Limiting case analysis in an electricity and magnetism course

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Limiting case analysis (LCA) is important to practicing physicists. Yet, there is little concrete guidance for physics educators, and a lack of consensus in the research community about how to help students learn, and learn from, limiting case analysis. In this study, we first review existing literature to find commonalities and variations in how instructors encourage and assess students' limiting case analysis and to highlight how it has been used by practicing physicists. Then, we examine written work from successive cohorts of physics students, all of whom have completed a course with the same instructor who emphasizes limiting case analysis in his teaching. We frame our analysis largely in terms of the theoretical framework of "adaptive expertise," finding support in the literature for the view that it is the nonalgorithmic and even playful aspects of LCA that are instrumental to its alignment with adaptive expertise rather than routine expertise. Analysis of students' commentary about how they decide which limiting cases to examine when evaluating the reasonableness of an equation provides new insights into how LCA might be better supported in the classroom so that more students can access this important tool of physics.

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I. INTRODUCTION

Limiting case analysis (LCA) is a widely recognized tool of professional physics. Checking behavior of equations in limiting or special cases helps physicists evaluate the reasonableness of an expression; move from specific equations to more generalizable or complete mathematical models; extract new information about behavior under conditions that are unobservable or outside one's intuition; and persuade themselves and others that a result is (or is not) valid and trustworthy [1–3]. Because of its utility in professional physics, limiting case analysis is also becoming part of the "canon" of undergraduate physics problem-solving instruction. A wide variety of textbooks and instructional guides, for example, include reminders for students to check limiting cases of their answers [4–7]. There has also been some research into how to encourage students to check if equations make sense, with limiting case analysis as one of many possible checks. Yet, very little is known about how students actually use LCA to make sense of mathematical equations. This study addresses that gap.

We describe here the results of a design-based research study of learning limiting case analysis in undergraduate physics. The study took place over four successive years of a junior-level electricity and magnetism course, exploring how students engage with written prompts intended to develop their capacity for limiting case analysis. In addition to providing new empirical insights about how students conduct limiting case analysis, the study articulates the instructional target of LCA in theoretical terms, using the concept of adaptive expertise, as articulated by Hatano and Inagaki [8] and expanded upon by Kuo *et al.* [9] and others.

For clarity, we present here (see Fig. 1) a typical example of the kind of questions and student responses that make up the bulk of our data. Students received a scenario, a formula

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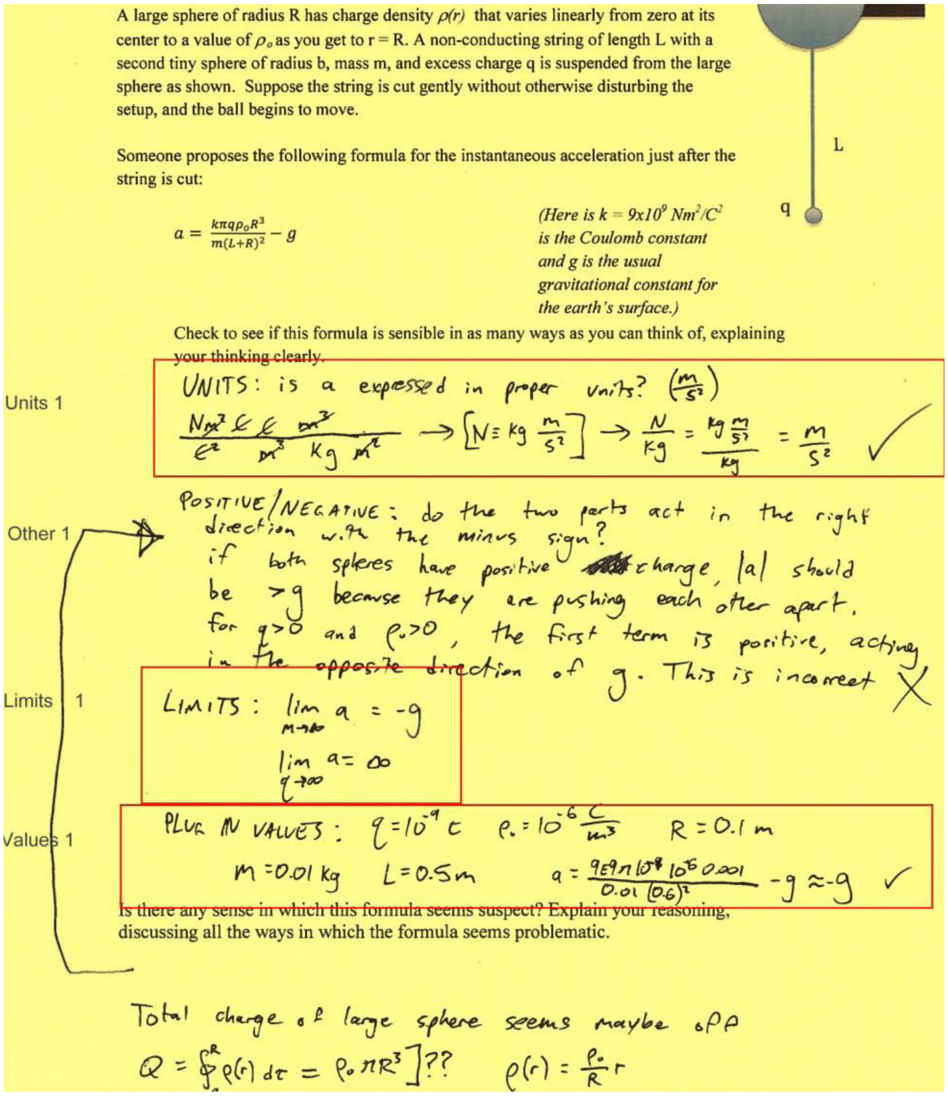


FIG. 1. Example prompt, student response, and researcher notes.

associated with that scenario, and instructions to check the sensibility of the formula. Various versions of the example shown here were used over the course of the study, along with other examples involving similar prompts, but focusing on magnetic field interactions with moving charged particles.

