrix U can effectively take is a group coset, the Grassmannian space

$$\mathcal{G}_{L,M} = \frac{\mathrm{U}(N)}{\mathrm{U}(L) \times \mathrm{U}(M)}. \tag{36}$$

Let us present our parametrization explicitly. We decompose $U \in \mathrm{SU}(N)$ in two rectangular matrices, arranged columnwise,

$$U = (W\,|\,X), \quad U^\dagger = \begin{pmatrix} W^\dagger \\ X^\dagger \end{pmatrix}, \tag{37}$$

where $X = X_{Ai}$ $(A = 1,\dots,N,\ \ i = 1,\dots,L)$ is a $N \times L$ rectangular matrix (a collection of L column vectors of height N),

$$\{X\} = \begin{pmatrix} X_{11} & \dots & X_{1L} \\ X_{21} & \dots & X_{2L} \\ \dots & \dots & \dots \\ X_{N1} & \dots & X_{NL} \end{pmatrix}, \tag{38}$$

and $W = W_{Aj}$ $(A = 1,\dots,N,\ \ j = 1,\dots,M)$ is a set of M column vectors of height N,

$$\{W\} = \begin{pmatrix} W_{11} & \dots & W_{1M} \\ W_{21} & \dots & W_{2M} \\ \dots & \dots & \dots \\ W_{N1} & \dots & W_{NM} \end{pmatrix}. \tag{39}$$

Unitarity of the matrix U imposes

$$X^\dagger_{iA}X_{Aj} = \mathbb{1}_{ij}, \quad W^\dagger_{nA}W_{Am} = \mathbb{1}_{nm}, \quad W^\dagger_{nA}X_{Aj} = 0,$$
$$i, j = 1,\dots,L, \quad n, m = 1,\dots,M. \tag{40}$$

With these choices the non-Abelian gauge and scalar fields can then be written

$$A^{\mathrm{SU}(N)A}_{\ell,B} = \partial_\ell\theta\, f_N(r)\left(\frac{L}{N}\mathbb{1}^A_B - X^{\ Ai}X^\dagger_{iB}\right),$$
$$\Phi^A_B = \delta^A_B\,\Phi_M(r) + [\Phi_L(r) - \Phi_M(r)]X^{Ai}X^\dagger_{iB}. \tag{41}$$

The matrix W drops out of the solution, only X remains as an orientation moduli matrix which should, all constraints and invariances enforced, point in a specific direction in the Grassmannian space. Had we imposed that the upper block of Φ^k_A in (22) of size $M \times M$ experienced winding, but not the lower block, then conversely X would drop out and W would become the orientation moduli matrix. Curiously, in either case both W and X will be constrained to live inside the same Grassmannian space.

Let us count the number of the internal degrees of freedom: the Grassmannian space has real dimension

$$\dim \mathcal{G}_{L,M} = 2LM = 2L(N-L), \tag{42}$$

while X's dimension, as a rectangular matrix with the above constraint, is

$$\dim\{X\} = 2LN - L^2 = L(2N-L). \tag{43}$$

These are not the same, however, this is an illusory discrepancy, we have an overcounting of the actual degrees of freedom in $\{X\}$. There is an unaccounted for gauge invariance that we must include. To see this we can turn on dynamics for the X coordinate, and observe gauge invariance of the effective theory that orchestrates its dynamics, but thinking geometrically about the space gives us a hint of why there must be a gauge invariance at hand.

The Grassmannian space is the space of all L-dimensional planes inside $\mathbb{C}^N$ and each point in the space is an individual plane. By choosing a specific X coordinate, we identify this as specifying an orthonormal basis of vectors for such a L-dimensional plane. However, this does not describe a unique plane: many such bases, even when constrained to be orthonormal, span the same space. They are all related by change of basis formulas, involving rotation matrices that are elements of U(L). In field theory terms, this is a continuous symmetry and we expect U(L) to be a symmetry of the Lagrangian. Now, since it is the physical translation of an overcounting of the degrees of freedom corresponding to physically distinct states, we expect that U(L) symmetry on the worldsheet is in fact a gauge symmetry. We can prove that this is the case, as mentioned, by explicitly constructing a gauge-invariant action for the X fields.

