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Editors' Suggestion

Making context explicit in equation construction and interpretation: Symbolic blending

Benjamin P. Schermerhorn [1,*] and John R. Thompson [2,3]

[1] *Department of Engineering Science and Physics, Monroe Community College, Rochester, New York 14623, USA*

[2] *Department of Physics and Astronomy, University of Maine, Orono, Maine 04469, USA*

[3] *Maine Center for Research in STEM Education, University of Maine, Orono, Maine 04469, USA*

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Much of physics involves the construction and interpretation of equations. Research on physics students' understanding and application of mathematics has employed Sherin's symbolic forms or Fauconnier and Turner's conceptual blending as analytical frameworks. However, previous symbolic forms analyses have commonly treated students' in-context understanding as their conceptual schema, which was designed to represent the acontextual, mathematical justification of the symbol template (structure of the expression). Furthermore, most conceptual blending analyses in this area have not included a generic space to specify the underlying structure of a math-physics blend. We describe a conceptual blending model for equation construction and interpretation, which we call symbolic blending, that incorporates the components of symbolic forms with the conceptual schema as the generic space that structures the blend of a symbol template space with a contextual input space. This combination complements symbolic forms analysis with contextual meaning and provides an underlying structure for the analysis of student understanding of equations as a conceptual blend. We present this model in the context of student construction of non-Cartesian differential length vectors. We illustrate the affordances of such a model within this context and expand this approach to other contexts within our research. The model further allows us to reinterpret and extend literature that has used either symbolic forms or conceptual blending.

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I. INTRODUCTION

One of the fundamental drives of physics education research is to interpret the way students understand, reason with, and use mathematics in physics. Mathematical models form the underlying foundation for the representation of physics content. Physicists apply mathematics to construct expressions that allow us to relay information efficiently, manipulate expressions to further advance our understanding, and interpret derivations to gain new insight into physical systems. From kinematics equations like $v_f = v_o + a\Delta t$, to the divergence of an electric field in electricity and magnetism, to Dirac notation and linear algebra in quantum mechanics, mathematics provides a fundamental language for physics.

Researchers in physics education have previously described mathematics as the language of physics [1], and researchers in physics and science education have developed broad theoretical models to frame the ways in which mathematics and physics interact in problem solving [1–4]. A common feature of these models and others is a step labeled "mathematization," in which a physical system is abstracted into a mathematical representation, or the "real model" of a system is abstracted into a "mathematical model" [4].

The symbolic forms framework was developed by Sherin [5] specifically to address how students construct and understand the mathematical structure of equations. Building from a *knowledge-in-pieces* approach [6], symbolic forms account for students writing an equation from a "sense of what they wanted to express" [5]. The purpose of identifying the underlying mathematical-based structures through which students understand equations speaks to the larger goal of how mathematics is used by students and ties to their understanding of mathematization in physics. At their inception, symbolic forms were designed as acontextual constructs with explicit focus on the mathematical justifications for equations and were not intended to address students' understanding of the associated physics concepts. As a theoretical perspective, it has been taken up to address student understanding and construction of integrals [7,8], construction of differential length vectors in electricity and magnetism, [9] and understanding of boundary conditions in electricity and magnetism and quantum mechanics [10].

*Corresponding author: bschermerhorn@monroecc.edu

In order to analyze the interactions between mathematics and physics, other researchers have incorporated conceptual blending [11], a theoretical framework from linguistics that describes the connection and combination of elements from separate domains of knowledge (referred to as mental spaces) into a blended domain. Conceptual blending has served as a means to describe the ways in which mathematics and physics are woven together, both at the introductory level [12,13] and upper division [14,15]. Notably, most previous adaptations of conceptual blending to discuss the interaction of mathematics and physics have not included a *generic space*, which was presented by Fauconnier and Turner as the underlying space that connects the two input spaces and determines which elements are "compressed" or blended. Previous research has instead focused on dividing concepts into "mathematics" and "physics" spaces.

Drawing on the depth of the theoretical work presented here and the various applications in physics and mathematics education literature, an equation emerges as a statement in a physical-mathematical language where meaning is embedded in the way variables and procedures are expressed in specific forms. Much in the way that the rules of writing a sentence govern structure, punctuation, and clauses, and inevitably convey a certain meaning, the way an equation is written conveys relationships between physical quantities. Because of these rules and structures in place, we are able to derive meaning from an equation and also write information into an equation.

We present a model for analysis of students' construction and interpretation of equations by connecting students' use of symbolic forms [5] with their physics conceptual understanding through the use of formal conceptual blending theory [11]. In this model, which we call *symbolic blending*, aspects of symbolic forms serve as the underlying structure for the blending of mathematics and physics, while the incorporation of conceptual blending brings contextual understanding to an acontextual symbolic forms analysis. This model makes use of the generic space to detail the mechanism of equation construction and the blending of mathematics and physics. To fully explore this theoretical model, we use data from our research in upper-division electricity and magnetism, which deals with students' construction of a differential length vector for an unconventional spherical coordinate system [9]. However, this model can be extended to analyze students' connection of structural/mathematical understanding to any physics context.

We first review the development of previous models for mathematization in physics to situate our work within the realm of physics education research on student understanding of mathematics. As a continuation of a review of relevant literature, we include detailed overviews of the symbolic forms and conceptual blending frameworks and discuss each of the instantiations of these frameworks in physics and mathematics education research. We then introduce and review previous work seeking to connect symbolic forms and conceptual blending [16].

In Sec. III, we present the proposed symbolic blending model for students' construction of equations. We argue that the aforementioned frameworks are complementary in that aspects of each framework fill analytical gaps in the other. Extending this, we present the affordances of the symbolic blending model by further connecting various analytical pieces of each framework as a means to show the scope and reach of the model. Finally, we summarize the symbolic blending model and discuss future work, in line with Sherin's suggestions for extending symbolic forms literature to account for further physics contexts and other types of mathematical representation.

II. THEORETICAL PERSPECTIVES AT THE MATHEMATICS-PHYSICS INTERFACE

The following section presents an overview of the relevant theoretical lenses for interpreting students' use and understanding of mathematics in physics as background for the development of the theoretical model described in Sec. III. The first subsection describes the large-scale models that have been developed to describe student work at the mathematics-physics interface. Section II B introduces the specific perspective of the symbolic forms framework [5] as it has been used to describe students' construction of equations as mathematical objects. Section II C. introduces the conceptual blending framework [11] as an additional means to describe the interaction between physics and mathematics. Finally, in Sec. II D, we draw attention to previous work within the literature that has used a conceptual blending framework to describe students' use of symbolic forms in physics.

A. Review of models for students' mathematization within physics

The incorporation of mathematics in physics goes beyond calculation, as mathematics plays a role in reasoning about relationships between physical quantities and in conveying these relationships with graphs or equations. Several physics education researchers have sought to describe and represent the way students incorporate mathematical concepts throughout physics. These models involve a number of common elements, suggesting key areas of mathematics understanding necessary for physics. One instantiation separated the mathematics and physics domains into two distinct spaces that students cycled between the physical system and mathematical representation [1]. This cycle involves four processes, starting with the act of modeling which moves from the physical system into a mathematical representation space (e.g., setting up an integral). This representation is then processed within the mathematical domain (e.g., integrating and possibly computing the final result). Interpretation of this new representation is a move from the mathematical domain back into the physics domain, where the result can then be evaluated in comparison to the initial physical conditions.

Uhden *et al.* developed a different framing for these processes based on additional literature that includes the idea of a blended space of mathematics and physics, with vertical levels portraying degrees of mathematical modeling, referred to as mathematization [2]. The closer to the bottom of the vertical axis the current representation is, the more grounded it is in the real, physical world. As students model the physical system by defining proportionalities, writing equations to connect variables, or using various laws, theorems, or physics relationships, the level of mathematization increases. Interpretation of these results corresponds to a lesser degree of mathematization and moving down the vertical axis. The third aspect of this framing includes technical mathematical operations (e.g., computing an integral or algebraic manipulation), where the student steps out of the blended mathematics-physics space to "pure mathematics" for the purposes of carrying out the calculation. Following calculation, the student then returns to the same level of mathematization.

A third model of students' use of mathematics resulted from work in upper-division electricity and magnetism [3]. The ACER framework places greater emphasis on the actions of the previous two diagrams, turning them into the steps by which problem solving occurs. This framework designates spaces for the "activation of a tool" (e.g., the choice of an equation), "construction of the [mathematical] model," "execution of mathematics," and "reflection on the results."

One model of a modeling cycle from the mathematics education community is from Blum and Leiß [4]. They label specific points in the cycle as well as processes and distinguish between the world of mathematics and the rest of the world. The former includes the mathematical model and the mathematical results, with the process between them being labeled "working mathematically." The latter includes the "real situation," "situation model," "real model," and "real results," with the distinction between "real" and "situation" being what is abstracted and assumed by the modeler. The process that takes the modeler from the real world to the mathematical world is mathematization. Recent work in undergraduate mathematics education has used this model to frame the actions of engineering students in a differential equations course [17].

While each of the above models represents students' reasoning with and use of mathematics in a different way, they all include features to account for modeling, calculation, and interpretation. These three areas have been the focus of research looking at the connections between mathematics and physics [17–28]. The plethora of models suggests that identifying students' interaction with mathematics in physics is nontrivial. However, the presence of common features (mathematizing, processing, interpretation) suggests these are key aspects of student understanding and use of mathematics in physics. The work presented here focuses primarily on mathematization as a means of creating mathematical representations, specifically during the process of equation construction. We further use the analysis from the construction of equations to describe students' interpretation of equations as they read information out from these abstracted representations.

B. Development and use of symbolic forms to address students' understanding of physics equations in terms of mathematical structures

Analysis using symbolic forms [5] provides a means to address student understanding of the mathematical representation used in equations. In this section, we provide an overview of symbolic forms and describe its use in the literature. Finally, an overview of the use of symbolic forms within our work is provided to lay the groundwork for the presentation of the model.

1. Overview of symbolic forms

In an effort to explore the mathematical structures students use to construct and interpret equations, Sherin [5] asked junior physics majors to solve several conceptual physics problems relating to introductory topics. Sherin found that rather than trying to derive an expression or manipulating known equations, students attempted to build equations from a sense of what they wanted to express. Students could additionally read information out of an equation based on the equation's structure. Motivated by this analysis, Sherin developed the idea of symbolic forms as a lens for the investigation of students' construction and sensemaking of equations in terms of mathematical understanding.

A symbolic form, in line with a *knowledge-in-pieces* model [6], is an element of a mathematical expression defined as a pairing of two parts. The main element of a symbolic form is the symbol template, the externalized structure of the equation. For example, $\square + \square + \square$ would be a template in which the students would place terms, numbers, or variables in order to add them. The particular associations underlying or motivating the template are what Sherin refers to as the conceptual schema. For $\square + \square + \square$, one associated schema identified by Sherin is "amounts of a generic substance contributing to a whole." Together, this template and schema are referred to as *parts of a whole*.

Sherin argues that students learn to associate meanings with structures in equations. Thus, the conceptual schemata are acontextual, meaning that they do not rely on a particular physics context, but on an underlying mathematical understanding of how the equation is written. *Parts of a whole* can be seen in a student's writing of an expression for the total energy of a system in terms of kinetic and potential energy, $\frac{1}{2}mv^2 + mgh$, or in an attempt to express the total surface area for a cylinder of radius, r, and length, l, as a sum of the end caps and shell, $2\pi r^2 + 2\pi rl$. While these equations contain drastically different variables and physical meanings, they share the symbolic structure of *parts of a whole*. Sherin illustrates *parts of a whole* through the construction of an equation by two students, Mark and Karl, around an (incorrect) idea of the coefficient of friction. The students write the expression as "$\mu = \mu_1 + C\frac{\mu_2}{m}$." Karl explains the structure of the expression:

Karl: …the frictional force as having two components. One that is constant and one that varies inversely with weight. ([5], p. 489)

Parts of a whole is illustrated by the addition of the two terms on the right side of the equation.

Sherin notes that symbolic forms can be used correctly by an appropriate pairing of a template and mathematical justification, even when students do not invoke normative physics ideas [5]. In the example above, the students invoke *parts of a whole* because it is consistent with their underlying idea that two quantities need to be added to produce a complete quantity.