It should be mentioned that early in the study, the prompts were much more specific: "Check to see if this formula is sensible in the three usual ways," where the usual three ways were identified as (1) unit analysis of the formula, (2) LCA of the formula, and (3) exploring characteristic numerical values of the formula. However, we discovered from examining the responses that this kind of prompt, while perhaps having some merits, often resulted in more rote replies from the students and gave us little insight into what kinds of checks that the students would perform if given more leeway [10]. In later iterations, we began asking students what kinds of checks they preferred to perform and about their reasoning. We saw rich variations, especially in the kinds of commentary that accompanied the students' approach to LCA, which resulted in the focus of this paper.

This study captures details, which are not reported in other literature, about how students conduct limiting case analysis. The results include how students determine which checks to conduct, the details of how they conduct LCA, and what they say about the purpose of LCA in written surveys. We find that students choose which limiting cases to check based on a variety of considerations, including their ability to anticipate what a sensible result would be, their assumptions about how quantities can be manipulated, and their expectations about which checks will be most informative. The study also reveals at least two different ways that intuition features in students' LCA. Some students clearly articulate an intuition in the early stages of analysis, while other students waited to articulate an intuition until after the result of the limiting case is known.

Finally, we observe at least one instance of a student developing new intuition about what a particular quantity means physically in an equation. Collectively, these results provide further evidence that teaching students limiting case analysis might "boost" problem-solving performance by encouraging students to develop and blend physical and mathematical intuition [9,11–14]. We also see in these student responses occasional indicators of what Hatano *et al.* call "adaptive expertise," particularly in their innovation in implementing LCA, in their flexibility in interpreting what they find, and in their expectation that something new can be learned from performing LCA. By providing this detailed account of how students implement limiting case analysis, and how they describe its purpose and usefulness, we aim to contribute new knowledge about how to better support learning of this important tool of physics.

II. DEFINING LIMITING CASE ANALYSIS

We use the phrase "limiting case analysis" as an umbrella term for nine closely related tools described in professional physics and in physics education (see Table I). The meaning of LCA is unpacked throughout the remainder of the paper, starting with existing literature on the topic. We examine meanings and uses of LCA within three categories of literature: historical and philosophical perspectives on LCA in professional physics; textbooks and teaching materials that have explicitly called for limiting case reasoning as part of problem solving, but with little or no student data; and investigations of students' use of LCA. Our anecdotal conversations with colleagues and searching of professional physics literature suggest that LCA is widely recognized and highly valued in professional physics. Yet, few studies specifically focus on how students learn and learn from LCA, suggesting a disconnect between the practices valued in professional physics and what is taught in classrooms [15]. Our study aims to bridge this gap in part.

Nersessian's work provides among the most well-known accounts of limiting case analysis in history and philosophy of science literature. Nersessian analyzes the writings of famous figures in the Western history of science, such as Einstein, Galileo, and Maxwell, to argue that limiting case analysis is a recurring feature of scientific discovery [3]. She defines LCA as follows:

> *Limiting case analysis is a form of idealization employed frequently in thought experimenting. In this species of thought experimenting, the simulation consists in abstracting specific physical dimensions to create an idealized representation. Isolating the physical system in thought allows us to manipulate variables beyond what is physically possible and this creates data we did not possess before. [26] (p. 298)*

This definition of LCA is rooted in an important concept from cognitive psychology—the mental model. A mental model is an imagined representation. LCA is a specific type of mental modeling where aspects of the representation are taken to extremes, beyond what one might be able to measure [27]. Thus, one way that LCA supports scientific progress is by helping scientists generate data, so to speak, about scenarios or conditions that are difficult to create in the physical laboratory. As one of the students in our study explained, checking limits, values, and units is akin to conducting a "rapid test without a physical experiment."

A second way that LCA contributes to the advancement of science is by helping scientists create quantitative representations of the world. Nersessian presents Galileo's use of LCA as a critical example of the shift toward *mathematization* in physics. To study free fall motion, Galileo developed an idealized world that is easily quantified and represented symbolically. This idealized representation allows Galileo to extrapolate what happens to motion of an object in a successively less dense medium. When students check limiting cases, they sometimes present the result as a statement of the value at the limit, e.g., "as density goes to zero, resistance goes to zero." But, Galileo's use of LCA includes quantification of *how* the "differential lifting effect" changes as the limit is approached, leading to a more complete mathematical representation. Students in our study were reminded to pay attention to not only the value at the limit but also how the limit is approached. When asked to describe LCA in an end-of-semester survey, one student explained succinctly, "what are limits as your independent variables get very large and very small and how does your value approach them. Understanding how the limits are approached helps to visualize the physical properties or behavior of a problem." Mathematization is a core component of professional physics but also one that is difficult to teach [28,29]. We suspect that more careful treatment of LCA in the physics classroom might help in this regard.

Nersessian's accounts of conceptual change in electromagnetic theory highlight yet a third important use of limiting case analysis—to check that a proposed model is "consistent under the appropriate conditions with established mathematical results" [30] (p. 34). The "appropriate conditions" could include limiting cases. For example, Maxwell required that his developing unified model of electromagnetism be mathematically consistent with previously derived results. In the case of his "vortex medium" model, this meant that the model must generate Coulomb's

TABLE I. Related limiting case analysis terms in the literature.

• Boundary condition [16]	• Limiting behavior [21,22]
• Degenerate case [17]	• Limiting case [14,23]
• Extreme case [18]	• Limiting condition [24]
• Functional dependence [19]	• Special case [11,12,25]
• Leading order analysis [20]	

law in the limiting case of a uniform magnetic permeability and no current in the medium. (Bohr also invokes similar consistency requirements through his correspondence principle).

So far, we have discussed three important ways that LCA supports scientific progress: generating new data about unobservables, representing phenomena quantitatively and symbolically, and developing unified models that are mathematically consistent with past results. Nersessian highlights a fourth critical function—to communicate. When physicists describe the limiting case as a thought experiment, they "assist readers in constructing their own mental simulation, thereby creating an understanding of what they have not actually witnessed themselves" [31] (p. 207). In this way, the arguments constructed from LCA can be very compelling as a way to communicate discoveries to the rest of the scientific community. In the writings of Bohr, Einstein, and Galileo, limiting cases are represented mathematically and in the form of a narrative. The narrative, according to Nersessian, is what adds to the

TABLE II. Some purposes and uses of limiting case analysis.