To realize it we can assume that X, previously a constant matrix, depends on the worldsheet coordinates of the string $\tilde{\ell} = (x^0, x^3)$ and generalize the derivation of the worldsheet effective theory for the minimal non-Abelian string with unit flux, see Ref. [7]. Gauge invariance of the bulk theory is then conserved so long as we turn on some extra gauge components, to wit

$$\begin{aligned}\left(A^{\mathrm{SU}(N)}_{\tilde{\ell}=0,3}\right)^{AB} &= -i\{X_{Ai}(\partial_{\tilde{\ell}}X^\dagger_{iB}) - (\partial_{\tilde{\ell}}X_{Ai})X^\dagger_{iB} \\ &\quad + X_{Ai}(X^\dagger_{iC}(\partial_{\tilde{\ell}}X_{Cj}) - (\partial_{\tilde{\ell}}X^\dagger_{iC})X_{Cj})X^\dagger_{jB}\}\rho(r) \\ &= -i\{X(\partial_{\tilde{\ell}}X^\dagger) - (\partial_{\tilde{\ell}}X)X^\dagger + X(X^\dagger(\partial_{\tilde{\ell}}X) \\ &\quad - (\partial_{\tilde{\ell}}X^\dagger)X)X^\dagger\}\rho(r) \qquad (44)\end{aligned}$$

for some arbitrary radial profile ρ. The latter should obey the following boundary conditions:

$$\rho(0) = 1, \quad \rho(\infty) = 0, \qquad (45)$$

the second of which is obvious, though the former will be justified later. Note that the second line makes use of a notational shorthand that will help alleviate the equations we write in the future; by enforcing the rectangular nature of X quite strictly, by always writing $X = X_{Ai}$ and $X^\dagger = X^\dagger_{iA}$ with the column and row indices ordered this way, the dimensions of multilinear objects composed of $X, X^\dagger$ should never be ambiguous and the matrix products all form intuitively in a neighbor-to-neighbor fashion.

Inserting this full ansatz in the four-dimensional action and performing the integration over the coordinates transverse to the string axis produces a two-dimensional worldsheet effective action for the field X, and the addition of the above gauge field not only preserve gauge invariance in the bulk but produces an action which is also gauge invariant on the worldsheet, namely,

$$S = \frac{4\pi I}{g_2^2}\int dtdz\left(\left|\partial_{\tilde{\ell}}X\right|^2 - \frac{1}{4}|X^\dagger(\partial_{\tilde{\ell}}X) - (\partial_{\tilde{\ell}}X^\dagger)X|^2\right), \qquad (46)$$

where X is still assumed to be an orthonormal set of vectors [imposed at the level of the functional integration measure, see also Eq. (53)], and the radial integration constant I is defined by

$$\begin{aligned}I = \int_0^\infty rdr\Bigg(\left(\frac{d\rho}{dr}\right)^2 + \frac{1}{r}f_N^2(r)(1-\rho(r))^2 \\ + \frac{g^2}{2}\rho(r)^2\left(\phi_{\mathrm{w}}^2 + \phi^2\right) - g^2(1-\rho)(\phi_{\mathrm{w}} - \phi)^2\Bigg), \qquad (47)\end{aligned}$$

which is the same expression obtained for the minimal non-Abelian string [7].

We see here that $1-\rho$ should vanish as r at the origin in order to cancel the singularity in the term $\frac{1}{r^2}f_N$ (in our gauge f_N does not vanish at 0). The integral I should be seen as an action for $\rho(r)$ and therefore be varied to determine a minimum of this quantity. This produces an equation of motion for ρ which ties it to the other field profiles. An extremal solution for ρ can be written in closed form in terms of the other profiles

$$\rho = 1 - \frac{\phi_{\mathrm{w}}}{\phi}, \quad \text{for which } I = 1. \qquad (48)$$

It is not immediately obvious that this is a solution and requires some algebraic tedium to derive, notably using all of the first-order BPS equations [Eqs. (31)–(34)] as well a judicious use of integration by parts. It is worth noting that this normalisation constant is *not* L, i.e., the total amount of flux running through the string.

The action in Eq. (46) has a peculiar and unobvious property. X transforms as a bifundamental of U(L)× SU(N), but in fact, through the particular shape of the self-interaction terms, the U(L) symmetry is made local, despite the absence of any tree-level gauge field. There are several ways of seeing this: one preliminary way of observing this phenomenon also happens to shine light on the $L \longleftrightarrow M$ symmetry that we expect to observe.