Sherin labeled a closely related addition form, *base plus change*. *Parts of a whole* and *base plus change* both describe an identical mathematical operation: addition. While *parts of a whole* describes the addition of independent quantities, *base plus change*, $\square + \Delta$, is a specific case where the first term is a fixed quantity augmented by a variable second term. While this may seem to be cued primarily by a physics understanding, as seen in kinematics equations, it is also the form for the equation of a line ($y = mx + b$) and thus can be recognized in many other physics equations and connected explicitly to graphical representations.

Mark and Karl constructed an additional incorrect expression of a kinematics equation, "$v_f = v_o + \frac{1}{2}at^2$," after accounting for acceleration as a change in velocity.

Mark: 'Cause we have initial velocity [circles v_o] plus if you have an acceleration over a certain time [circles $\frac{1}{2}at^2$]. Yeah I think that is right [5] (p. 515).

As before, a student's conceptual schema is illustrated during the construction and connected explicitly to the associated structures in the *base plus change* template: one term represents the initial or "base" quantity and the other term represents some amount of change in that quantity.

Returning to Sherin's coefficient of friction example, the conceptual schema for the *parts of a whole* template is described as "seen behind Karl's statement that the coefficient of friction consists of two components" [5] (p. 491). This further supports the idea that despite an incorrect physics understanding, students can show the correct use of a symbolic form and that symbolic forms are divorced from physics understanding. For Sherin, the conceptual schemata are simple acontextual structures similar to diSessa's phenomenological primitives (p-prims), which are intuitive knowledge elements that are not learned but intrinsically held by individuals, such as "more means more," a notion that can map onto specific contexts (e.g., larger objects are heavier) [6]. While addition is certainly a learned mathematical skill, the idea is built up by years and years of association so that students arguably develop an intrinsic sense of what it means to express two quantities contributing to a whole. In this sense, the conceptual schemata of symbolic forms can be thought of as the intuitive knowledge elements through which students intrinsically understand the written structures in an equation.

Beyond this, it is important to note that equation construction or interpretation involves the invocation of several symbolic forms, which when used together carry the associated meaning of the symbols. Students' construction of an expression to describe the coefficient of friction invoked the inverse proportionality, (prop$^-$, $\frac{\cdots}{\ldots\square\ldots}$), *coefficient* ($\square\ldots$), and *no dependence* ([…]) symbolic forms [5] to express the full mathematical meaning students attached to the variables in the equation. Symbolic forms can thus be nested within each other in whatever manner is deemed necessary to convey the full meaning of the equation. In order to interpret or convey this meaning beyond reading the mathematical structures, we must tie symbolic forms to understanding of the context or the relevant extramathematical concepts, which is the aim of Sec. III.

2. *Previous application of symbolic forms in related literature*

Given the nature of symbolic forms to describe student construction and understanding of equations, it has been adopted in physics education research, as well as education research in chemistry and mathematics. Working in an electrostatics context, Meredith and Marrongelle [8] adapted the conceptual schemata of symbolic forms to account for the features of electrostatics problems that cued integration among introductory physics students. They found students invoking the conceptual schema of the *dependence* form, a symbolic form that establishes the need for a particular variable that an expression "depends on." Students invoked this conceptual schema when eliciting the reliance of an integral on a particular variable. Students invoked the *parts of a whole* idea when acknowledging the need to sum up multiple small charges along a charged rod. While Meredith and Marrongelle did not identify the invocation of the accompanying symbol templates for these schemata, the underlying ideas of *parts of a whole* and *dependence* were revealed as aspects driving students' choices to integrate.

Ryan and Schermerhorn incorporated symbolic forms analysis to address student understanding of the structure of boundary conditions in the contexts of a potential step in quantum mechanics and an electromagnetic wave at the boundary between two media [29]. Researchers discovered students invoked similar symbolic forms, starting with the *same amount* ($\square = \square$) symbolic form [5]. Subsequently, students would invoke other symbolic forms to further describe other nested aspects of the equations. The flow of students' invocation of symbolic forms as well as the added conceptual complexity to equation construction in upper-division physics is in line with our prior findings [9].

Using symbolic forms in an effort to develop theory around mathematical sensemaking, Dreyfus *et al.* address student reasoning in quantum mechanics with the eigenvalue equation for the Hamiltonian and suggest two new symbolic forms derived from an expert perspective [30]. The authors hypothesize the *transformation* symbolic form, with a template of $\widehat{\square}|\square\rangle$ and a schema of "reshaping," and the *eigenvalue-eigenvector* symbolic form, with a template of $\widehat{\square}|\square\rangle = C|\square\rangle$, and a schema of "transformation that reproduces the original," as a means to aid interpretation of student understanding of eigenvalue-eigenvector equations. The *transformation* symbolic form accounts for an operator acting on a specific state. The *eigenvalue-eigenvector* symbolic form is made up of the *transformation* symbolic form as well as a *coefficient* symbolic form to incorporate the processes of the quantum mechanical operation. In both cases, student understanding of these structures in terms of connecting mathematical sensemaking and physics concepts is an important part of upper-division quantum mechanics.

Within the mathematics community, the ideas of symbolic forms were used to analyze calculus students' ideas when making sense of integral expressions [7]. Jones identified variations in students' conceptual understanding when interpreting the various structures associated with (mostly definite) integrals given as part of the tasks. This led to the creation of several distinct symbolic forms, some of which possessed the same template to distinguish between *Riemann sum*, *area and perimeter*, and *function-matching* interpretations. Some of these forms were duplicated to account for an integral expression without limits on the integrand, while others had more varied templates to account for different types of integration, the area between two functions or integration over a physical shape. Students' exposed conceptual understandings often led to graphical representations of the given functions and the use of these graphs to explain the integration. An additional study by Dorko and Speer [31] investigated student difficulties with units in area and volume computations, noting that students associate a measurement with such a computation. They identified a "measurement" symbolic form with a template that included a box for an amount and a box for the unit, with a variant template including a box for an exponent; the conceptual schema would be "measurements have magnitude and units." White Brahmia [32] then modified this form to include a box for sign in front of the magnitude box.

Beyond mathematics and physics education research, the symbolic forms framework has been used for the analysis of physical chemistry students' use of partial derivatives in a thermodynamics context [33]. This work illustrated the ways in which students understood and applied symbolic forms of reasoning when working with common mathematical expressions in physical chemistry. In several cases, students recalled specific processes, such as that of taking the total derivative, or invoked rules or conditions, such as $dx = 0$ when x is constant. This showed the specific role of recall in mediating student construction of and reasoning about expressions when working with upper-division content, consistent with findings of analyses of student construction of differential vector elements.

3. *Symbolic forms analysis of differential length vector construction*

As part of a broader study of student understanding of mathematics and mathematical methods in upper-division physics, we identified symbolic forms appearing in students' construction of differential length vectors for an unconventional spherical coordinate system we called "schmerical coordinates" [9]. The functional difference in the coordinate systems was the change in placement

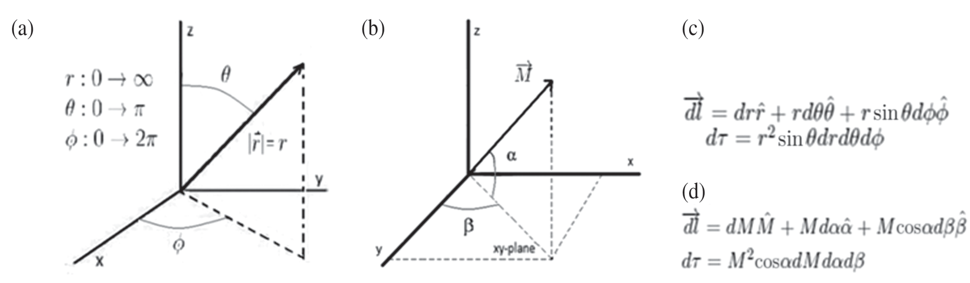

FIG. 1. (a) Conventional spherical coordinate system; (b) the "schmerical coordinate" system, an unconventional spherical coordinate system with radial component M and angles α and β, that was presented to students in the task discussed. The correct differential length vectors for each system are shown in (c) and (d), respectively. Image from Ref. [9].

of the angle measured from the z axis; we changed the angle as being measured up from the xy plane consistent with latitude measurements on a globe (Fig. 1). This change necessitates the use of cosine rather than sine when constructing the arc length associated with motion in the azimuthal direction. In this section, we provide a brief overview of the original data analysis before specifically expanding on this work to develop our theoretical model later in the paper.

Differential length and area elements appear in vector and scalar integration involving electric and magnetic fields. Due to the symmetry of physical situations such as the electric field due to a point charge or the magnetic field due to a line of current, much of vector calculus in physics uses non-Cartesian (spherical and cylindrical) coordinate systems. The development of schmerical coordinates allowed us to assess students' underlying understanding in terms of arc lengths and differential changes without relying exclusively on explicit recall of the differential quantities in spherical coordinates.

Pair interviews were conducted at two universities, facilitating more student-driven interaction with less input or influence from the interviewer. Interviews were videotaped and later transcribed for analysis. After providing the coordinate system, the pair of students were asked to construct a differential length vector in schmerical coordinates. Preliminary analysis identified students' concept images [34] associated with the differential length vector as a means to identify the specific ideas or properties that students associated with such elements. A concept image perspective was first used in mathematics education research and is similar in many aspects to resources [35] or knowledge in pieces [6]. Where a concept image is a set of related, resourcelike ideas, the evoked concept image would be the resources activated in a given context. While students focused on several key aspects, such as a need for appropriate dimensions or multiple components, other aspects relevant to coordinate system geometry were not employed by all students. With a further desire to understand the construction process and the terms with which students wrote their expressions, the secondary analysis involved the identification of symbolic forms by attending to the structures students expressed in equations and their understanding of that structure. Symbolic forms provided a lens to investigate students' structural understanding of differential length vectors as they constructed a generic differential length vector for a non-Cartesian coordinate system [9].

Our analysis identified several symbolic forms from the original literature (*parts of a whole, coefficient, no dependence*), as well as new symbolic forms that emerged due to the increased sophistication of mathematics in upper-division physics (*differential, magnitude-direction*) [9]. With the concept of image analysis in mind, we noticed that the inclusion of specific structures in student expressions sometimes resulted from an alternate understanding of a specific term; symbolic forms analysis did not account for this differing understanding. For example, Carol and Dan often motivated the inclusion of a differential as a change in a particular quantity, without reference to size.

Carol: …you have a change in your $\hat{M}$ is going to be your dM, it's your change in your M.

Elliot and Frank, however, emphasized the infinitesimal aspect of the differential, with Elliot saying "I think it's M times $\Delta\beta$, a small beta, because it is like you take r times a small θ." Elliot's initial mention of $\Delta\beta$ is accompanied by a discussion of differentials as "small" quantities.

Despite different perspectives on the meaning of a differential, both pairs arrived at the same symbolic structure. Varying representations of a differential make sense, given that literature has identified several ways in which students use and understand differential quantities [23,26,36–38], but our interpretation of symbolic forms is that of acontextual constructs. Symbolic forms were designed to account for student recognition of the need for specific mathematical structures but do not explicitly account for the contextual motivation for that structure; the context of the problem drives the choice of conceptual schema and associated symbol template. If, indeed, symbolic forms accounted for contextual analysis, they would have to be described in a way that distinguishes the structural components between different physics contexts (e.g., adding constituent energies in a conservation of energy context would require a different form than that for adding constituent vector components in a net force context). This distinction would inevitably confound analysis and obscure the understanding of the underlying mathematical reasoning for symbol arrangement and representation for which symbolic forms were designed.

A simpler depiction of how varying contexts can motivate the same symbolic structure can be seen when looking at students' reasoning about the inclusion of the scaling factors. Given the curvature of non-Cartesian coordinate systems, the differential length components in the angular directions are arc lengths. For spherical coordinates, this yields $rd\theta$ for the component in the $\hat{\theta}$ direction and $r\sin\theta d\phi$ for the $\hat{\phi}$ component. As students constructed differential length vectors, one pair recognized the nature of the component as an arc length using geometrical reasoning, while others often only reasoned about the inclusion of the radius terms as necessary to give the appropriate dimensions.