Purpose or use	Illustrative reference
Practice physics problem solving	"A useful class of problem for encouraging students to work with symbols is the limiting case problem…" [19] (p. 4)
Extend perception or intuition	"In both cases, thinking about extreme cases is used for purposes of overcoming the limits of human intuition, with the aim of finding some 'deeper truth.'" [33] (p. 152)
Generate models, solutions, equations, or answers	"They [students] can generate new and creative uses for extreme-case reasoning, here recasting it from a tool for evaluating answers to one for generating them." [34] (p. 108)
Explore or interpret models, solutions, equations, or answers	"Let's now examine some limiting cases of the general expression (8). One interesting special case occurs when $\mu_{s,\mathrm{floor}} = 0$. In that case, the argument of the arctangent function diverges, and we see that $\theta_{\mathrm{crit}} = 90°$, i.e., equilibrium is not possible if the ladder leans. This occurs because if the ladder leans, the floor's static friction force is necessary to balance the normal force from the wall if equilibrium is to be maintained." [35] (pp. 567–568)
Create quantitative representations of phenomena (e.g., mathematization)	"Galileo repeatedly used thought experiments and limiting case analyses in tandem…both in constructing a quantifiable representation of bodies in motion and in attempting to convey this new representation to others." [27] (p. 433)
Communicate results	See above.
Establish a correspondence between theories or phenomena	"…the idealized representations of the thought experiments and limiting case analyses often facilitated Galileo's recognition of analogies between different phenomena, such as the motion of falling bodies and the motion of a pendulum. That is, idealized representations form abstract schemata common to different problems." [36] (p. 175)
Check results for reasonableness or consistency (including critique and peer review)	"…as with checking units, checking limiting cases (or special cases) is something you should always do at the end of a calculation. But as with checking units, it won't tell you that your answer is definitely correct, but it might tell you that your answer is definitely incorrect." [37] (pp. I–6)
Generalize or infer solutions from known or specific cases	"This type of analysis, known in physics as limiting case analysis [1,19], allows for generalizing inferences by evaluating or estimating values of algebraic functions based on its extreme inputs." [1] (p. 40)
Identify relevant variables and identify relationships between them	"…If you're having trouble figuring out how a given system behaves, then you can imagine making, for example, a certain length become very large or very small, and then you can see what happens to the behavior. Having convinced yourself that the length actually affects the system in extreme cases (or perhaps you will discover that the length doesn't affect things at all), it will then be easier to understand how it affects the system in general…" [37] (pp. I–6)

"logical and empirical force" [27] (p. 431) of LCA as a communicative tool; it helps readers imagine the simulation represented by the mathematical form of the limit. In our study, students were reminded in class to include some explanation of how they determined whether the results of a limiting case check were sensible. Students who initially presented only the mathematical form of the limit would typically include a bit more narrative explanation as the semester went on [32]. Still, one student admitted, "I don't always explicitly write it out, rather just do it in my head." We did not investigate students' coming to appreciate LCA as a tool for scientific communication in this study but that could be an interesting direction for future work given the importance of communicative functions of LCA in Nersessian's account.

More recent professional physics literature shows LCA being used to check one's own or other's results; this function is not emphasized in Nersessian's work but lies at the core of our pedagogical intervention and is related to the communicative function previously discussed. We searched the databases of professional physics journals for use of phrases like "limiting case" and "in the limit of" in published articles. We were particularly inspired by the errata and comments articles, where physicists can be observed critiquing or disputing claims in a published paper by way of a limiting case analysis. Our prompts encouraged students to use LCA to evaluate the sensibility of a result in a mathematical limit. We present students with a situation and a formula and then ask students to check whether that formula makes sense. Limiting case analysis is one tool that students have at their disposal to critique the formulas; in this way, we hope they learn to recognize LCA not only as a tool for checking their own answers but also as a practice invoked by the physics community as part of the construction and verification of knowledge.

We have argued that physicists invoke LCA "locally" for many purposes, including to generate data, extend intuition, communicate results, and critique results. LCA has played an important role in the advancement of physics knowledge, allowing physicists to explore regimes that could not otherwise be investigated experimentally, and also supported mathematization of physics. Table II summarizes some of the ways we have encountered physicists describing the uses of LCA in their research and teaching of physics, with one or two illustrative references for each.

III. LITERATURE REVIEW

If LCA is a useful tool of physics, then as physics educators, it is important to understand different ways that students might learn and use this tool. Previous studies of how students learn and use LCA indicate widespread agreement that typical physics teaching is not effective for teaching LCA. For example, Loverude [38] reports that upper-level students from a math methods course do not spontaneously use limiting cases to make sense of equations after traditional instruction and calls for explicit instruction to promote this kind of "nonprocedural" learning. One way that instructors try to encourage students to check limiting cases is by providing students with a problem-solving process that includes an answer-checking step. However, this approach does not seem to be particularly effective. Wilcox *et al.* [39] (p. 3) found that only 8% of students in a junior-level electricity and magnetism course attempted to check their solution in the fourth "reflection" step of the ACER protocol, and while some of these students examined limiting cases, most merely "made superficial statements about whether the solution looked familiar". Lenz *et al.* [40] indicate that in postcourse interviews, none of the 11 introductory students invoked special case analysis, whether they experienced traditional classroom instruction or reformed instruction where special case analysis was demonstrated. We suspect that the low uptake of LCA in such studies might be partially explained by the lack of consistency between the instructor's statements about what is valued in the class and how points are actually awarded [41]. In situations where students are rewarded for quickly generating a correct answer, sometimes called "answer-making," there is little incentive for students to check limiting cases; such checks take time, and it is not immediately clear to students that LCA will help get to "the answer," and perhaps even worse from the student's perspective, a limiting case might actually further complicate their thinking about a problem or result or hinder their attempts to get to the next problem.