Indeed, as we mentioned previously our choice of winding resulted the extra components of the unitary matrix acting on our string solution to vanish: recall that U got split into W and X, the latter of which became our basic degree of freedom. We can rewrite the Lagrangian we obtained in a way that uses both fields along with a constraint. By using integration by parts, we can first rewrite the Lagrangian in Eq. (46) as

$$(\partial X_{Ai})(\partial X^\dagger_{iB})(\mathbb{1}_{BA} - X_{Bj}X^\dagger_{jA}). \qquad (49)$$

Recalling the constraints that unitarity imposes on these matrices, we have that

$$W_{Aa}W^\dagger_{aB} + X_{Ai}X^\dagger_{iB} = \mathbb{1}_{AB}, \quad W^\dagger_{aB}X_{Bi} = \mathbb{0}_{ai} \qquad (50)$$

alongside orthonormality of W, X individually as bases. Assuming we impose all of these constraints in the path integral, we can substitute W back in the Lagrangian above: we obtain

$$(\partial_\mu X_{Ai})(\partial^\mu X^\dagger_{iB})(W_{Ba}W^\dagger_{aA}) = (W^\dagger_{aA}\partial_\mu X_{Ai})(\partial^\mu X^\dagger_{iB}W_{Ba}). \qquad (51)$$

It is then a matter of using the mutual orthogonality of W, X to shift derivatives onto the W variables, providing the required rewriting with W as the dynamical variable.

In addition to shedding light on this issue, this Lagrangian is gauge-invariant under unitary U(L) transformations acting on X: perform a gauge transformation

$$\begin{aligned}&\partial^\mu X_{Ai} \to \partial^\mu X_{Ai} + X_{Aj}\alpha^\mu_{ji}, \\ &W^\dagger_{aA}\partial^\mu X_{Ai} \to W^\dagger_{aA}\partial^\mu X_{Ai} + W^\dagger_{aA}X_{Aj}\alpha^\mu_{ji} = W^\dagger_{aA}\partial^\mu X_{Ai}. \qquad (52)\end{aligned}$$

The local group variation disappears due to mutual orthogonality of W, X. Similarly, shifting derivatives onto W we could also discover an U(M) gauge symmetry so long as it is the dynamical variable. Choosing one of the two matrices to have a quadratic kinetic term will hide one of the two gauge invariances. It is surprising that gauge invariance occurs in a theory with no tree-level gauge fields, but we can introduce one to that effect.

We can, in fact, make this accidental gauge symmetry explicit by introducing an auxiliary gauge field $(A_{\tilde{\ell}})_{ij}$ in the adjoint of U(L) for the minimal action for X (assuming X is taken to be the fundamental degree of freedom, without loss of generality),

$$S = \frac{4\pi}{g_2^2}\int dt dz\, |(\mathbb{1}\partial_{\tilde{\ell}} - iA_{\tilde{\ell}})X|^2. \tag{53}$$

This gauge field $(A_{\tilde{\ell}})_{ij}$ on the worldsheet has no kinetic term (classically). Eliminating it via its equation of motion correctly produces the effective action in Eq. (46) obtained by reduction of the 4D theory. In addition, the X fields are constrained to obey certain orthogonality relations, which so far have been assumed to be enacted in the path integral measure. We can exponentiate this constraint and introduce it to the action as a Lagrange multiplier term: this produces a "Gauged linear sigma model" (GLSM), i.e., where the degrees of freedom are allowed to exist in a vector space rather than a more complicated manifold, but whose total degrees of freedom are constrained at tree-level by gauge invariance and auxiliary fields

$$S = \frac{4\pi}{g_2^2}\int dt dz\, |(\mathbb{1}_{ij}\partial_{\tilde{\ell}} - i(A_{\tilde{\ell}})_{ij})X_{Aj}|^2 + D_{ij}\big(X_i^{\dagger A}X_{Aj} - \mathbb{1}_{ij}\big). \tag{54}$$

The rotation over the i index in X_{Ai} becomes gauged. Since X is now a linear field, unconstrained in the path integral, it is good to canonically normalize its kinetic term. By rescaling D at the same time, this produces

$$S = \int dt dz\, |(\mathbb{1}_{ij}\partial_{\tilde{\ell}} - i(A_{\tilde{\ell}})_{ij})X_{Aj}|^2 + D_{ij}\left(X_i^{\dagger A}X_{Aj} - \frac{4\pi}{g_2^2}\mathbb{1}_{ij}\right). \tag{55}$$

This verifies our previous assertion: despite the fact that X now exists in a linear representation of SU(N)$\times$ U(L), U(L) is in fact not a global invariance of the ansatz we presented above but a local one. In addition to the orthonormality constraint, this diminishes the number of real degrees of freedom contained in X by just the right amount: Eq. (43) effectively reduces to (42). The exact same procedure, in the case when W as the basic degree of freedom, proves that W and X do indeed live in the same space and have the same number of degrees of freedom after gauging the corresponding symmetries; in the case of W, we gauge the U(M) index m so that $2LM + M^2 \to 2LM$.