Adam: This doesn't have any units of length, so it needs to have some M term.

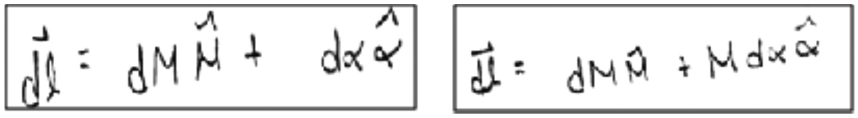

FIG. 2. Adam added "M" in his second term to include units of length when constructing the differential length vector. We identify this as the use of the *coefficient* symbolic form.

Here, Adam recognizes that the differential angle component does not have the units of length and thus fills the blank space in front of $d\alpha$ with an M (Fig. 2).

Others engaged in a third line of reasoning recognized that the coefficient box needed to be filled; but as the groups did not draw on the requisite knowledge to derive the terms, these students used a process of recall to a more familiar spherical coordinate system and mapped quantities to schmerical coordinates.

Bart: so now we have just to compare so we have r it is M, θ is α, ϕ is β.

Students in each of these groups recognized that an extra term was needed in their expressions. We identify their treatment of this space before the differential angle terms as an invocation of Sherin's *coefficient* form (□…). The associated conceptual schema describes the coefficient form as a factor or constant multiplied on the left of an expression that attenuates the value of the quantities. In the case of Sherin's coefficient of friction task, the constant, C, was added "almost as an afterthought" [5]. In our case, students reasoning geometrically can easily see how increasing the radius would attenuate the value of the arc length, while those using dimensionality express the inclusion of M as just a factor that contributes needed units to the term without explicitly accessing the underlying idea of arc length. Students using recall display little underlying conceptual reasoning, only arguing that some term needs to fill the spot because it needs to bear resemblance to an earlier problem. While each of these cases invokes the *coefficient* symbolic form, the reasoning for the invocation is distinct and not addressed with attention to the underlying mathematical schema.

Recall presents a uniquely interesting mechanism for the invocation of symbolic forms, as it sidesteps attention to the underlying conceptual schema. Yet recall of specific ideas is relevant to equation construction at the upper level [33]. Utilizing a conceptual blending framework [11], we later address the role of recall as it is connected to the students' construction of expressions or equations.

C. Connection of mathematics and physics through conceptual blending analyses

As a means to address the integration and networking of contextual ideas with students' understanding of the symbolic structures in an equation, we draw on the theory of conceptual blending [11].

1. Overview of conceptual blending

Conceptual blending originated from the study of linguistics as a way to discuss the interaction of form and meaning in the development of language and human understanding. At its most basic, it explains the origin and understanding of metaphor and analogy as they are conceived in human interaction. A conceptual blend describes the combination, or *compression*, of ideas from two distinct *mental spaces*, which often contain information connected to one's previous experiences. The result of the combination is referred to as a *blended space* where new meaning emerges.

One of the more accessible examples of blending from the original literature involves two rival CEOs in a business competition:

> We say that one CEO landed a blow but the other recovered, one of them tripped and the other took advantage, one of them knocked the other out cold [11] (p. 126).

This example represents a compression of two input spaces: the business space, which contains the CEOs and market strategies; and the boxing space, which contains two competitors engaging in fisticuffs. Each input space represents a collection of individual ideas that do not inherently belong to one narrative. It is not until we connect a CEO to a boxer or a blow to an effective business strategy within the blended space that we can make sense of "one knocked the other out cold," as the CEOs are not engaged in actual physical combat or being rendered unconscious by shifts in the market.

The typical figure presented to illustrate blending shows the compression of individual input spaces into the blended space, as well as a *generic space* (Fig. 3). The generic space provides the underlying structure to the two input spaces, identifying the commonalities within each space. Starting with an element of one input space, the generic space is selected for the connection to another element in a second input space. Once the appropriate connections are established, there is an active compression of elements (solid line) into the emergent blended space. Using this representation, we can develop a conceptual blending diagram

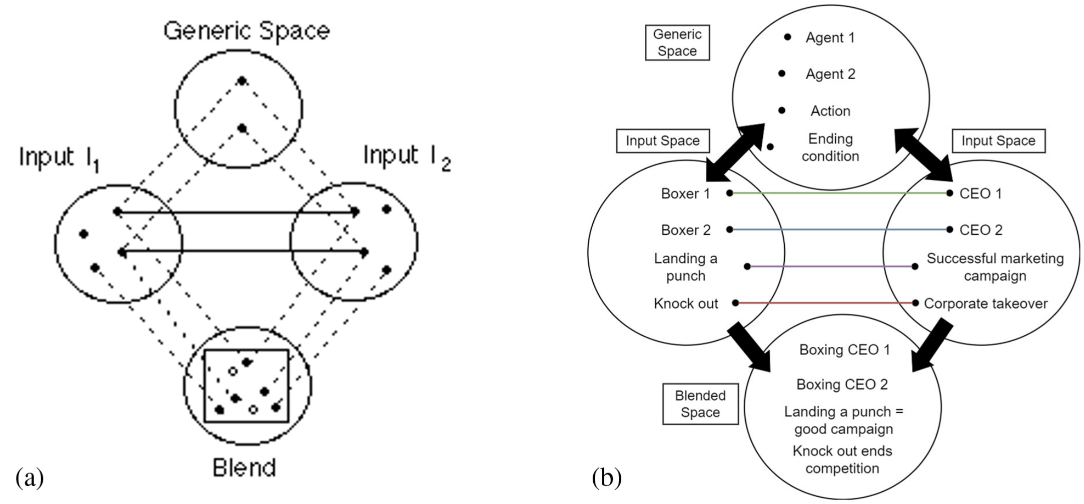

FIG. 3. Basic diagrams depicting conceptual blending. (a) Generic model of a conceptual blend. Image reproduced from Ref. [11]. (b) Model for the boxing CEO blend. Adapted from Ref. [11].

for the boxers or CEO blend (Fig. 3) with the generic space identifying which elements in the input spaces are combined. Here we see the connections laid out as the conception of boxing CEOs emerges as an amalgamation of the two different spaces.

The boxing CEOs example represents a specific type of blending network identified as a single-scope blend. In such a blend, the frame of one space (boxing) provides the organization of the blend, bringing the two CEOs into spatial and temporal proximity. The boxing input space is mapped entirely onto the business frame to provide a lens of physical combat onto the business competition. As such, single-scope blending provides the prototypical network for most conventional metaphors [11].

The other commonly cited type of network is identified as a double-scope blend, in which the organizing frame of the blended space is integrated from both spaces. Drawing on conceptual blending literature, when one describes foolish investments as “digging your own grave,” there is a conceptual blend of grave digging and foolish actions [11]. While the grave digging provides most of the framing, presenting “you” as the gravedigger and “your actions” as the shovel sinking into the earth, the causality is projected from the foolish action space since the completion of one’s grave plot does not immediately imply death within the space of grave digging. Yet the implication is emergent in the blended space, as the causality of “foolish action leads to failure” is brought into the blend. Whereas in a single-scope blend, one input space contributes the entirety of the organizing frame, in a double-scope blend, the other input space provides aspects of the structure as well, such as causation, and time and space compressions [11].

In some cases, with either conceptual blending network, *backward projection* can occur, in which the blended space provides guiding information back to an initial input space. For example, the blending of mathematics and physics ideas may provide insight into the meaning of a particular mathematics operation or physics concept [14]. In a study of student understanding of vector calculus operations in electricity and magnetism, while reasoning about the curl of a given field a student had difficulty connecting the symbolic interpretation of Maxwell-Ampère’s law to the graphical representation of the field. Bollen *et al.* [14] describe how a fluent calculation allowed the student to reinterpret the curl of the field at a given location without needing further intervention.

> This makes sense, if the changing electric field vanishes, the curl of the magnetic field should vanish as well. However, the magnetic field itself is nonzero. […] the drawing confused me at first, but now I can see that a paddle wheel would not spin here [14] (p. 7).

In this case, the students’ calculation and subsequent interpretation of the equation led them to reevaluate the nature of the physical system and arrive at the correct expression. The student then recognizes the curl is 0, by invoking the paddle wheel heuristic (a common visual heuristic used for quickly determining the existence and direction of the curl at a point). The backward projection is the use of the blended mathematics-physics space to make sense of the physics input space.

2. Previous application of conceptual blending in related literature

Given the focus of conceptual blending on providing a lens for understanding how ideas are connected and combined in the learning process, conceptual blending has been specifically brought into physics education research to explain how students connect mathematics and physics [13–16,39]. Researchers have additionally used conceptual blending to explore the use of mathematics in chemistry courses [40]. Blending has been used more generically in physics to explain the interplay of various physics principles in terms of wave mechanics [41] and energy [42].

At the introductory level, Bing and Redish [13] adapted the language of conceptual blending to discuss the ways in which students combine mathematical and physics knowledge using two examples of air drag and kinematics. In these examples, the two input spaces are defined as "mathematical machinery" and "physics world." An example of a blend here takes "positive and negative quantities" as mathematical machinery and maps it with "up and down directions" to arrive at a typically defined one-dimensional coordinate system, with "+" meaning up. In the single-scope example, the student mapped a mathematical template for equating two fractions onto the numeric values of a given velocity and distance, without regard to the physical meaning or units of the quantities. Since the student focused on the mathematical process without attention to units, Bing and Redish identified this as a single-scope blend. Furthermore, the researchers distinguish this from double-scope blending, in which a student used the signs as algebraic rules that encoded the physical direction of the forces.

Other researchers have adapted conceptual blending to upper-division physics in order to explain how students connected physics concepts in electricity and magnetism to the mathematical concepts of integration [15] and vector differential operators [14]. The blending structure used here separates out three spaces as "math notation space," "symbolic space," and "physics space." Across the students' conceptual blends, the physics space and symbolic space remain uniform lists of quantities (electric field, charge density, etc., for the physics space) or equations (e.g., $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_o$ in the symbolic space) [14]. The only variation in the blends comes from the mathematics notation space, which accounts for the ways in which students understand or express mathematical concepts of integration, differentials, or divergence of a vector field. By separating out various realms that function together to establish a student's conceptual understanding, these results demonstrate several cases where a student's conceptual understanding of an equation or mathematical idea leads to an incorrect response.

Blending has been further incorporated into the study of chemistry students' connection of a mathematics space and a chemistry space in the context of chemical kinetics at both the introductory and upper divisions [40]. Chemical kinetics involves mathematical ideas such as graphs, algebraic expressions, rates, and exponentials to describe various reaction orders of chemical reactions. The researchers categorized students into groups based on how often, or whether, the students blended mathematics and chemistry. Students were labeled as high-frequency blenders, low-frequency blenders, or nonblenders. High-frequency blenders were able to succinctly express the relationship between algebraic expressions and graphs, while nonblenders were unable to connect the relevant mathematics to the associated chemistry. Additionally, researchers found that students receiving a mathematics prompt prior to the prompt within a chemistry context were more likely to engage in the blending of mathematics in chemistry when compared to students who were given a chemistry context first.

Wittmann adapted conceptual blending to explain the origin and intricacy of introductory students' emergent conceptualizations of wave pulses with intuitive ideas related to throwing a ball [41]. Depending on the aspect of the physical system that students attended to, he identified a "wave-ball blend," where a faster flick corresponds to faster movement in the way a harder throw means a faster ball, and a "beaded-string blend," where the nearest-neighbor interactions are responsible for pulse speed. The blends here are depicted with concise, one-to-one compressions by connecting elements directly between input spaces and then subsequently to an element representing the blend (e.g., "wrist flick" from the "observed wave input" and throw from the "ball input" to a "wave-toss"). This representation is similar to that in work depicting the integration of location and substance metaphors for energy into a coherent picture of "absorbing energy makes things go up" [42].

None of the examples presented here make use of the generic space described in the conceptual blending literature. For the latter two examples, the generic space is arguably tacit and redundant (as in the boxing example): the compressions of the two input spaces are concise in that elements that share analogous aspects in other input spaces are explicitly connected by a dotted line (representing a compression in the original blending literature [11]). In the examples connecting mathematics realms to physics realms, the input spaces represent three distinct areas from which students draw knowledge without structure or connection among the input spaces. Without the generic space identifying connecting elements, the active nature of a student's blending process is obfuscated. We argue that the generic space, or depiction of compression, is necessary to the invocation of blending, especially in cases where the blending is not clear and students' combination of ideas is unclear from a conceptual standpoint in order to highlight underlying ideas that drive the compression of two elements.