In studies where limiting case analysis is rewarded explicitly and treated as important content in the course, some positive results are observed. Warren [12] aimed for his introductory physics students to learn to assess their own work, rather than having to rely on an outside authority to tell them whether or not their work is correct. To achieve this goal, students were introduced to a range of "evaluation strategies," including special case analysis in class. Students were assigned problems that required the use of these evaluation strategies for homework and during recitation sessions. Further, students received a rubric that clarified the expectations for how each evaluation strategy was to be used, and students received descriptive feedback on their performance throughout the course. Under these conditions, Warren reported an increase in students' use of LCA on a variety of problems. Warren also reported further that "use of evaluation strategies, particularly…special case analysis, help[ed] generate a significant increase in performance on multiple choice exam questions…" but only when those questions cover the same topics as the evaluation strategies (p. 020103–10). Chasteen *et al.* [42] also report that targeted instruction can encourage answer checking in physics education research (PER)-aligned, junior-level electricity and magnetism students, such that these students can describe limiting behavior for their solutions better than students in standard courses. In earlier work [10], the authors argue that upper-level electricity and magnetism students on their campus readily adopt limiting case analysis when instruction was explicit and reward structure apparent, but lamented a certain rote-ness in their adoption. Eichenlaub *et al.* [34] treat LCA as a variation of an epistemic game, specifically documenting a case wherein the student is guided through extreme case reasoning. This pedagogical approach "can cause *significant* shifts to students' epistemic frames and lead to new, creative sorts of blended mathematical and physical cognition" and even result in successful application to problems previously not solvable by the students (p. 111). Following their discovery that instructor demonstration and homework prompts were not sufficient to encourage special case analysis in introductory physics courses, a team at Oregon State developed an upper-level *Techniques of Theoretical Mechanics* course that explicitly addressed "sensemaking strategies," including special case analysis, in nearly all course meetings and assignments [43]. The team found that when sensemaking tools, such as LCA, are "treated on equal footing as the content goals of the course," students will embrace these tools and even use them on homework without explicit scaffolding [43,44]. Students reported "special case" and limiting case as the most common "sensemaking tool" they use on homework [44]. Student Charles reported that he "probably do[es] the dimensions without prompting most of the time and then one or two limiting cases" [45] (p. 115). Thomas also reported using limiting case analysis throughout the course but found it particularly helpful for a topic introduced toward the end of the course:

> *Limiting cases I think, yeah, especially since we just started special relativity. I think limiting cases have kind of helped a lot, especially when you're just like 'oh, if the velocity is really small compared to c then it ends up being kind of what you'd expect classically'. That help me a lot in the homework that's due on Friday because it's just a really quick check. [45] (p. 110)*

Finally, further evidence of the effectiveness of a comprehensive approach to teaching LCA can be found in student quotes like those below:

> *I think looking back on some of the [other physics classes], they'd give us a couple of the limiting cases and then just give us the solution to those and then run the fit, but it wasn't more of the holistic kind of thing that we do I guess. [45] (p. 118)*

We conclude from these studies that students are more likely to implement and find value in LCA if it is woven throughout the course, not treated simply as an add-on step of problem solving. As described in Sec. IV B, we used a similar approach as Hahn and Lenz, attempting to elevate LCA as a central tool of physics and to cultivate its use throughout an upper-level electricity and magnetism course.

The previous studies identify instructional conditions that support students' learning of LCA. Another line of work attempts to model how and what students learn from checking limiting cases. Zietsman and Clement [18] propose "extreme cases" as a pedagogical tool for learning physics. During a series of tutoring sessions about levers, 7th grade students were presented with two diagrams and asked to consider which showed a scenario that would be "easier" to maintain. The diagrams were identical except for one feature. Both showed a person holding one end of a board, with the other end of the board resting on a fulcrum. In one diagram, a mass was located directly over the fulcrum; in the other diagram, the mass was located next to the person's hand. The researchers found that these extreme cases were effective in drawing students' attention to a variable that they overlooked in preassessments—the effort arm. Further, students were able to infer a "general direction of change relation" between the effort required to keep the board level and the effort arm (p. 63). Vidak *et al.* similarly observed that introducing extreme cases helped students use their intuitions about thermodynamic expansion in one dimension to answer questions about thermodynamic expansion in two dimensions [46]. Together, these studies indicate that extreme case reasoning may be beneficial because it requires learners to activate intuitions and develop an explanatory model and also facilitates construction of new causal relationships by drawing attention to variables that may have been overlooked [18]. Both of these studies introduced extreme case reasoning to students; however, Stephens and Clement have also observed students spontaneously invoking extreme case reasoning during physics class discussions [47]. In our study with college physics students, we also find evidence that while conducting LCA, students articulate "intuitions"; however, as discussed in the findings, we are at times unsure whether these are truly physical intuitions in the sense of being grounded in students' everyday ideas about the physical world.

Limiting case analysis involves examining an expression and determining which limiting cases might be useful to examine. We were unable to find very much information about this aspect of LCA in the existing literature, other than a note from Hahn about future plans to attempt to teach students "how to choose advantageous cases to analyze" because "while students often used these strategies, they did not always choose cases that would yield the most insight into the problem" (p. 163). In this study, we did examine which variables students preferred to vary in their implementations of LCA, finding that there was some resistance to considering variations in things considered constant (such as Coulomb's constant or the speed of light or the gravitational constant), even while an expert might have fewer such inhibitions [19]. These points are further elaborated on in the findings. We conclude here by noting that in reviewing the literature we did not find any articles with a singular focus on students' perceptions and use of limiting case analysis; this paper is a unique contribution in that regard.

IV. THEORETICAL FRAMEWORK

In this study, we aimed for students to use limiting case analysis more often, and in the process, discover potential ways that limiting case analysis can be epistemically valuable in physics. While some students will spontaneously check limiting cases on their own, and a few may even do so systematically, we did not think that our classes would reach our pedagogical goals related to adaptive expertise without some instructional support. As previously described, including a "checking" step at the end of a problem-solving protocol is not sufficient for learning LCA [38–40]. Students are unlikely to complete this step, and even if they do, it is not clear whether they actually get better at LCA as a result. A more concerted approach is required, in our view.

Over the course of the study, we aimed to change students' use of LCA through targeted instruction, in a design-based approach (see Sec. V. B). We conceptualized an end goal of instruction as students' development of *adaptive expertise* in checking limiting cases. The concept of adaptive expertise holds that "innovative uses of knowledge" [48] are the hallmark of professionalism in fields that require innovation and problem solving (p. 217). Though our instruction included scriptlike elements as a starting point, rote implementation of these scripts does not constitute expertise. As Nersessian explained, LCA is "nonalgorithmic"; it requires innovation, choice, and context-dependent reasoning.