In this process, we may remark that setting $L = 1$ produces the minimal non-Abelian string [2,7], which has a moduli space based on $\mathbb{CP}(N-1)$, a special case of the Grassmannian space. It is in this sense that we call the construction we have outlined a composite string: we can then view the above setup as a synthetic object obtained by fusing L minimal non-Abelian strings (each of string tension $T = 2\pi\xi$, the lowest attainable) each with a different color of magnetic flux. Each comes with its own $\mathbb{CP}(N-1)$ internal degrees of freedom, but once the strings fuse and are superposed, these become mutually indistinguishable: this reproduces another possible definition of the Grassmannian manifold [17]

$$\mathcal{G}(L, M) = (\mathbb{CP}(N))^L /\!\!/ S_L, \tag{56}$$

where S_L is the discrete symmetric group freely interchanging the L copies of $\mathbb{CP}(N)$. It is important to emphasize that this is *not* the S_L orbifold of L copies of $\mathbb{CP}(N)$, since the latter does not have the same dimension as the space that we consider. Rather, it is the set of all maximal orbits under the action of S_L, it removes from the space any set of points bearing any definite symmetry included in the symmetric group S_L: a point left invariant under any transformation of S_L will necessarily have a shorter orbit and is eliminated from this construction.

From this Lagrangian, we can move to a genuine nonlinear sigma model, free of constraints and gauge symmetry at the expense of introducing fields evolving in a curved manifold. This involves solving the constraint equation above and produces a metric most analogous to the $\mathbb{CP}(N)$ Fubini-Study metric. We will briefly detail it here.

We explicitly solve the constraint equation and fix a gauge condition by writing

$$X_{Aj} = \begin{pmatrix}\varphi_{mi}\\ \mathbb{1}_{ki}\end{pmatrix}\left(\frac{1}{\sqrt{\mathbb{1}+\varphi^\dagger\varphi}}\right)_{ij}, \quad A = 1, \dots, N,$$
$$m = 1, \dots, M, \quad k = 1, \dots, L, \tag{57}$$

introducing ML complex scalars $\varphi = \varphi_{mi}$. These degrees of freedom now directly specify a unique point in the Grassmannian space. Substituting this decomposition into the ungauged form of the action, Eq. (46), we obtain after some algebra very reminiscent of the $\mathbb{CP}(N)$ model the following Lagrangian, a nonlinear sigma model with a metric which generalizes the Fubini-Study metric of $\mathbb{CP}(N)$:

$$\mathcal{L} = \partial_{\tilde{\ell}}\varphi_{im}^\dagger\partial_{\tilde{\ell}}\varphi_{mj}\left(\frac{1}{1+\varphi^\dagger\varphi}\right)_{ji} - (\varphi_{im}^\dagger\partial_{\tilde{\ell}}\varphi_{mj})(\partial_{\tilde{\ell}}\varphi_{jm}^\dagger\varphi_{mk})\left(\frac{1}{1+\varphi^\dagger\varphi}\right)_{ki}. \tag{58}$$

This can be rewritten in a more symmetric form that treats the L-sized and M-sized indices equivalently as the following:

$$\mathcal{L} = \left(\frac{1}{\mathbb{1}+\varphi^\dagger\varphi}\right)_{ji}(\partial_{\tilde{\ell}}\varphi^\dagger)_{im}\left(\frac{1}{\mathbb{1}+\varphi\varphi^\dagger}\right)_{mn}(\partial_{\tilde{\ell}}\varphi)_{nj}. \tag{59}$$

In these forms, the target space geometry is made explicit and its properties can be explored in all the usual ways. This manifold is Kähler, the metric results from the Kähler potential

$$K(\varphi, \bar{\varphi}) = \mathrm{Tr}\ln(\mathbb{1} + \varphi\varphi^\dagger). \tag{60}$$

Given the Kähler potential above, it is straightforward to write a supersymmetric extension for this nonlinear formulation. Our theory, in fact, should be $\mathcal{N} = (2, 2)$ supersymmetric, since it is a BPS object. For field theory purposes, however, we would like to keep the linear gauged presentation of the action if possible, which it eminently is.

IV. SUPERSYMMETRIC GRASSMANNIAN MODEL

A. Introducing the full Lagrangian

The action we have derived in the previous section, see Eq. (53), is the bosonic, nonsupersymmetric version of the Grassmannian model. In practice, many extra fields (including bosonic ones) need to be added in order to get the actual supersymmetric action of the worldsheet theory which should preserve four supercharges.