D. Previous representations of symbolic forms as elements of a conceptual blend

Recognizing the role of symbolic forms in the construction of equations within physics, Kuo *et al.* framed symbolic forms as a conceptual blend of the symbol template and conceptual schema [16]. They addressed students' qualitative reasoning or "processing" of equations by presenting two contrasting case studies in which students interpreted the kinematics equation $v = v_o + at$. While one student reasons formulaically, the other is said to engage in a blended process of mathematics and physics as they interpret the mathematical structure of the equation in terms of the physical situation.

The authors then discussed students' reading out or failure to read out a *base plus change* symbol template, $\square + \Delta$, from the given equation and connected this to students' responses to the second prompt. One of the students, "Pat," explained their reasoning as follows:

Pat: Because I mean, if you look at it from the unit side, it's clear that acceleration times time is a velocity, but it might be easier if you think about, you start from an initial velocity and then the acceleration for a certain period of time increases or decreases that velocity. [16]

Pat's attention to the "*at*" term as changing the velocity is the key aspect of the reasoning that evokes the *base plus change* formalism.

The authors identify this as conceptual blending of the symbol template and conceptual schema of the *base plus change* symbolic form, labeling the change in velocity as the underlying conceptual schema. According to the original use of symbolic forms, though, they are acontextual: the conceptual schema of *base plus change* only accounts for the summation of terms in which "one is the base value; the other is a change to that base" [5]. Ideas of velocity and acceleration are not included in the definition of *base plus change*. It is only through an understanding of the physics principles in the task that we recognize that the context for the base and change is velocity and acceleration. Depending on how the relationship is presented, this in turn shares the same underlying conceptual schema as the Δ in the $\square + \Delta$ template. With introductory physics concepts, a symbolic form's conceptual schema and physics conceptual understanding can often be closely related; however, the conceptual schema is not the content idea itself, but the underlying mathematical expression of the idea.

Likewise, the *parts of a whole* template appear in equations when there is a need to add components of a larger quantity together. As an argument in semantics, this does not stipulate why such quantities need to be added. In another example of blending, Kuo *et al.* present the conceptual schemata of *parts of a whole* with an example of how guests at a wedding belong to multiple groups: close relatives, close friends, business contacts, and others [16]. The idea that wedding guests can be split into various groups that can be summed to give the guest list is a property of the wedding in the same way vectors can be represented as a sum of components. In both cases, the conceptual schema appears buried within the property of the target quantity but is defined by neither. The schema, "substances contributing to a whole," is acontextual so that it may be applied across multiple physical laws. The representation of vector quantities using equations is related to the mental integration of the properties of vector quantities with the appropriate mathematical template, while guest lists for weddings are (typically) devoid of mathematical symbols—but *parts of a whole* is applicable as a conceptual schema for both situations. This distinction between context and underlying schema is the essence of what drives the theoretical work in this paper.

Kuo *et al.* suggest symbolic forms as an act of conceptual blending to illuminate the way students work at the mathematics-physics boundary but provide little attention to the actual blending process or the associated formalism since this is not the focus of their work. As a result, an underlying structure to the blend is not addressed. In the model that Kuo *et al.* present, blending is adopted as a broader process. This leaves room for deeper interpretation and further efforts to connect students' conceptual understanding to symbolic forms in general.

In the next section, we present an argument as to why a symbolic form is not a full blend in and of itself. We address the missing analytical aspects in previous literature, such as the underlying generic space, and provide theoretical structure for how blending occurs when constructing equations. In particular, we present students' interpretations of equations, such as in the task presented by Kuo *et al.*, as an act of backward projection rather than of forward blending.

III. SYMBOLIC BLENDING: STRUCTURING THE ROLE OF SYMBOLIC FORMS IN A CONCEPTUAL BLEND

In the same way conceptual blending was used to attach meaning to form in the development of language, our goal for the analysis of differential length vector construction was to connect *contextual* meaning (understanding) to symbolic forms as students develop equations. Writing an equation in physics is a blend of mathematical representation and measurable or quantifiable entities. There is a need to understand both the physical system or variables and the mathematical representations. In the analysis of student work, these mathematical relationships appear as symbolic forms.

The literature utilizing symbolic forms often bypasses the contextual reasons for invocation by equating the student's mathematical understanding of the expression with the understanding of the physics content, such as the ideas of velocity and acceleration describing the *base plus change* symbolic form in the previous section. While a strict symbolic forms analysis reveals students' structural understanding and associations related to the mathematics context, it only tacitly, if at all, attends to the students' understanding within the context that dictates the need for the specific form. That is, the content basis for choices made as to the symbolic arrangement of expressions is neglected within the formal theory.

In this section, we present the symbolic blending model for equation construction that situates the two aspects of symbolic forms as two spaces within a conceptual blend. We present supporting examples of this model in the context of earlier work investigating students' construction of differential vector elements in upper-division electricity and magnetism [9]. Upper-division physics provides several affordances in regard to parsing students' conceptual understanding and expression of equations. By the time a physics student has entered the upper division, they have encountered and used symbols for addition, notation for vectors, and calculated numerous integrals and derivatives in both mathematics and physics courses. Therefore, we expect that the symbol templates used during the construction of a differential vector equation are fairly ingrained in what we could call a student's conceptual toolbox. We can then think of the construction process as a blending of these templates with physics understanding rather than a spontaneous creative process.

We show the affordances of symbolic blending in terms of other analyses, both from our own work and in previous literature. We further elucidate the importance of the generic space in conceptual blending in terms of accounting for variation in conceptual understanding. We also show how such a model can account for variations in structural representation for the same conceptual information. Next, as symbolic forms can also be used to describe students' "reading out" of equations, we focus discussion on the role of recall and backward projection in symbolic blending and show how this can also contribute to equation construction. Finally, we elaborate on the utility of this particular model in interpreting when students are struggling with constructing equations by isolating difficulty to either structural or conceptual understanding.

A. Proposed model of symbolic blending

Possessing some level of conceptual understanding, students can express their understanding of a physical situation as an equation, choosing from various symbolic representations. With symbolic forms to account for mathematical structure, two large input spaces appear. One of these spaces includes a selection of mathematical representations, which we identify as the symbol template piece of Sherin's symbolic forms. The remaining input space contains the sum of students' content understanding regarding a specific topic, including associated variable representations. For simplicity, we consider the use of a variable to be identical to the content idea it represents to allow for a smoother depiction of physics ideas and equation construction. We believe that most upper-division students are more like expert physicists and have more familiarity with treating a physics concept and the variable used to represent it as one and the same. As students combine aspects of these input spaces, the equation is constructed. The equation is then a sentence in a physics-mathematics language given form by the understanding of mathematical relationships and meaning because of the physics conceptual understanding. This leads to a final representation or emergence of an equation within the blended space.

In the symbolic blending model, given how the conceptual schema was designed to describe the justification for the mathematical structures of an equation in symbolic forms, we identify it as the underlying generic space in a conceptual blending framing of students' construction of equations. The conceptual schema is preserved as the underlying mathematical schema of a template but now also appears as the generic essence of students' ideas. For example, the underlying idea behind some total energy as being a sum of both kinetic and potential energies is the addition of two parts to make a whole. When the underlying meaning of the idea that the student wants to express aligns with the conceptual schema of the symbol template, the elements from the physics space and the template space are blended together in an equation. The conceptual schema, as the generic mental space, provides the structure for connecting the template to the context—the quantities being mathematized.

As in our prior example, we could also write an equation for the surface area of a cylinder. This second example is also fundamentally a sum of multiple parts to equal a whole. Defining the generic space as the conceptual schema allows us to account for students representing identical structures in equations that represent different contextual information. We discuss the deeper role of the generic space in the next section.

The mapping of symbolic forms and contextually relevant knowledge onto conceptual blending creates the format for a blending diagram that can later be used to parse students' construction of equations. Figure 4 shows the model of symbolic blending; Table I provides abbreviated descriptions of each space. Blending of this sort, involving the connection of physics and mathematics, can take either a single- or double-scope form. The distinctions are discussed by Bing and Redish [13], who present two cases discussed in Sec. II. C. 2., one in which the mathematics structures the physics and another where mathematical and physical statements provide structure to one another (e.g., $+/-$ signs behave as algebraic rules but also convey physical meaning). Interpreting this model into work with symbolic forms means in some cases the conceptual understanding may entirely drive the construction of an equation (single-scope). In other cases, the symbol template may also serve to guide students' incorporation of physics ideas into an equation (double scope).

The final equation is the product of the blend. Similar to the statement "the CEO knocked out his competition," which only makes sense in a space where business and

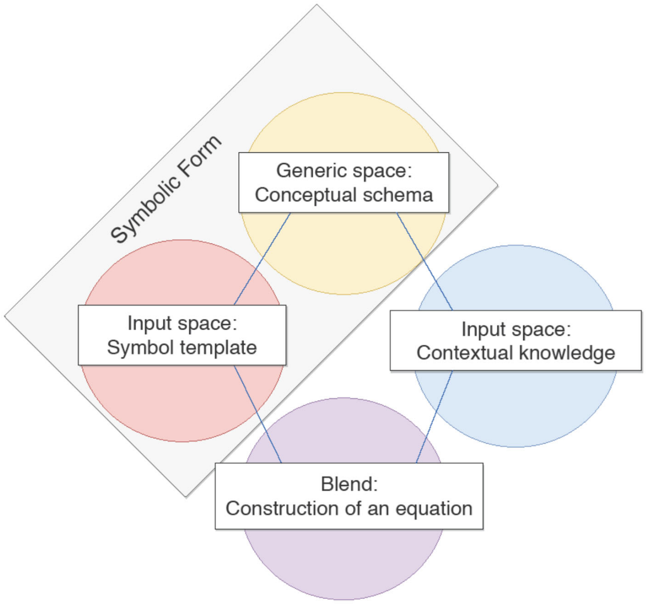

FIG. 4. Symbolic blending model for student equation construction and interpretation that incorporates symbolic forms elements. The contextual knowledge to be expressed as an equation forms one input space. The symbol template is a second input space, while the conceptual schema is the underlying generic space. The resulting blended space is the represented equation.

boxing are blended, a contextually relevant equation only has interpretable meaning when there are symbolic and contextual spaces from which to draw information.

This combination gives a focus on content knowledge to extend symbolic forms, in a way that students' varied conceptual understanding can be tied to explicit representations in their equations. This allows us to look at the physics context as the justification for the representation of terms, which is independent of the structural focus of a symbolic forms approach.

As noted earlier, the generic space, which further structures the blending of elements within the input spaces, has typically been absent in previous analyses of students' blending of mathematics and physics. The incorporation of symbolic forms establishes this underlying structure for the blending of mathematics and physics in terms of constructing and interpreting equations.

B. Supporting examples

To illustrate the symbolic blending model, we turn to earlier work discussed in Sec. II B 3, where we applied symbolic forms in upper-division electricity and magnetism [9]. The addition of an input space to address contextual knowledge allows us to deepen our analysis by looking at how the aspects of the context motivate the creation of the equation and specifically, the selection of different symbol templates used by students when constructing differential length vectors. As an example of how students blend contextually relevant information with symbolic representation, consider a pair of students, Elliot and Frank, as they started their construction of a differential length vector for schmerical coordinates.

Frank: Yeah, so like there, dl, there are three different dl's. There is dl with respect to M, dl with respect to α, and dl with respect to [β].[construct each component individually]

Elliot: You sum them, so it is those added together.

Elliot and Frank focused on the component nature of a vector; specifically, on the idea that a differential length vector has three components for each of the three coordinate directions. They further recognize that these components (or parts) need to be summed to completely represent the full vector. Likewise, students associated each component with a given variable direction, which is expressed in the final magnitude-direction pairing of a vector. Elliot and Frank articulated the "with respect to" later as they specifically stated things like "now you're going to have a length component in the beta-hat direction." With a symbolic forms perspective, observations of students' written work and discussion of the expression reveal two main structures: *parts of a whole* to account for

TABLE I. Description of each space in a conceptual blending [11] framework as it is used in our symbolic blending model. The two components of a symbolic forms [5] framework, the conceptual schema and symbol template, take the role of the generic space and an input space, respectively. The additional input space contains the relevant contextual knowledge that a student brings to the construction of an equation.