A. Conditions for development of adaptive expertise

Hatano and Inagaki [8] identify three factors that are important for the development of adaptive expertise. The first is built-in randomness that necessitates trying out variations of the target skill, monitoring the results, and then inferring conceptual relationships between the variation and the result. They provide the example of cooking. When following a recipe, you may not have all of the ingredients or exactly the same cookware as the recipe requires. Therefore, you make an adjustment, maybe substituting one ingredient for another, and now you have the opportunity to see how that adjustment impacts the results. If you cook only once, then you may not build adaptive expertise, but if over many opportunities following recipes, you make many adjustments, eventually you will build up a conceptual model including many variations and how they impact the results. However, if you always follow the recipe exactly, you are unlikely to gain these kinds of insights. Nersessian noted that LCA and related "modelling activities" are "nonalgorithmic, and even if used correctly, may lead to the wrong solution or to no solution" (p. 13). The nonalgorithmic nature of LCA discouraged philosophers of science from attempting to account for its role in scientific discovery [3]. This nonalgorithmic property of LCA also poses a challenge for instructors.

In our study, we began with a more "follow the script" like approach, G. W., the course instructor, introduced three "easy to remember" checks (checking units, using LCA, and plugging in numerical values) as a way to scaffold students' use of limiting case analysis. Students were routinely asked to check answers in "the three usual ways." While the script was effective in helping students remember which checks to conduct, we found early on in the study that it also constrained students' sensemaking [10]. Even though students had repeated opportunities to conduct the checks over the semester, over time, they conducted only the three checks introduced in class, pruning out of their repertoire other useful checks (such as sign dependence or symmetry arguments) that did not seem to fit the script [49]. While routines may help students remember to conduct a limiting case check, a purely algorithmic approach is not consistent with the ways that Nersessian observes this tool's use in professional physics. Thus, we began modifying the prompts to allow for more flexibility and experimentation.

A second critical factor in the development of adaptive expertise is the social context in which the skill is being developed. If the skill is developed in a community where ingenuity is valued, even when it comes at the cost of efficiency, learners are more likely to be "playful" and try variations [8]. However, if the community upholds efficiency as the main value, then learners are unlikely to be willing to take the risk of trying something new that may not pay off. In a classroom, however, both of these norms are present. For example, during class, G. W. provided metacognitive commentary about the use and value of limiting case analysis. In this way, the instructor's tacit epistemic commitments were made public to students. For example, one sequence of questions involved ascertaining the electric field at the center of a square fashioned from charged rods, then generalizing beyond the square to the equivalent n-gon problem, and finally then determining whether the limit as n goes to infinity matches what was previously obtained for the electric field due to a circular loop of charge. Clearly, this would not be the most efficient way to derive the electric field at the center of a circular loop of charge, but we maintain that there is a certain satisfaction in seeing that the results match those obtained before despite the inefficiency and that this acknowledgment helps to foster a culture that is more "understanding oriented" than "promptitude oriented" [8] (p. 35). However, students did express that on tests, they were unlikely to have time to try many different checks, so they would focus on those that were likely to help them make a determination of “sensible” or “not sensible” as quickly as possible. Students noted that unit checks were especially helpful in this regard.

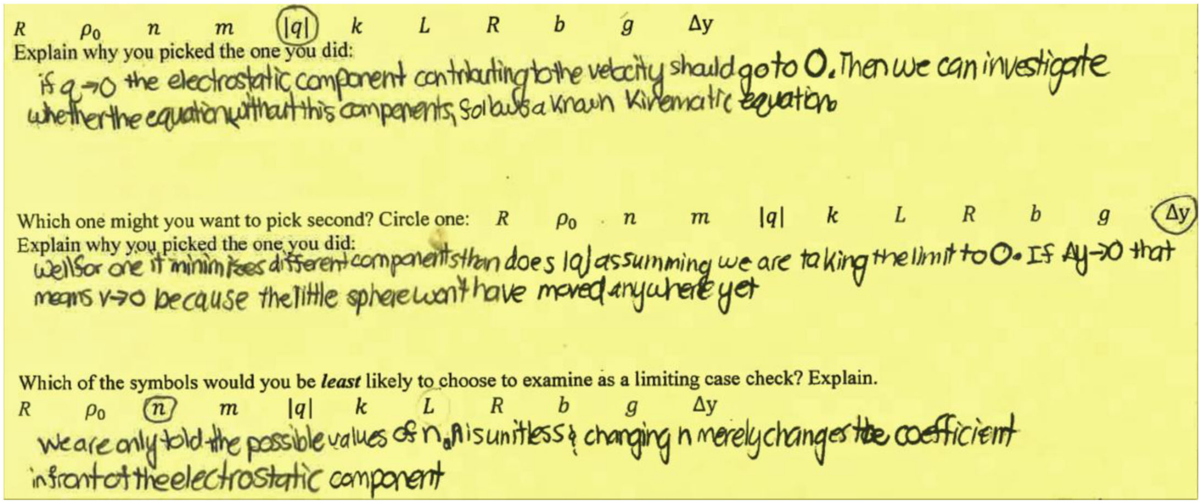
The third factor that Hatano and Inagaki posit is required for adaptive expertise again relates to how the social context encourages, or discourages, individuals from trying variations of the target skill. If individuals are performing the skill to receive a reward, such as a grade, then they are more likely to employ “conventional” versions of that skill, as they do not want to risk missing out on the reward by trying something new. However, if the skill is not being performed for a reward, then people are more willing to make adjustments and monitor the results. A number of studies support the negative impact of grades on student learning:

> *Inagaki (1980), reviewing these studies, points out the possibility that the expectation of reward may prevent learners from understanding things deeply: It changes the “goal structure” of the activity and thus leads learners to shift their strategy from “heuristic,” such as “examining possibilities of alternative solutions” or “seeking a more universal solution beyond the present successful one,” to “algorithmic,” such as strategy ensuring steadier and often quicker solutions within a given time. [8] (p. 34)*