Let us introduce two superfields, Ξ_i^A and $V_{ij} = V\mathbb{1}_{ij} + V^a T^a_{ij}$, respectively, the matter and gauge multiplets, the latter is valued in the Lie algebra of U(L). Then, schematically,

$$\Xi_{Ai} = X_{Ai} + \theta\xi_{Ai} + \theta^2 F_{Ai},$$
$$V = \cdots + \bar{\theta}\theta(\sigma^1 + i\sigma^2) + \theta\sigma^\mu\bar{\theta}A_\mu + \bar{\theta}^2\theta\chi + \bar{\theta}^2\theta^2 D, \tag{61}$$

see Eq. (65) for the definition of the σ fields. We combine these in the following superspace action:

$$\int d^2x d^2\theta d^2\bar{\theta} \quad \left\{ \mathrm{Tr}((\Xi^A)^\dagger e^V \Xi^A) + \frac{4\pi}{g_2^2}\mathrm{Tr}V \right\}. \tag{62}$$

The last term is a Fayet-Iliopoulos term. Out of superspace it produces the following Lagrangian:

$$\mathcal{L} = (D_\mu X_{iA})^\dagger (D^\mu X_{Ai}) - D_{ij}\left((X^\dagger_{iA} X_{Aj}) - \frac{4\pi}{g_2^2}\mathbb{1}_{ij} \right) + \bar{\xi}_{iA}(\not{D}\xi)_{Ai}$$
$$+ \left((i\sqrt{2}\bar{\chi}X\xi) + i\sqrt{2}\bar{\xi}^A_i (\sigma^1_{ij} + i\sigma^2_{ij}\gamma^5)\xi^A_j + \text{h.c.} \right)$$
$$- 2(X_{Ai})^\dagger (\bar{\sigma}\sigma)_{ij} X_{Aj}. \tag{63}$$

Note that g_2^2 is the four-dimensional coupling constant. It occurs in our two-dimensional model in the form $4\pi/g_2^2$.

The D coupling is now entirely fixed by supersymmetry, since this auxiliary field is no longer introduced by hand but exists as the top component of the gauge superfield. We can project D into a trace and traceless component, transforming the potential in the following way: define T^a_{ij} to be the generators of SU(L), then

$$D_{ij}\left((X^\dagger_{iA} X_{Aj}) - \frac{4\pi}{g_2^2}\mathbb{1}_{ij} \right) = D\left(X^\dagger X - \frac{4\pi L}{g_2^2} \right) + D^a\left(X_{Aj} T^a_{ji} X^\dagger_{iA} \right) \tag{64}$$

Additionally, an extra bosonic field has been introduced: the gauge multiplet scalar field σ_{ij}, a matrix of real dimension L^2, also in the adjoint representation of U(L) = U(1) × SU(L), which we expand in real and imaginary components

$$\sigma_{ij} = \sigma^1_{ij} + i\sigma^2_{ij}. \tag{65}$$

Note also the structure of the coupling of the gauge scalar to the matter fermions: the appearance of a γ^5 coupling will yield an anomaly which breaks chiral symmetry.

The auxiliary F terms in the matter multiplet Ξ vanishes since no superpotential is included. The entirety of the vector superfield is auxiliary—it has no tree-level kinetic term.

As was explained in Sec. I, in the present paper, we address a simplified problem describing the subspace of dimension $2LM$ of the full moduli space which has dimension $2LN$, see Eqs. (3) and (4). In the Hanany-Tong approach based on the brane picture [1], the U(L) gauge theory on the worldsheet contains an additional matter multiplet, namely, an adjoint multiplet Z_{ij}. In particular, the D-flatness constraint includes this multiplet and takes the form

$$D_{ij}\left((X^\dagger_{iA} X_{Aj}) + [Z^\dagger_{ik}, Z_{kj}] - \frac{4\pi}{g_2^2}\mathbb{1}_{ij} \right). \tag{66}$$

The additional adjoint scalar Z_{ij} describes relative separations of L component strings and their relative orientations. Our worldsheet theory (63) can be obtained from the Hanany-Tong construction in the limit $Z_{ij} = 0$. This limit ensures that all L strings share one and the same axis and they are all "orthogonal" to each other i.e., each of the component strings has a flux of a different color.[2]

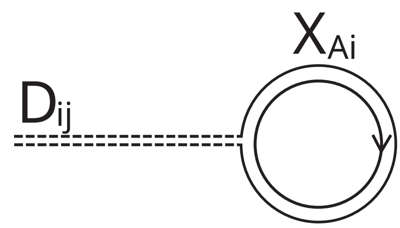

FIG. 1. The tadpole diagram leading to running of the coupling.