Conceptual blending space	Role of space within the model
Generic space	Conceptual schema from symbolic forms [5]: underlying mathematical justification of the symbol template. Provides framing for which elements in the associated input spaces are connected.
Input space 1	Symbol template from symbolic forms [5]; includes students' externalized structural representation of equations.
Input space 2	Contextual knowledge to be expressed. Geometric, physical, etc. meaning used to provide context for the mathematical structure. Informs which schema and template to use in other input spaces.
Blended space	Final expression or equation; carries condensed conceptual information from the combination of input spaces. Individual elements are compressed to form new meaning in the blended space.

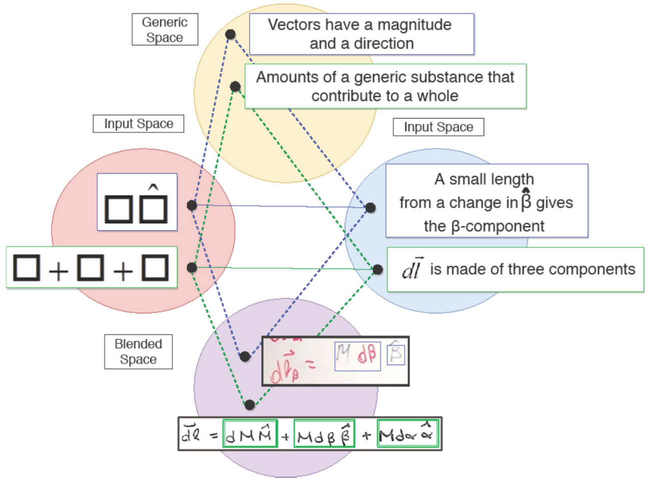

FIG. 5. Symbolic blend for Elliot and Frank. They first constructed the individual components using *magnitude-direction*, then added each component together, since the differential length vector included multiple terms for each dimension.

students' addition of the multiple components and *magnitude-direction* to account for the specific instantiation of the vector notation [5].

We argue that these specific combinations of conceptual knowledge and symbolic representation can be treated as a conceptual blend of the two domains since they result in the construction of complete or partial expressions, which only have meaning when understood through both of the initial input spaces. Figure 5 shows the blend for this student pair. The generic space corresponds with the conceptual schema of symbolic forms. In the symbolic forms literature, the conceptual schema of "amounts of a substance contribute to a whole" is seen as behind the use of the template [□ + □ + □]. As before, the conceptual schemata of symbolic forms are the underlying *mathematical* understandings of those external structures. In the contextual knowledge input space, we see that "amounts being added to make a whole" is also associated with the understanding of the vector component property of a differential length vector. This understanding of three-dimensional vectors is connected to the symbol template through the generic space and the two are then compressed in a conceptual blend resulting in an equation that depicts the summation of individual components of the differential length vector. Put another way, combining the knowledge that a vector in three dimensions can be represented as three magnitude-direction pairings pursuant to the coordinate system ($\hat{\alpha}$, $\hat{\beta}$, and $\hat{M}$ in schmerical coordinates), with the understanding of the template for the addition of substances that contribute to a whole, results in a final equation that is the sum of vector components.

While the previous example depicts Elliot and Frank's broader characterizations of the differential length vector, this model for conceptual blending can be mapped onto students' processes of construction, connecting the pieces of the template to the physical reasoning and discussion as the template is filled out.

In a second example, Carol and Dan begin their interview by calling forth the need to have three unit vectors to account for each component and leave space between each to fill in the magnitudes.

Carol: So, we're going to have, um, we're going to have $[\hat{\alpha}]$, $[\hat{\beta}]$, and some $\hat{M}$. That's what we usually do and then they each need to be a length (boxes each component with hands). You need a length vector… This is, there is going to be a plus here.
Dan: (writes M with the $\hat{M}$)
Carol: Okay, yup, so some M in the $\hat{M}$. Isn't this dM?
Dan: Yeah, because it is dl, yup.
Carol: Right. So, you have a change in your $\hat{M}$ is going to be your dM, it's your change in your M.

Following the structuring of their differential length vector, Carol articulated that each component needed to be a length and then curved her hands into a parenthetical shape to isolate each magnitude and unit vector pair. This cued Dan to write an M in the space before the $\hat{M}$. In terms of symbolic forms, they attended to the *magnitude-direction* template nestled in the *parts of a whole* structure and identified that each needs to contain an element of length (Fig. 6). Carol emphasized the existence of structure of this template at this moment by articulating "yup, some M in the $\hat{M}$." Once again, we see where the contextual need for expressing multiple components is blended with a symbol template for *parts of a whole*. For Carol and Dan, this results in written expression that holds space for future variables.

While Carol and Dan did not elaborate on the specific underlying reasoning as they hybridized the *parts of a*

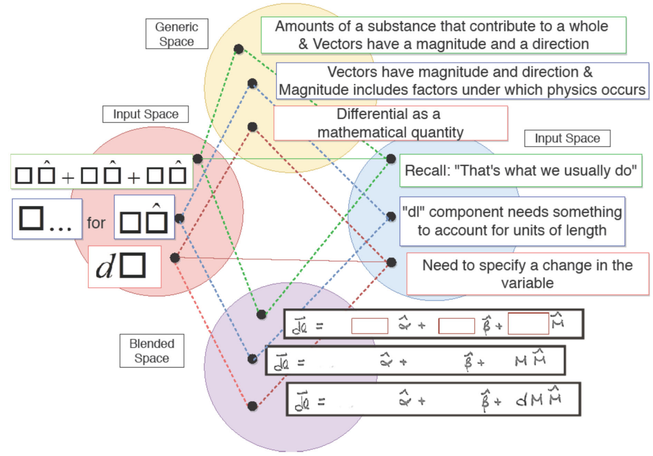

FIG. 6. Symbolic blending diagram for Carol and Dan as they began construction of the differential length vector. They started with a combined *parts of a whole* and *magnitude-direction* template. Next, they used the idea of length to fill in the magnitude term before they represented it as a differential.

whole and *magnitude-direction* symbolic forms, the statement “that’s what we usually do” suggests a level of recall moderating the construction. This is an idea consistent with previous blending literature, which argues that invoking forms together, rather than each independently, is not unexpected for upper-division students [15]. Using this perspective, it also makes sense that Carol’s and Dan’s dual invocation was accompanied by a level of recall. The students had become familiar with these quantities and representations to a specific extent and they believed they recognized how the differential length vector needed to be structured. Here, Carol and Dan were correct with the structures that they had carved out from memory. While recall has been shown to mediate students’ construction of equations and use of symbolic forms [33], here recall played a role in the students’ contextual input space (Fig. 6). Students accessed the underlying mathematical understanding of vectors needing multiple components through this recall and blended the requisite elements of the coordinate system with the symbol template. Had the students been asked to elaborate on why they had written the expression this way, we can imagine that they would say something similar to what Elliot and Frank said above. The further role of recall in symbolic blending will be discussed later.

The next step for Carol and Dan was to add an M before the $\hat{M}$ terms to account for terms of length. At this point, Carol and Dan are attending the magnitude portions of the *magnitude-direction* symbolic form and including an M in the magnitude portion of the template (Fig. 6). Emphasis on dimensionality in other places in the interview appeared as an invocation of the *coefficient* symbolic form [9]: pairs of students were building angular components and recognized that a differential angle did not carry the needed units of length, as Adam identifies below.

Adam: …This doesn’t have any units of length, so it needs to have some M term.

These represent manifestations of the *coefficient* because students explicitly argued that something else needed to be included just to account for the units of length. With the *coefficient* form representing a constant or static factor that “defines the circumstances under which physics is occurring,” [5] we can see the placement and treatment of M within this light. In a blending diagram, this would show up in the contextual input space. In the case of the angular components, M is a constant radius at which the differential length would be traced out in an angular direction.

When considering motions in the $\hat{M}$-direction, the variable M is no longer static but needs to account for variation in the length of the coordinate vector. Carol and Dan invoked a new symbolic form of representation upon recognizing this. They represented this as a dM, as the differential length vector component in the $\hat{M}$ direction

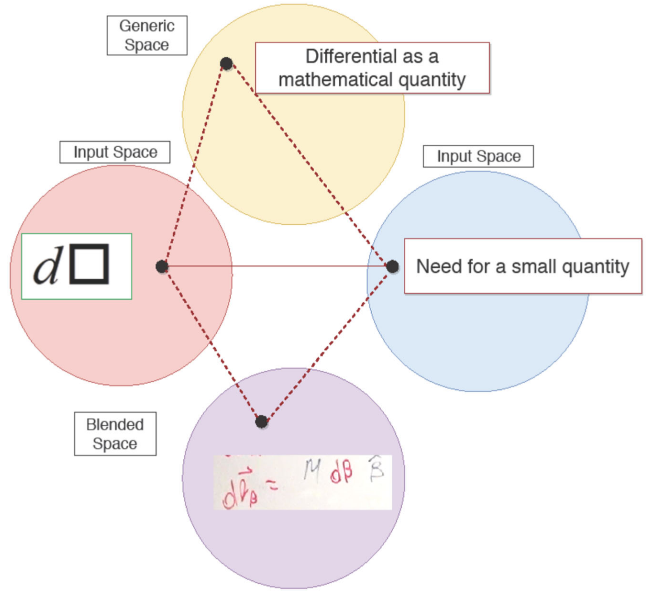

FIG. 7. Symbolic blending diagram for Elliot for the *differential* template with varied conceptual understanding. Here Elliot used the differential as a small quantity rather than a change in a quantity.

needs to account for the change in M. This is represented in the blend with the invocation of the symbol template $d\square$ for the variable M (Fig. 6). The differential concept image aspect and *differential* symbolic form identified in previous work [9] go hand in hand.

The conceptual blending template now allows the connection of these two ideas from different theoretical lenses and dually allows us to model variations in students’ ideas related to the differential. For example, Elliot and Frank invoked the *differential* symbolic form but did so by attending to the infinitesimal nature of the differential. Rather than a contextual space that focused on a change in a variable, the context driving the use of the differential template was the need for a small quantity (Fig. 7):

Elliot: So, it’s M times some Δ. I think it’s M times $\Delta\beta$, a small β.

The pairs CD and EF invoked the differential with “change in quantity” and “small quantity,” respectively. While both interpretations are appropriate in the given context, we consider these distinct evoked concept images, consistent with the varied treatment of the differential between disciplines and the different meanings that students attribute to a differential quantity [7,23,24,36]. Given the variation, we identify the conceptual schema or essence as “differential as mathematical quantity.” The connection of multiple contextual concepts to the same symbol template highlights the importance of including the generic space, which is discussed in greater detail later as an affordance of our model.

The last of the symbolic forms identified in this study was the *no dependence* form, which accounts for the absence of a variable or quantity in an expression after a student explicitly dictates that the expression is independent of said quantity. This appeared in two sets of interviews, where students attended to components in the angular directions. When constructing the $\hat{\alpha}$ component, Adam and Bart correctly decided that the term should not include any aspect of the other angle. This *no-dependence* form appeared because of a comparison to the $\hat{\beta}$ component, which scaled with the placement of the azimuthal angle.

Adam: (sweeps arm vertically) For [motion in] α, it doesn't have any dependence on this other angle.

As with the other symbolic forms, we can now elaborate upon students' use of the *no dependence* form and connect it explicitly to student reasoning about the geometric motions using conceptual blending (Fig. 8). Again, Sherin's conceptual schema "a whole does not depend on a quantity associated with an individual symbol" takes the role of the generic space. The whole is the α component nested inside of the summation of components. Then Adam's explicit exclusion of a β term in the α component can be compressed with the symbol template that shows the absence, because of the connection through the underlying schema in the generic space.