However, here, we diverged from the recommendations of Inagaki and Hatano. Because students receive a grade at the end of the semester, the grade “goal structure” is always operating, even if a particular assignment is not graded. Turpen *et al.* found that grading practices can influence how students perceive the instructor’s messages about classroom norms. In encouraging students to conduct LCA, but then not awarding them credit for it (while still awarding points for other aspects of the course), we risked sending the message that LCA was not really that important to physics. Instead, points were used as a way to put LCA on the same footing as other components of the class for which students received credit [32,44]. G. W. rewarded limiting case checks in class through verbal encouragement and by assigning points to students who attempted checks on quizzes and tests, even if those checks were not completed fully or accurately. Wilcox and Pollock [50] found that awarding partial credit in this way can be a helpful way to acknowledge the internal consistency of students’ reasoning and their attempts to connect ideas together in unexpected ways. Warren [12] was able to increase the quality and quantity of usage of LCA (he calls it special case analysis) by algebra-based intro physics students over the course of the academic year by the use of feedback to students on homework and tutorials using specific formative rubrics and then summative assessments to measure whether the students invoke LCA to solve open-ended questions. While our approach is at odds with Hatano and Inagaki’s guidance, by assigning points to checks, we hoped to consistently message that checking answers is a valued part of students’ physics learning, thereby encouraging exploration and search for understanding required to develop adaptive expertise.

Hatano and Inagaki’s model of adaptive expertise development requires that learners have many opportunities to practice the target skill. As previously mentioned, the typical “solve and check your answer” approach on its own does not seem to be effective for teaching LCA. We suspect that one shortfall of this approach is that students do not have much opportunity to actually perform the target skill. They may run out of time at the end of a problem, or get stuck generating a solution, and never have an opportunity to conduct the checks. The types of problems students are reasonably able to solve also limit the kinds of variations they can experience. To address these shortcomings, we decided to create prompts that are designed specifically for students to practice checking solutions for sensibility. To get around the issue of students being unable to solve a problem, we sometimes provided students with a situation, and a possible solution, and then asked students to check that solution for sensibility. We are not the first to use this approach [11,51]. Students were positioned more in the role of a peer reviewer, checking to see whether someone else’s solution holds up under scrutiny. For example, in the response below (Fig. 2), the student indicates that they pick their second choice because it “minimizes a different component than does” their first choice, suggesting that they are choosing which limit based on an ideal that allows them to learn as much as possible by choosing a parameter that addresses a “different component”…Next they say they do not choose “n” because it is “unitless” and it “merely changes the coefficient” suggesting that understanding it is not as important for full understanding as are their first and second choices…One might quibble with the assertion about the relative importance of these parameters’ limits giving reasonable results, but it seems clear that the student values fuller understanding over achieving a successful (but less meaningful) limit analysis. These prompts were provided to students repeatedly throughout the semester, to ensure multiple opportunities to practice.

Thus, the pedagogical approach aligns in many ways with Hatano and Inagaki’s requirements for the development of adaptive expertise summarized in Table III (built-in randomness, community emphasis on understanding, and encouragement to try variations of the target skill) [8,52]. But, these are features of the context that support adaptive expertise, not features of adaptive expertise itself. To formulate potential indicators of adaptive expertise in LCA, we reviewed Hatano and Inagaki’s original writings, as well as PER literature citing those writings, to identify three potential indicators of adaptive expertise.



R ρ_0 n m $|q|$ k L R b g Δy

Explain why you picked the one you did:

if q→0 the electrostatic component contributing to the velocity should go to 0. Then we can investigate whether the equation without this components follows a known kinematic equation

Which one might you want to pick second? Circle one: R ρ_0 n m $|q|$ k L R b g Δy

Explain why you picked the one you did:

well for one it minimizes different components than does |q| assumming we are taking the limit to 0. If Δy→0 that means v→0 because the little sphere won't have moved anywhere yet

Which of the symbols would you be ***least*** likely to choose to examine as a limiting case check? Explain.

R ρ_0 n m $|q|$ k L R b g Δy

We are only told the possible values of n, n is unitless & changing n merely changes the coefficient in front of the electrostatic component

FIG. 2. Student response describing why they picked which variables for LCA. The response indicates a desire to explore variables and cases that may generate interesting or meaningful results.

B. Indicators of adaptive expertise in LCA

The first indicator derives from a consensus in the literature that adaptive expertise is uniquely characterized by innovation [9,53]. Routine experts “are outstanding in terms of speed, accuracy, and automaticity of performance” [8] (p. 31); they typically approach novel problems as “opportunities to use their existing expertise to do familiar tasks more efficiently” [54] (p. 46). Like routine experts, adaptive experts find ways to be more efficient, but they also sacrifice immediate efficiency in order to “explore a variety of possibilities and try to make sense of their actions” [55] (p. xii). In our prompts, we asked students to check a range of limiting cases and thus explored a “variety of possibilities” [55]. However, adaptive expertise is not characterized by the number of different limiting cases a student checked. Rather, a better indicator of adaptive expertise is when a student willingly checked a limiting case because the results were not immediately obvious to them, and they wanted to see what they could learn, even if it turned out the limit was difficult to interpret or not particularly insightful, a sort of willingness to go down “rabbit holes.” Because adaptive experts invest time in developing a conceptual understanding of how variation impacts their performance, they are better able to determine which procedures to implement in a new situation or even develop new procedures [56]. In other words, adaptive experts will sacrifice efficiency in the short term in order to explore and develop knowledge that might prove useful in the future.

A second possible indicator of adaptive expertise, arguably at the core of what differentiates adaptive from routine expertise, is students’ attempts to glean new insights by conducting a check. Some evidence can be found in students’ expression of the purpose and principles underlying their implementation of LCA. When asked “What are some reasons you might give to someone new to physics about why it is a good idea to “check your answer in the usual three ways”? Students often respond with some version of “it keeps you from making a mistake.” However, students sometimes say something that suggests that they learn from the limiting case analysis, as a student in cohort 4 did, “Limiting cases are important for understanding not just the equation, but the actual physical concepts behind the equation.” or another who wrote, “I’ve found that, as I said earlier, you can learn a lot about what the equation is actually predicting from utilizing the three checks: limiting cases, numerically, and units…Studying the variables and seeing what happens to them in the equation tells you what the process is that is happening, and if you’ve characterized it correctly in this form of equation.” The language they use reflects that they are not really surprised that they learn something from this procedure,

TABLE III. Conditions for adaptive expertise.

• Built-in randomness that necessitates trying out variations of the target skill, monitoring the results, and then inferring conceptual relationships between the variation and the result.
• A social context that values ingenuity over efficiency.
• A goal structure that allows individuals to try out variations without fear of missing out on a reward.