The β function of Grassmannian spaces, like all other homogeneous coset-type spaces, is known exactly in the supersymmetric case as it is saturated by one-loop diagrams: while it is computed with full precision in the nonlinear representation of the theory (particularly since Kähler manifolds have very regular curvature tensors), in which it appears as a result of target space geometry, a convenient shortcut can be obtained in this representation of the theory by computing the tadpole corrections to an external D insertion, since $\frac{4\pi}{g_2^2}\delta_{ij}$ is the operator coupling to D_{ij} in the Lagrangian. We show in Fig. 1, drawn in the t'Hooft double-index prescription, how it comes about.

The diagram produces the following loop integral:

$$-N\delta^{ij} \int \frac{d^2q}{(2\pi^2)} \frac{1}{q^2}, \tag{67}$$

which leads to the following result

$$\beta(g_2^2) = -\frac{N}{4\pi} g_2^4. \tag{68}$$

This correctly extends the result for $\mathbb{CP}(N-1)$, since it is independent of L. Due to this result, the theory is asymptotically free and is expected to generate a mass scale dynamically, namely,

$$\Lambda = M_{\mathrm{UV}} e^{-\frac{8\pi^2}{N g_2^2}}. \tag{69}$$

The dynamical scale parameter Λ on the left-hand side is renormalization-group invariant.

B. Vacuum structure of the Grassmannian string

We can count the corresponding vacua in the classical approximation by deforming the theory with masses, assumed large compared to the dynamically generated scale. Giving masses to the four-dimensional fields Φ produces masses for the worldsheet degrees of freedom [3,4]: we introduce a set of

[2]Unlike in other gauge theories that are suggested to arise as low-energy open string modes on stacks of branes, such as $\mathcal{N} = 4$ Super Yang-Mills on stacks $D3$ branes, the expectation values of the σ field are not related to distances between branes, since it is not a dynamical object, hence the occurrence of an extra field which fits this purpose.

(complex) masses m_A for each flavor X_A at hand, leading us to us the following bosonic Lagrangian:

$$\mathcal{L} = (D_\mu X_{Ai})^\dagger (D^\mu X_{Ai}) - D_{ij}\left((X^\dagger_{iA} X_{Aj}) - \frac{1}{g_2^2}\mathbb{1}_{ij}\right) - 2\sum_{A=1}^{N} (X_{Ai})^\dagger (\bar{\sigma}_{ik} - \bar{m}_A \delta_{ik})(\sigma_{kj} - m_A\delta_{kj}) X_{Aj}, \tag{70}$$

where

$$\{m_A\}, \quad A = 1, 2, \ldots, N \tag{71}$$

is a set of N complex twisted mass parameters. We assume that the masses m_A are all different, $m_A \neq m_B$ for all $A \neq B$, so that SU(N) is broken down to $\mathrm{U}(1)^{N-1}$. This lifts the orientational moduli and allows this theory to isolate its vacua, see review [7] for a similar deformation for the CP($N-1$) model on the minimal non-Abelian string.

The vacua have to satisfy two constraints simultaneously: the orthonormality relations due to the D-term potential and the σ equations of motion. The two are inextricably linked.

The former of the two has the following expression:

$$X^\dagger_{iA} X_{Aj} = \frac{4\pi}{g_2^2}\mathbb{1}_{ij}. \tag{72}$$

Needless to say, all fermion fields, as well as the kinetic terms, vanish in the vacuum. The vanishing in the vacuum of the last term in (70) is the dynamical requirement.

Inequivalent classical vacuum solutions are described by the expectation values of the fields X and σ. They are obtained as follows. From the set of N masses let us choose L of them,

$$m_{A_1},\ m_{A_2},\ \ldots, m_{A_L}. \tag{73}$$

These will provide vacuum expectation values for the σ field. In the vacuum, only the diagonal elements of the field σ are nonvanishing, and they are

$$\sigma_{11} = m_{A_1}, \quad \sigma_{22} = m_{A_2},\ \ldots, \sigma_{LL} = m_{A_L}. \tag{74}$$

It is essential that not only the trace component of the scalar field σ gains a VEV, but also that all the components corresponding to the Cartan subalgebra generators of SU(L) do so as well. The nonvanishing elements of X_{Ai} must be taken as

$$X_{A_1,1} = \frac{\sqrt{4\pi}}{g_2}, \quad X_{A_2,2} = \frac{\sqrt{4\pi}}{g_2}, \quad \ldots, X_{A_L,L} = \frac{\sqrt{4\pi}}{g_2}, \tag{75}$$

which corresponds to different choices of winding flavors in (21), see the solution for the squarks fields (41).