Importing a conceptual blending framework allows for a sense of the contextual mechanism through which symbolic forms are activated as students make sense of the mathematics used in physics. A depiction of a deeper understanding of physics and mathematics concepts, as well as the relationship between them, emerges, which is essential for students in upper-division physics.

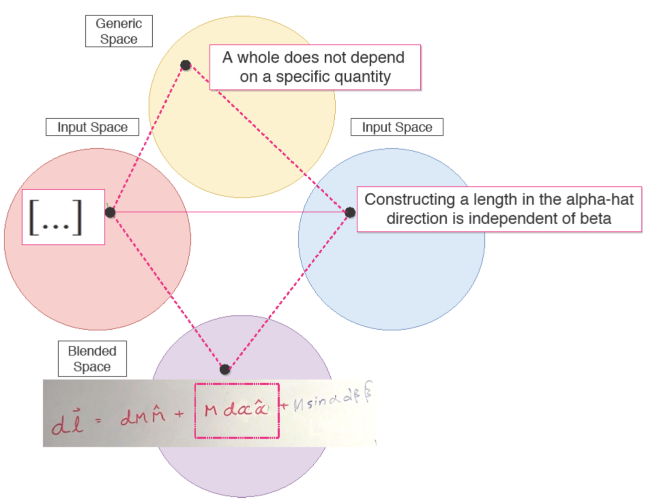

FIG. 8. Symbolic blending diagram for Adam and Bart including the *no dependence* symbolic form, with which they acknowledged the alpha component does not include a beta term.

The introductory kinematics context described earlier involved connecting acceleration to changing velocity, which could be interpreted as an incomplete understanding of the definition of acceleration. As such, it is difficult to distinguish between the conceptual schema of "change in base quantity" and the contextual understanding of "acceleration as a change in an object's initial velocity." By distinguishing these two quantities as different within our model, we recharacterize the work of Kuo *et al.* [16], which treated blending as between the two components of symbolic forms (conceptual schema and symbol template), as instead exploring students' blends of contextual understanding and symbolic expression. In this section, we have clarified and fully articulated such a model by representing the conceptual schema as the generic space in a blend of contextual understanding and symbol template input spaces.

In physics, the expression of an equation often involves a substantial understanding of physics concepts; in upper-division physics, both the mathematics and the physics are generally more sophisticated and/or abstract. Expressions of vector calculus connect to various coordinate systems, vector fields, and charge or current distributions, which are built into students' expressing of equations and in turn can be interpreted from the expressions. As shown above, variations in students' conceptual understanding of quantities, such as the differential, are now present. The symbolic blending model presented here accounts for such variation by separating the conceptual schema and contextual conceptual understanding in the analysis of students' in-the-moment construction of equations, which becomes increasingly important to develop an understanding of students' work as they move beyond algebraic contexts to those that require scalar and/or vector calculus.

Dreyfus *et al.* conjecture that there are "new" symbolic forms that are "a step farther removed from physical interpretations" [30] (p. 9) than Sherin's original symbolic forms and are necessary for mathematical sensemaking with advanced physics content such as quantum mechanics. Our work supports, and even formalizes, this notion of symbolic forms being removed from physical interpretations, especially at the upper division; however, we disagree with their characterization of Sherin's original forms and maintain that these forms, even at the introductory level, had a similarly separated relationship with the physical world that was underappreciated by subsequent literature. We hope to demonstrate this throughout the remainder of this work.

C. Affordances of the symbolic blending model

We identify some key benefits of the symbolic blending model throughout the following section. This model acknowledges that contextual meaning is not part of the symbolic form, but a part of a blend of mathematics and physics, consistent with literature on the blending of mathematics and physics [12,43]. The affordances outlined in this section provide additional insights and considerations for future researchers looking to explore the blending of mathematical representation and science contexts.

First, the inclusion of the generic space allows us to disentangle the justification for the template from the contextual knowledge. We review prior research using symbolic forms through the lens of symbolic blending to illustrate the role and constancy of the generic space as the applied contextual knowledge changes. This aspect of the model allows future researchers to incorporate symbolic forms analysis in complex contexts where multiple meanings are associated with a single representation. Second, we show how the symbolic blending model provides a mechanism for rote knowledge recall. Rather than recall being a piece of what happens, the symbolic blending model documents how recall influences equation construction. Finally, symbolic blending provides a way to explore errors in equations when the contextual understanding is technically correct. Rather than errors being delegated to context or template, symbolic blending can be used to illuminate errors within the generic space that connects the context and template.

1. The role of the generic space

The underlying connections outlined by the generic space drive the compression of the input ideas and the emergence of the blend. To analyze how students engage in the construction of equations, the conceptual schema of symbolic forms becomes the generic space. Just as before, it is important to note that Sherin's conceptual schema is not a stand-in for physics conceptual understanding. This is even more true in the upper division, where students' conceptual understanding pertains to more sophisticated mathematical and physical ideas.

In order for a conceptual schema that underlies a symbol template to occupy the generic space in a conceptual blend, it must also be consistent with the student's contextual knowledge or understanding. In line with Sherin's depiction of the underlying conceptual schema as consistent with phenomenological primitives [6], we see the conceptual schema as the fundamental "behind-the-scenes"[5] understanding of the conceptual input of the blend. We elaborate upon this by returning to the discussion of varying concepts being attached to the representation of a differential element $d\square$. By the time students make it to upper-division physics, the differential has become a fundamental quantity involved in everyday calculation, but the meaning of the quantity can vary: as Carol and Dan worked on constructing their differential length vector, they only referenced the differential as a change in a quantity, while Elliot and Frank were mostly focused on the size of the quantity, invoking the differential as part of a need for a small bit of a variable.

Other research has identified other ways in which students treat or conceptualize the differential in electricity and magnetism: as a small amount, a dimensionless point, a cue to integrate, and an identification of what to integrate with respect to [23,38,44]. Investigations of calculus students' interpretations of integrals revealed interpretations of the differential related to the width of a Riemann rectangle, shape in space, and "way to obtain the original function" [7]. Small quantities or changes are often the more prevalent understanding of the quantity for students using mathematics in physics problem solving [21,23,36,45] but that does not prevent the other ideas from appearing in physics students' problem solving. From a concept image perspective [34], students can have multiple ideas about a certain topic that may vary from the formal concept definition. Then depending on the context, students may evoke some of the aspects of the concept image over others.

In the sense of a symbolic form, the box of the template for the differential is not large enough to encapsulate the entirety of those ideas. Instead, we put forth that there is some underlying conceptual schema, a fundamental essence of a differential, that exists beneath these ideas. This idea is consistent with Sherin's association of the conceptual schema with p-prims. However, this essence becomes difficult to define, given the difference in conventions and pedagogical emphases between disciplines. For the sake of this work, we describe the conceptual schema as "differential as a mathematical quantity" so that it may be extended to multiple conceptualizations for the given context. This schema is broad because the context specifies the meaning of the differential.

Isolating Sherin's conceptual schema in such a way now allows a reengagement with prior literature utilizing symbolic forms without detracting from the value of that work. Meredith and Marrongelle [8] originally identified the conceptual schema of *parts of a whole* and *dependence* as cues for integration. Our conceptual blending model of equation construction and interpretation identifies these cues as the underlying mathematical understanding of the generic space connected to students' contextual conceptual understanding, not necessarily the conceptual schema of an invoked template, since students would likely use a different template when constructing integrals.

Separating the contextual and symbolic input spaces using conceptual blending also suggests a different definition of what Jones identified as integration symbolic forms [7]. Jones presents several of these forms, most of which are built from the same symbol template, $\int_{\square}^{\square} \square d\square$. We argue that what Jones actually defined were not different symbolic forms using the same template, but different conceptual blends for the same generic (definite) integral symbolic form. In addition, we believe one or more of these templates for integrals (e.g., $\int_{\square}^{\square} [\square - \square] d\square$) are rather a nesting of smaller units of symbolic forms, in the way that students often combine multiple templates to express more complex physical relationships among numerous quantities. The way that Jones distinguishes what he referred to as symbolic forms is via the contextual, conceptual meaning of the integral elements—typically but

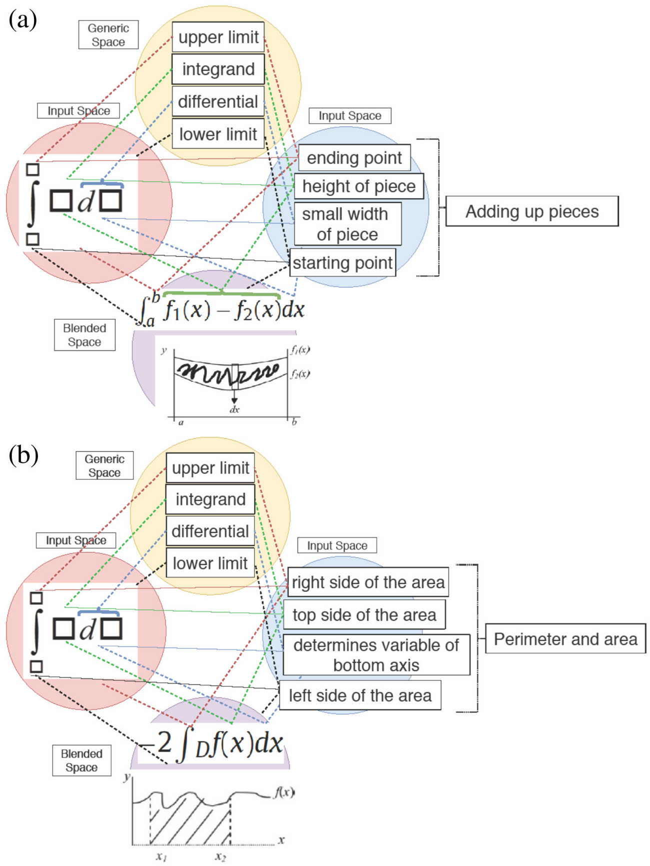

FIG. 9. Interpreting symbolic forms for integration using conceptual blending, based on Ref. [7]. Depending on the conceptualization of integration by the student pair as either (a) adding up pieces or (b) area and perimeter, students attached different meaning to each part of the integral template.

not exclusively involving graphical representations—rather than their generic meaning.

We argue that the conceptual schema, as defined in the context of symbolic forms, for the definite integral symbol template is the same for all of Jones's "forms" and serves as the generic space for the blend (see the examples in Fig. 9). In this case, rather than there being multiple symbolic forms tied to the same symbol template, each of Jones's "forms" is described here as the manifestation of a single symbolic form (*definite integral*) blended with distinct contextual meaning. The role of each box in the symbol template is the same in all these blends, e.g., the box at the top of the integral sign is always the upper limit, representing the highest value involved in the integral. Much like the ideas associated with the differential, the ideas of adding up pieces, adding up the integrand, perimeter and area, and function matching—which all utilize the same definite integral template—are now multiple departures from a more representational understanding of what the arrangement of symbols within the integration means.

In Jones's study [7], Devon employed an "adding up pieces" model, treating the integral symbol as a cue to accumulate small pieces from one end of the function to the other [Fig. 9(a)]. Devon stated the following as he wrote out his integral:

Devon: I would imagine it as, you slice it [draws thin rectangle], like very small pieces and each of them is dx. And this part here [puts finger along the height of the thin rectangle] is the, is this part right here, [points to $f_1 - f_2$ inside] [7] (p. 126).

Within the same study, Chris is described as having the "area and perimeter" model as he discussed an integral that was given to him.

Chris: So if you want to draw a graph [draws axes], um, we have f of x [draws squiggly graph above the x axis] and then since we're saying over the domain D….so we can assume D is from some point x_1 to some point x_2….And then we'd take the integral from x_1 to x_2 [said as he shades the area] [7] (p. 128).

Despite starting with an integral that used D to represent a domain, Jones used a symbolic form with limits as Chris isolated specific positions x_1 and x_2 for his diagram [Fig. 9(b)]. For the purposes of drawing comparison, we can imagine that the integral did indeed include x_1 and x_2 as the limits of the integrand.