TABLE IV. Indicators of adaptive expertise.

• Sacrificing immediate efficiency to explore and develop knowledge that might be useful for the future (e.g., student checks limiting case because the results were not immediately obvious to them and they wanted to see what they could learn).
• Inferring "new" conceptual relationships between variation tried and the result (e.g., student discovers value of checking distance dependence for scenarios involving electrostatics).
• Flexible application of knowledge, such as inventing, modifying, or breaking from a procedure (e.g., deciding a scenario warrants checking limiting cases of constants).

indicating that they did not know what they were going to find before embarking on the LCA. Students' acceptance of "unanticipated" learning is perhaps reflective of the discovery nature of LCA described by Polya [17] and adaptive expertise more broadly. We uncovered further evidence of students' inferring new conceptual relationships during the fourth year of the study when we intentionally modified prompts to frame LCA as a way to discover physics (in addition to checking if a formula or answer makes sense).

The third indicator derives from Inagaki and Hatano's suggestion that adaptive experts possess conceptual knowledge that allows for the flexible application of procedures or even the development of new procedures. A routine expert operates with extreme efficiency under a fixed set of conditions; if these conditions change too dramatically, the routine expert may not be able to adapt. In contrast, adaptive experts are not constrained to high performance under fixed conditions. They can apply their conceptual understanding of procedures gleaned through many iterations of practice to make informed choices about what procedures to apply (or develop) in an entirely new context. Thus, when studying students' development of how to use LCA, we were particularly interested in this study in understanding how students decide which limiting cases to check when judging the reasonableness of an equation. All of the students in this study, when prompted, provide some sort of rationale for why they chose to examine a specific limiting case. Sometimes, students' rationales indicate a rigid scriptlike implementation that is not characteristic of adaptive expertise, e.g., "we don't check constants." Though adaptive experts do develop these sorts of procedures, they are careful to "avoid the over-application of previously efficient schema" [56] (p. 3). Rather, they consider a range of possibilities, such as checking constants, and select among those possibilities through careful analysis of the situation at hand. Thus, when we saw students provide a rationale that indicated they were *flexibly* applying insights gleaned from previous problems, we interpreted these responses as evidence of adaptive expertise.

The adaptive expertise literature provides a framework for the pedagogical goals of the study, i.e., for students to *learn* and *learn from* LCA. Though we did not set out to refine the theory of adaptive expertise in this study, our efforts to apply this framework required novel synthesis and application of prior work on the conditions and indicators of adaptive expertise, as summarized in Table IV.

V. STUDY DESIGN

A. Context and origin of study

This study occurred at a midsized, private university and predominantly white institution. Participants in the study enrolled in an upper-level electricity and magnetism class required for all undergraduate physics majors. Twelve (12) students participated in research during the first year of the study, 10 students in the second year, 16 students in the third year, and 8 students in the final year. We did not collect demographic data from individual participants. However, the institutional dataset indicates that the composition of physics majors changed each year of the study. Majors ranged from 44% to 54% White; 8% to 20% Hispanic; 0% to 15% Asian; 0% to 14% unknown ethnicity; 0% to 11% Black; 7% to 11% two or more races. During this period of the study, no physics majors at the institution identified as Pacific Islander or American Indian. International students comprised 8%–14% of physics majors. In terms of the reporting categories available for gender in the institutional dataset, 56%–67% of physics majors were male and 33%–44% were female. The institution enrolled between 7 and 16 physics majors per year.

The study emerged from anecdotal evidence gleaned from conversations with former students. Some of these students suggested that they learned an especially valuable skill in the electricity and magnetism class—how to check if answers are reasonable. In general, the instructor's approach, prior to the study, could be characterized initially as a rather rote, but consistent, element introduced into the curriculum in an effort to move students toward more expertlike behavior. Specifically, the instructor routinely would ask the students to "check whether their answer is sensible in the usual three ways" on graded homework assignments, as part of in-class group work, and on tests. The word "usual" was used intentionally as was the omission of what exactly the three ways were (after introducing them once early on), as the instructor wanted the students to internalize them, to remind each other what they were as a way to initiate peer discussions, and draw upon them reflexively as expert physicists do. (For reference, we include them here: unit or dimensional analysis, limiting case analysis including whether the LCA matches physical intuition, and computing numerical values to see whether they match expectations)….The word “limit” is also used intentionally (as opposed to extreme cases or “special cases,” for example) so as to remind students more explicitly of the word limit and how it was used in their math classes, leveraging the wealth of math class ideas that they should bring to bear (L’Hospital’s rule, indeterminate forms, etc.). Feedback from a few high-performing former students about this aspect of the course was particularly striking (and mostly positive) but helped reveal that perhaps not all students were benefitting from this more cryptic approach regarding LCA in the same way. Thus, began the attempt to study how students take up checking and particularly LCA. This led to a more nuanced introduction to the LCA ideas such as providing more explicit scaffolding early on about what the instructor means by limiting case analysis and giving students purported algebraic answers to some problems (rather than always insisting that they derive the results themselves) and asking them to check whether this answer is sensible in as many ways as possible.

The research team consisted of G. W., the electricity and magnetism course instructor; T. Sikorski, a science education researcher at the same institution; and two undergraduate research assistants (M. Ahmed and J. Landay) who participated in data logging and analysis. The research team met approximately weekly in the fall of each year to design the prompts, analyze data, discuss related literature, and prepare for conference presentations. A system was designed whereby G. White would not know which students were participating in the study while they were enrolled in the course. The undergraduate research assistant J. Landay participated in year 1 of the study. J. Landay was a physics major, a former physics learning assistant, and had previously taken the electricity and magnetism class that was the focus of this study. J. Landay assisted in analyzing cohort 1 data to identify which check(s) students conducted. The second assistant, M. Ahmed, joined the project in year 4. M. Ahmed took a class on pedagogy with T. Sikorski, was a biology learning assistant, and had no prior physics training at the time of the study. M. Ahmed looked across all four cohorts of data to create data tables showing which limiting cases students checked, and in what order, for each prompt. We presented our ongoing analysis regularly throughout the project, at AAPT [2,49], APS [57], and PERC [10,58] in order to gather critical feedback from the physics education research community.