All other components of the matrix fields σ_{ij} and X_{Ai} are put to zero. The above solution broadly behaves like L copies of the construction of the isolated vacua of $\mathbb{CP}(N)$ model with twisted masses: this is to be expected, since as we explained previously,

$$\mathcal{G}(L, M) = (\mathbb{CP}(N))^L /\!\!/ S_L. \tag{76}$$

In fact, this construction of the Grassmannian space forbids us from taking two different diagonal elements σ_{ii} to equal the same mass, a condition without which the counting of the vacua fails to produce the right answer. Indeed, with this criterion the number of the classical vacuum solutions is then obviously

$$\nu_{L,M} = \binom{N}{L} = \frac{N!}{L!\,M!}. \tag{77}$$

given we are choosing L distinct masses for the eigenvalues of σ.

This is to be compared with the Witten index of the theory [18]. It is a topological invariant defined by

$$I_{\mathcal{W}} = \mathrm{Tr}((-1)^F e^{-\beta H}), \tag{78}$$

where we trace over all states in the theory, H is the Hamiltonian derived from the action and F is the fermion number operator (i.e., it weights fermionic states and bosonic states with a sign difference). It was shown, most generally, that for the Kählerian (Einstein) nonlinear sigma models the Witten index is exactly the Euler characteristic of the manifold. By using the explicit target space geometry in Eqs. (58) and (59), the characteristic can be computed and be shown to match our result quoted in Eq. (77). Since this index is a topological invariant we can hypothesize that the number of vacua remains the same in the quantum theory.

The exact result for the vacuum values of the σ fields generalizing the classical expression (74) can be inferred, e.g., from Ref. [19]. In the full quantum theory, the corresponding equations can be written in the form

$$\prod_{A=1}^{N} (\sigma_{jj} - m_A) = \Lambda^N, \quad \text{no summation over } j, \qquad j = 1, 2, \ldots, L. \tag{79}$$

As in the classical approximation all off-diagonal values of σ_{jk} ($j \neq k$) can be put to zero. The above system of L equations can be readily solved in the $\mathbb{Z}_N$-symmetric twisted masses,

$$m_k = m_0 \exp\left(2\pi\, i\frac{k}{N}\right), \quad k = 1, 2, \ldots, N. \tag{80}$$

In this case,

$$\sigma_{jj} = \left|m_0^N + \Lambda^N\right|^{1/N} \exp\left(2\pi\, i\frac{k_j}{N}\right), \tag{81}$$

which matches Eq. (74) in the limit $m_0 \gg \Lambda$. When m_0 is set to zero, this formula also provides the mass spectrum for the low-lying excitations of the theory. Indeed, the entire action can be put into Landau-Ginsburg form [17], in which Eq. (81) appears as the equation that minimizes the superpotential appearing in this formulation. Thereby, the low-lying energy states are kinks interpolating between minima of this potential. We label each minimum by its set of masses

$$V_{\{k_1,\ldots,k_L\}} = \left|\sigma_{11} = \Lambda e^{2\pi\, i\frac{k_1}{N}}, \ldots, \sigma_{LL} = \Lambda e^{2\pi\, i\frac{k_L}{N}}\right\rangle. \tag{82}$$

This vacuum is independent of the ordering of the k_i, it is a function of the *set of values* rather than the values themselves.

Fundamental solitons will exist between two vacua whose set of indices differ in only one element: $V_{\{k_1,\ldots,k_L\}}$, $V_{\{k'_1,\ldots,k'_L\}}$ will connect if[3]

$$k_1 = k'_1,\ \ldots, k_j - k'_j = r \neq 0,\ \ldots, k_L = k'_L. \tag{83}$$

[3]Without loss of generality, since ordering of the indices does not matter, we take the unequal elements to have the same index by relabeling and reordering them.

The index difference r is defined modulo N, and the sign of r is irrelevant in what will follow, or, to put it another way, there is a $r \leftrightarrow N - r$ symmetry in the structure of vacua.

The mass of the object interpolating between these two vacua is therefore

$$m_r = \frac{N}{2\pi}\Lambda\left|\exp\left(\frac{2\pi i k_j}{N}\right) - \exp\left(\frac{2\pi i k'_j}{N}\right)\right| = \frac{1}{\pi}N\Lambda \sin\left(\frac{\pi r}{N}\right). \tag{84}$$

This formula is indeed invariant under the advertised symmetry.