Analyzing Jones's data with the goal of incorporating symbolic forms into a conceptual blending model means that, for Devon, the upper and lower limits of the integral template correspond to the starting and ending points of his summing of "small pieces" (Fig. 9(a)]. For Chris, Jones argued the limits are the left and right sides of the area that he created [7]. The empty box before the differential is the height of a small piece for Devon and thus the difference between the two functions but is described as the "top of this fixed region" for Chris. Symbolically the integrals carry the same components, which have underlying mathematical definitions. Yet, depending upon which concept image a student attaches to integration, the meaning of the components of the expression change.

The analysis of Jones's work within the new model raises further points of interest for our model. In one case, Devon constructed the integral representation from a graph they drew, while in the other, Chris was interpreting a written integral and constructing a graph. This raises questions about the role of graphs as students construct integrals by connecting their understanding to graphical representations. Just as with equations, graphs represent another means to relay conceptual information. Since the information is encoded by certain structural features of the graph and associated functions (i.e., axes, slopes, heights, concavity) and that those features relay specific conceptual information (i.e., how physical quantities relate), we argue that graphs are also elements of the blended space and result from the combination of conceptual understanding and symbolic, graphical representation. Devon read out the inherent meaning of the physical height of the graph and then connected the functional representation to the integral template to form a new blend in the form of an equation. Chris, likewise, interpreted the structures of the function to gain a conceptual understanding of the equation and then blended it with graphical structures as he drew his representation. The discussion of graphical representations here is a simplistic overview of the work conducted here; there is room for future work to identify these graphical features and how they would tie into this model. Rodriguez *et al.* began this work by expanding the symbolic forms literature to include graphical forms [46]. Analyzing undergraduate chemistry students as they make sense of equations and graphs related to chemical kinetics, the researchers identified students' attention to different features of the graphs (e.g., steepness of slopes, curves) as the students interpreted the nature of chemical reactions. Our model of blending would provide the structure for analyzing how these graphical representations are tied specifically to the provided chemistry contexts.

The symbolic blending model acknowledges that contextual meaning is not part of the symbolic form, but part of a blend of mathematics and physics. By utilizing the generic space as the underlying conceptual schema, distinct from a contextual knowledge input space, we can separate students' conceptual ideas from the more fundamental template understanding, the latter being the focus of the original symbolic forms literature [5]. Symbolic forms represent specific mathematical structures, where variations in contextual knowledge can be connected to the same underlying mathematical representation, there are not multiple symbolic forms. Future research applying symbolic forms can make use of the symbolic blending model to disentangle the representation with the contextual knowledge that students are bringing into the representation.

2. Recall, backward projection, and reading out

Being able to read out information from an equation is identified by Sherin as a demonstration of an implicit understanding of symbolic forms. In this section, we return to the context of constructing differential vector elements to connect reading out to a process in conceptual blending referred to as backward projection [11]. The idea of reading out information from equations and graphs is just as common as the construction of equations. Kuo *et al.* [16] analyzed students' interpretations of a kinematics equation to assess student understanding of the physics involved. Similarly, Jones [7] described how students Chris and Devon moved between graphical and symbolic representations.

Within the context of our study [9], students were able to produce the appropriate structural representations from repeated use and classroom exposure. Students across several interviews experienced difficulty in generating or applying the correct ideas as they constructed the $\hat{\beta}$ component. In order to fill in the template, students recall the similar spherical coordinates in order to make sense of the unfamiliar system. Consider this example from Adam and Bart:

Bart: You can, you can check from [spherical dl]
Adam: For α it doesn't have any dependence on this other angle over here, but when you're talking about β, um/
Bart: So this is dl (g. to spherical $\overrightarrow{dl}$ he wrote), okay, drr [adds ˆ to r], $rd\theta\theta$ [adds ˆ to θ], $= \ldots = r\sin\theta d\phi\phi$ [adds ˆ to ϕ], so now we have just to compare so we have r it is M, θ is $\alpha \ldots \phi$ is β. Go ahead [Adam]....
Adam: Yeah, I can see now, this α here is independent of whatever β is, yeah, so $M\sin\alpha d\beta$.

Here we see Adam working within the coordinate system to construct the differential length vector. In contrast, Bart immediately begins to map onto spherical coordinates, drawing on the spherical differential length vector to finish the construction. After an attempt to redraw the coordinate system, and some confusion between the mathematics and physics representations of spherical coordinates, Adam finally settled on the mapping of $\sin\alpha$ into the $\hat{\beta}$-component. For Adam and Bart, the recall of a spherical differential length vector took the place of conceptual understanding and neither student drew on the knowledge that went into the construction of the spherical differential length element (Fig. 10). Our conceptual blending analysis inserts the recalled element into the conceptual input space, regardless of its correctness.

In contrast, other interview groups attempted to use recall as a sensemaking tool. Carol and Dan recalled the spherical volume element as well as the Cartesian coordinate transformations to, as Carol states, "make sense of the new coordinate system." However, the pair struggled to find anything to dissuade them from a direct mapping of variables and thus settled on the $\sin\alpha$ term as part of the beta component. In contrast, Elliot and Frank acknowledged the differences between the two coordinate systems. Frank correctly initially dictated the comparable spherical component as $r\sin\theta d\phi$, but Elliot, unable to discover a conceptual basis for the inclusion of a trigonometric function, was hesitant to use recall as a justification.

Elliot: Yeah, because if it were spherical coordinates, you'd have a $\sin\theta$ somewhere in there, you know…which it's very similar, I agree, but I feel like we should just work only by what we see here and try not to fog our mind with preconceived notions of how this should work.

At this point the group settled on $Md\beta$, invoking their conceptual understanding of arc length but still missing the necessary projection aspect that motivates the

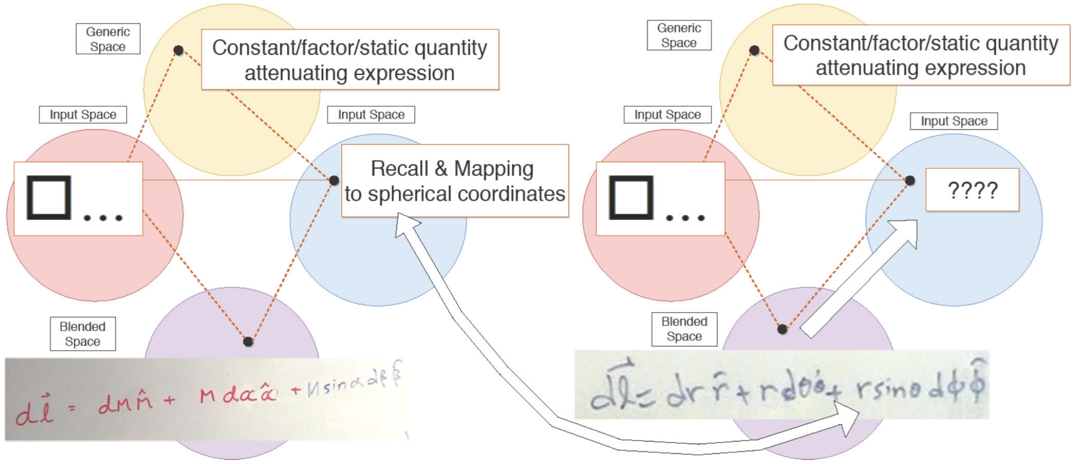

FIG. 10. An example of symbolic blending from Adam and Bart involving recall and backward projection during their pair interview. In place of conceptual understanding (right), Adam and Bart connected the coordinate system to spherical coordinates and attempted to deconstruct the spherical differential length vector (left) but did not know why the sine function was included. Without the underlying understanding, they pulled the sine function into their schmerical construction because of how the differential length is written in spherical coordinates (right).

trigonometric function. Later, the group returned to this idea, as Frank felt the need to have their differential length resemble the one in spherical coordinates without including the reasoning around the projection.

Frank: I mean, uh, spherical coordinates don't look like that. They have sines in there and I agree but if I can't find a reason to put it in there, you know, and there must be something wrong with the way I'm thinking. If that's true but I just don't, I don't see it yet, so why do you have $r \sin\theta$?

This statement of "I can't find a reason" is a marker in the blending literature of backward projection: The use of the blended space to interpret or look back at one of the input spaces [11]. We identify the use of the spherical differential elements by the other groups as an attempt to use spherical coordinates to draw out the associated conceptual understanding attached to the angular components. Neither group recognized the need for a $\cos\alpha$ in the $\hat{\beta}$ term but each took different paths: Carol and Dan directly mapped the elements into the schmerical coordinate elements, while Elliot and Frank stuck to the elements constructed within the realm of what they understand. In each case, our blending model accounts for students' difficulty with contextual knowledge despite having the correct structural understanding of the template.

Students "reading out" from structures to ascertain the relevant conceptual information from a previously constructed equation further connects conceptual blending to symbolic forms. Sherin identified symbolic forms not only as a way to analyze students' abilities to construct equations but also as a means to address their abilities to "extract implications from a derived expression," i.e., students' abilities to read out information from an equation based on the given structures [5]. While we see this to some degree in attempts to isolate the coefficient template of a spherical differential term, we suggest that this reading out more explicitly draws on the backward projection. Drawing again on *parts of a whole*, a student seeing an equation in which multiple things were being added together could recognize the *parts of a whole* template and then infer the nature of the relationship between the added quantities. In essence, the equation then carries this information, which is then projected into the larger conceptual space. This is seen in the earlier example from Bollen *et al.* [14] in which the interpretation of a calculation led a student to understand correct features of the physical system.

Within the symbolic blending model, rote knowledge recall is not only a reproduction of content but also a process or mechanism that can be mapped from the recalled information to the newly produced information. Recall is accounted for through reading outor backward projection. In the case where this illuminates contextual knowledge, that contextual knowledge can be mapped into the new construction. In the case where the contextual information is unknown, as with Adam and Bart, the recalled information takes the place of the contextual knowledge. With the symbolic blending model, we can identify recall as a process and determine which of the above forms it takes.

3. Interpreting template errors in equation construction

One of the benefits of applying conceptual blending in any context is the ability to isolate particular realms of ideas. In research on the use of mathematics in upper-division physics, this has manifested as the ability to isolate particular errors to difficulties with mathematics or physics ideas [14,15]. While symbolic blending has given us a means to assess errors in a final expression that can

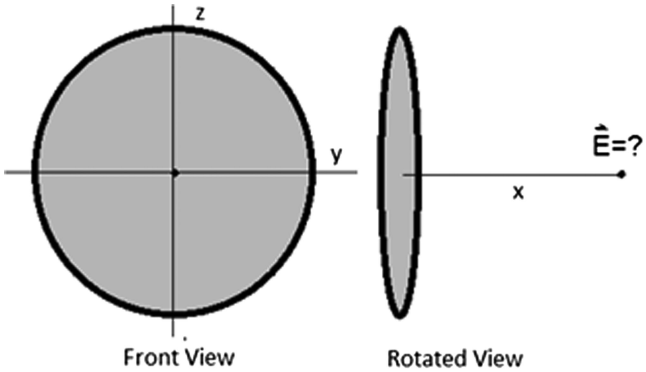

FIG. 11. Figure provided for the charged sheet task [47] in which students were asked to determine the electric field at a distance x from the center of a uniformly charged sheet. The task required students to construct a differential area to describe the surface.

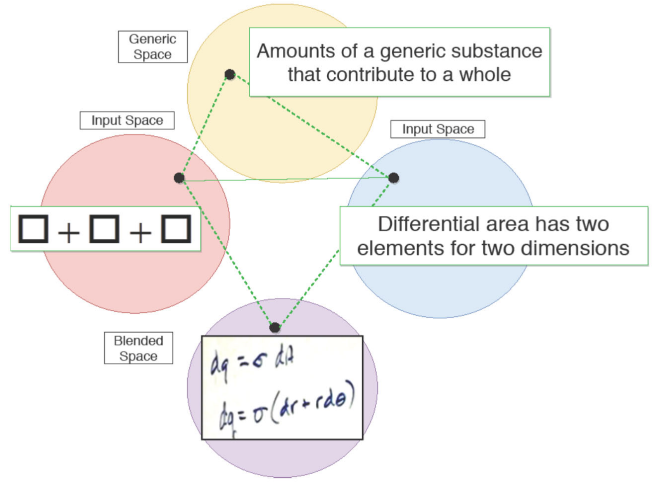

FIG. 12. Symbolic blending for charged sheet task where Jake added differential length terms instead of multiplying the terms. We classify this as Jake invoking the incorrect template.

be attributed to missing or unaccessed conceptual understanding, the benefits of this model extend to analyzing students' errors with mathematical representations based on a symbolic forms analysis within the blending structure.

In a different study [47], we conducted individual interviews to investigate students' understanding and construction of differential area elements within common electricity and magnetism contexts. One task in particular required students to construct a scalar differential area to solve for the electric field above a circular sheet with constant charge density (Fig. 11). In response, a student seemingly displayed the correct conceptual information but invoked the incorrect symbol template. After first attempting to ascertain the differential area by taking the derivative of the expression πr^2 with respect to r, Jake then recognized he could build a differential area from differential length components.

Jake: Actually no, it will be $drd\theta$ because it's a surface area so I'll need two dimensions… my $d\theta$ is probably going to come in from my dq. Because I should have a differential area shouldn't I, and a differential area should be $drd\theta$…

At this point, Jake wrote $dr + rd\theta$. At no point in his description did Jake say "dr plus $d\theta$" or reference addition at all. Instead, his language using paired differentials, "$drd\theta$," implied a product but was written as a sum, implying a template error. Jake made this error on an earlier task as well, despite having an otherwise appropriate concept image of a differential area as a small portion of area [47].

Within our proposed model for equation construction, Jake's conceptual understanding of input space for differential area contains a correct idea, yet it is blended with an inappropriate *parts of a whole* template (Fig. 12). From symbolic forms, we can hypothesize that Jake's underlying conceptual schema was skewed to that of *parts of a whole*. He could be seen approaching the idea of area as being made up of two lengths and thus used the incorrect template during the compression of ideas, combining the terms as a sum rather than a product. Much later in the interview, Jake was able to correct his differential area by reasoning about dimensionality, which shifted the representational form to the correct multiplication of lengths.

Likewise, Sherin noted instances of students accessing the requisite conceptual information but applying the incorrect template [5]. Within our work analyzing students' differential length elements, we noted that students had a general understanding of the symbol template in terms of the structural representation of the differential length vector but had more specific difficulty with understanding the geometry of the coordinate system and expressing it appropriately.

In another of our prior studies [48], students constructed differential length vectors during a calculation of change in electric potential around a curved path. During these interviews, we noted an incorrect encoding of vector notation, which was also seen commonly in students' work from course observations. The correct expression involves a differential length with two components to represent each polar direction of motion, as Molly demonstrates.

Molly: So first I travel in the r-direction so I go dr in the $\hat{r}$ and then I travel in the $\hat{\theta}$-direction and the arc length of a circle is the radius times the angle that you move so that is $rd\theta$, here in the $\hat{\theta}$.

Figure 13 shows Molly's blending diagram for this task. Here, we see her emphasis on the unit vectors and associated components, which she deftly represented using the magnitude-direction template.

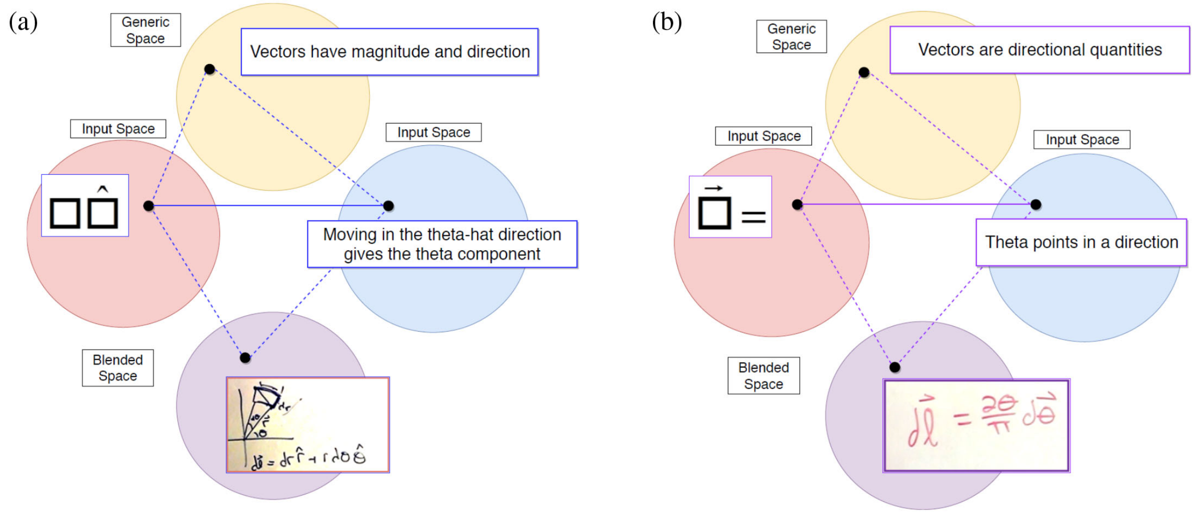

FIG. 13. Comparison of symbolic blending diagrams for Molly (a) and Lenny (b) for the student's invocation of different differential length elements. Molly correctly invoked the *magnitude-direction* form while Lenny focused on θ having a direction and used the vector symbol which is nested within the differential symbolic form $d\square$.

In contrast, Lenny only constructed a component in the $\hat{\theta}$ direction. Despite similar conceptual understanding, Lenny expressed his differential component as "$rd\vec{\theta}$." When asked to describe why he wrote the term in such a way, his response was absent of magnitude-direction reasoning.

Interviewer: What do you mean by $d\vec{\theta}$ there?
Lenny: So I guess, any differential shift in θ… because that's just the direction of the change in θ.

Mathematically speaking, the use of "$d\vec{\theta}$" makes the expression incorrect. While $d\vec{x}$ would make sense for a differential shift in the x direction, polar unit vectors are not static quantities and vary based on position in space. In our analysis (Fig. 13), Lenny's idea of representing a vector within this space is reduced to a representation of "the direction of change in theta." His emphasis on directionality without a separation of magnitude and unit vector leads to his encoding of this expression with a vector arrow template, $\vec{\square}$, rather than the *magnitude-direction* template, and thus makes sense within the presented model of conceptual blending and equation construction.

IV. SUMMARY AND CONCLUSIONS

In this paper, we presented the symbolic blending model by inserting the components of the symbolic forms framework [5] into a conceptual blending structure [11] in order to analyze students' mathematical sensemaking when constructing equations in upper-division physics. This builds on previous analysis [9] of students' construction of differential length elements in an unfamiliar spherical coordinate system using two different frameworks: concept image [34] and symbolic forms [5]. In that work, the concept image framework analysis identified specific properties students associated with a differential length vector in a non-Cartesian coordinate system, while students' structural understanding during equation construction was interpreted using symbolic forms. Because symbolic forms were designed to assess the mathematical understanding of the structures within an expression without explicitly accounting for the physics conceptual understanding, these frameworks afford complementary perspectives on both physical and mathematical aspects of the expression but still remain independent analyses. The lack of connection between these aspects in the individual frameworks runs counter to the perspective that, in a physics equation, mathematics and physics are inextricably coupled–blended [12,43]. By incorporating the two frameworks inside a conceptual blending lens [11] originally designed to describe the connection of meaning to form in the use of language, the symbolic blending model has the means to analyze students' construction of equations as an expression of a mathematical-physical language. The blend of contextual knowledge and structural expression creates emergent meaning in the final equation, making the result greater than the sum of its parts.

This approach to equation construction analysis uses the aspects of one theoretical framework to complement the missing analytical aspects of the other. Conceptual blending adds a component of conceptual understanding to a symbolic forms analysis, which becomes increasingly important within upper-division physics, in which the physics concepts connected to equations become more challenging. Likewise, incorporating symbolic forms into a conceptual blend provides structure to the blend of physics and mathematics centered on student reasoning with equations in physics contexts. To represent the union of these frameworks and illustrate the model, we designed a symbolic blending diagram in which the conceptual schema of symbolic forms [5] serves as the generic space in a blend; this model depicts the compression of conceptual and representational understanding into the final construction of an equation, structured by the mathematical justification in the conceptual schema. For example, the need to express three components for a differential length vector is fundamentally a sum of parts, which leads to the *parts of a whole* symbol template and eventually the final representation of a sum of three components. The symbolic blending model teases out the details of equation construction while preserving the components of both the symbolic forms and conceptual blending frameworks. The blending of conceptual information with mathematical structure leads to an equation that has meaning emerging from the blending process. Even though the conceptual schema of the symbolic forms is acontextual, the task context cues schema, thus providing the structure as a generic space in the blend.

A number of examples were given in which the symbolic blending model is employed within the context of the differential length vector study, as well as several other instances in our own work, which serve to illustrate the model as well as to show the utility of bringing conceptual blending to the construction of equations and symbolic forms. We have also provided discussion as to how this model is consistent with and reinterprets the use of symbolic forms within the current literature base where the conceptual schema of symbolic forms has become associated with the conceptual understanding of the contextual content [7,8,16] as well as with more theoretical work proposing "new" symbolic forms that are more "indirect" with respect to physical interpretation [30]. Similarly, we showed how the use of the generic space, which is generally absent from conceptual blending analyses of mathematics in physics [13–15], can provide deeper explanation of students' conceptual and representational choices when constructing equations.

Finally, several benefits of a symbolic blending model, as well as the full scope of its explanatory power, were outlined. The incorporation of the generic space as the underlying mathematical structure has provided the ability to connect varying contextual meanings to the same template. Symbolic templates as designed are very broad and can be used in many situations (e.g., *parts of a whole* can apply to energy, surface area, wedding guests, etc.); the symbolic blending model provides a means to connect symbolic forms to the context (which justifies addition). Further, the separate spaces of the model allow for the identification of a student's difficulty as either template error or incorrect/incomplete conceptual understanding. Finally, symbolic blending provides a mechanism for rote knowledge recall and explicitly demonstrates how backward projection, a feature of the conceptual blending model, connects to the reading of information out of an equation to gain conceptual understanding. The backward projection was useful in describing errors in recall in which students use previous ideas to make sense of new contexts.

The presented model provides the opportunity for obtaining a deeper and more complete understanding of students' construction of equations in situations that draw on sophisticated mathematical and/or physical understanding. The connection of analytical aspects across these theoretical frameworks allows for analysis on the level of both conceptual understanding and structural representation.

Given the role of physical context in driving the mathematical conceptual schema and thus the structure of the final expression, symbolic blending recognizes and makes concrete the need for coherence between mathematical formalism, mathematical concepts, and physical phenomena. This is consistent with, and a central tenet of, broader models of mathematical modeling and mathematization [2–4], the blending of mathematics and physics [12,49], and mathematical sensemaking in physics [50,51].

V. FUTURE WORK

With the understanding of the affordances of symbolic blending to the analysis of student construction of equations in terms of conceptual and representational understanding, we envision further applications of the model. Just as Sherin suggests the symbolic forms framework could be extended into other domains of physics, we believe that our model presents as a key analytical tool to the study of mathematics used in physics problem solving, especially in an upper-division context where, throughout the course of their academic track, students connect physics to concepts of vector calculus, partial derivatives, and linear algebra. In line with the goal of theoretical development of mathematical sensemaking in upper-division quantum mechanics [30], this model would prove useful in connecting the multiple conceptual steps to the construction or interpretation of equations and representations in quantum mechanics. Van den Eynde *et al.* [52] have extended the symbolic blending model to explore the dynamics and depict the line of reasoning for students' blending of mathematics and physics when looking at boundary conditions for the heat equation.

Sherin also suggests that "stretching farther still," symbolic forms could be generalized to discuss other representational forms that contain sets of meaningful structures. We hypothesize that the incorporation of conceptual blending takes a step in that direction by providing the generic space as a means to connect ideas by their underlying similarities. As such, we can extend the template space to a representational space and connect students' understanding of linear relationships and graphing knowledge to graphical representations, and additionally with concepts of wave vectors, wave functions, or probability density graphs. Researchers have recently begun to address students' understanding of the various representations of Dirac notation, wave function notation, and matrix notation [53]. Other researchers have explored students' metarepresentational understanding of these notations, finding when students make judgments about which notation is easier or better suited to a task [54]. More broadly, a model of conceptual blending as we have presented could be extended to analyze student work as they translate between various representations that effectively convey the same conceptual understanding.

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