The objects of truly minimal mass are therefore the ones interpolating between vacua where only one pair of indices are unequal and differ only by 1 (mod N), which we may call the "closest neighbors." Pairs of vacua differing in one index by more than one unit will have higher mass as a result, and vacua "further away" with multiple unequal indices are considered not to be fundamentally connected at all [20]. This is different from $\mathbb{CP}(N)$, where the latter case does not exist, all vacua are connected to each other in the sense of the above.

This notion of connectedness is considered to be an exact one, as this whole discussion can be derived directly from considerations of topological-antitopological fusion [17]. Using these methods, further information can be extracted, for instance the degeneracy of the solitons interpolating between two given vacua. In $\mathbb{CP}(N)$, the lightest solitons in the theory have multiplicity N [21], all of which having the same mass. Using the topological construction of the vacua, the multiplicity of the solitons with mass m_r between two specific vacua was shown to have multiplicity

$$n_r = \binom{N}{r}, \tag{85}$$

which, again, does have the advertised symmetry.

To summarize, the low-lying excitations around these vacua consist, much like in $\mathbb{CP}(N)$, of an N-plet of kinks interpolating "nearest neighboring" vacua (i.e., $r = 1$), all of which have mass given by the mass formula above in Eq. (84).

Let us present a graphical illustration of the structure of the theory in a simple example. Let us work in the smallest (nontrivial) Grassmannian space

$$\mathcal{G}(2,2) = \frac{\mathrm{U}(4)}{\mathrm{U}(2)\times\mathrm{U}(2)}. \tag{86}$$

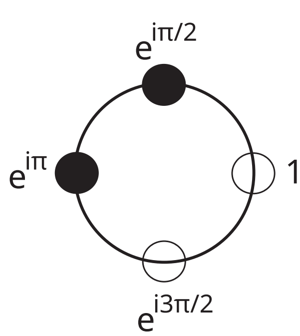

FIG. 2. A pictorial representation of a particular vacuum in $\mathcal{G}(2,2)$, one where the eigenvalues of σ are given by the roots of unity marked with black dots. The vacuum pictured here is $V_{\{1,2\}}$.

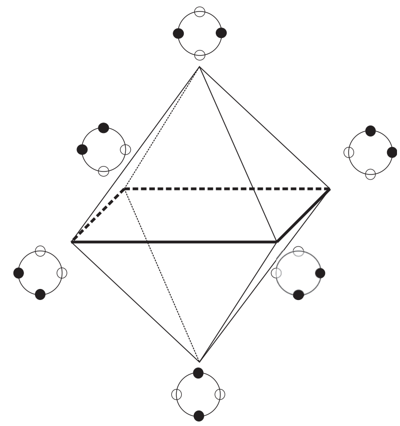

FIG. 3. The adjacency graph of the six vacua in the theory. Each vertex of the octahedron is associated to a vacuum state represented by a roots-of-unity diagram. The central square connects neighbors whose indices differ by more than one unit, thus have a higher mass, represented by a thicker line. No solution, elementary or bound, exists to connect the top and bottom vertices.

There are four possible values that the σ eigenvalues can take, and there are two of the latter, so there should be six vacua. Since these values are roots of unity, we can represent an individual vacuum using a unit circle on the complex plane and marking which roots of unity are used up by the σ fields, see Fig. 2. Then, we can draw a graph which connects neighboring vacua. For the simple example at hand, we obtain an octahedral structure as shown in Fig. 3. For a general review of these vacuum polytopes, see Ref. [22].

Even should we focus exclusively on objects of minimal mass, for which the neighboring vacua differ by one unit in one index, the connectivity of the vacua shows some nontrivial structure, and the polytopes required to display even these "closest neighbors" will become complicated very quickly.

V. CONCLUSIONS

In this work, we have constructed the composite non-Abelian vortex string solutions and investigated some of their properties. In particular, we derive the worldsheet effective theory for the reduced number of moduli living on the string. These moduli describe overall orientations of the composite string inside SU(N) group. Much like its more elementary counterpart, with $\mathbb{CP}(N)$ on its worldsheet, this string is topological in nature, is BPS protected, and possesses some leftover gauge degrees of freedom along its worldsheet. These fields live in a generalization of the $\mathbb{CP}(N)$ space usually seen in elementary non-Abelian vortices, the Grassmannian space, which nonetheless formally looks very similar to $\mathbb{CP}(N)$. The vacua of this theory were exhaustively justified through several different means and some aspects of the quantum behavior were touched upon.

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