B. Research process

Design-based research “simultaneously pursues the goals of developing effective learning environments and using such environments as natural laboratories to study learning and teaching” [59] (p. 200). We approached the study with the dual goals of interpreting and influencing how students check equations for sensibility. The study took place within the bounded context of an electromagnetism course. The course ran 4 times, once per year of the study, each time with the same instructor, but with students and learning assistants changing for each iteration. Thus, the classroom acted as a “living laboratory” allowing us to hone in on specific aspects of our pedagogical intervention “while maintaining the complexity of the local context” [60] (p. 32). Over the course of the four years of data collection, we made many systematic adjustments informed by our hypotheses about student thinking and our ongoing data analysis.

The study was iterative, and modifications in each iteration were principled and driven by the aforementioned instructional and research goals [61]. We began with a set of instructional goals and research questions at the start of each year. Based on what we learned from each round of data collection, we made modifications for the next round of data collection. These adjustments occurred within each year and also across years. For example, year 1 began with the instructional goal of students learning to check their solutions in three ways: to check if the units of the solution are reasonable, to check if the limiting cases of a solution are sensible, and to try plugging in numerical values into the solution. As already discussed, prior research had shown that students often skip the “answer checking” stage of problem-solving protocols. We wondered if requiring students to generate their own solutions got in the way of their being able to practice checking solutions. For example, if students spend the entire time allotted generating a solution, they may not have sufficient time or focus to try any of the checks. In the electromagnetism course, we also knew that students would face quite difficult problems that they may be unable to solve. From a pedagogical perspective, we thought at the time that in order to develop adaptive expertise with the checks, students must, at minimum, have adequate opportunities to practice using the checks. Thus, in order to make sure that students had adequate opportunities to practice the checks, we generated a type of “layered” exercise that focused exclusively on the checking aspect of problem solving. By “layers” we mean to indicate that the students were shown only one layer of the exercise or prompt at a given time (a layer might be one or two or even more pages), and students were not able to access responses on a previous layer after viewing and responding to that previous layer.) Students were presented with a scenario, such as a charged pith ball suspended near a charged Van de Graaf sphere and a proposed formula describing some aspect of that scenario. Students were asked to check if that solution is sensible, either explicitly by asking them to invoke the three checks (units, limiting cases, numerical values) or by asking them more generally to check the formula in as many ways as possible. Most of these data collection prompts occurred during the “high stakes” situation of quizzes and tests.

In year 1, we focused exclusively on students invoking the three checks explicitly taught (units, numerical values, and limiting cases). However, in analyzing the student responses to the prompts, we noticed that over time students stopped conducting other, potentially useful checks that were not part of the "usual three ways," such as looking for resemblances to other equations or trying to derive the origin of terms in the equation [49]. So, in year 2, we emphasized in the written exercises that students should check "in as many ways" as they could think of, and we began keeping track of the other types of checks students used. We also noticed differences in terms of which variables students would choose for their limiting case checks—an important observation that shaped our year 3 investigations.

Year 3 marked a critical phase of the research when we began to focus in greater depth on how students implemented limiting case analysis. We had noticed in prior datasets that for any given formula, students would check particular variables frequently and others rarely or not at all. We wanted to better understand how students were selecting which variables to check; students' rationales might provide evidence of their focus on efficiency, scripts, and routines, or playfulness, flexibility, and innovation. In year 3, we modified our prompts so that students could identify which variables they would check first, second, and never, and why. We learned for example, that students do not take limiting cases of constants because they perceive these as unchanging. Critically, year 3 also revealed that limiting case analysis could serve a potentially important role in students' development of physical and mathematical intuitions; invoking physical intuition is explicitly part of the check, which may explain its apparent greater effect on student problem-solving performance as described in the literature review [9,11–14].

Years 1 through 3 focused on students' implementation of the limiting case analysis to make a judgment about the reasonableness of a solution. A professional physicist might use limiting case analysis in this way, for example, to check a formula in peer review [62], or to check an intermediate result. In year 4, we shifted our attention to another very important role of limiting case analysis in professional physics: to develop new intuitions about the world [20,26]. Here, our instructional and research goals merged. Can we shift students' framing of limiting case analysis so they view it as a way to learn or discover physics? (as opposed to or in addition to checking if a formula or answer makes sense). We wondered whether, as Polya suggests, our students could develop new insights about a formula by conducting limiting case analysis.

To our knowledge, this study represents one of the first iterative and ongoing analyses of how students learn to use limiting case analysis in the context of an upper-level undergraduate electricity and magnetism course. This paper is a cumulative presentation of four iterations of data collection and analysis, exploring how students develop capacity for, and perspectives about, limiting case analysis in response to written prompts specifically designed for that purpose. Our findings yield new insights into what instructional approaches might support students' learning this tool of professional physicists. As well, we identify a number of promising directions for future research that considers more specifically the role of limiting case analysis in the development of physical intuition in the context of studying electromagnetism and the blending of their physical intuitions with mathematical formalism.

VI. FINDINGS

In prior work, we have reported on our efforts to increase students' use of LCA, checking units, and checking numerical values, detailing some success in increasing the use of LCA among undergraduate majors [2,10,49,57]. In addition, we have noted in earlier work that our less experienced students, while able to produce a robust list of kinds of checks that they might apply to a purported solution did, not readily generate the idea of checking limiting cases; this was in dramatic contrast to students in upper-level courses who had previously taken a course with the instructor [58]. This provides additional motivation for us to study LCA because it seems that we are able to see substantial changes in performance over a relatively short period, within a single semester.

Here, we report cumulative observations from all four cohorts of the electricity and magnetism class, addressing students' choice of which limiting cases to check, evidence of intuition in students' limiting case analysis, and examples of student learning from their use of LCA. We present these findings as emerging from the local conditions of the study. The instructor's way of explaining and introducing the checks, the students' prior academic history in the program and with the instructor, the design of the prompts and implementation, and the social context of the class all likely contribute to the way that students chose to respond to these prompts. Past studies of students' use of limiting case analysis present very different results, speaking to the importance of local contextual factors in how students may understand and utilize this tool.

A. Students' choice of which limiting cases to check

Part of what makes limiting case analysis interesting to study is that, in the words of Nersessian, it is not algorithmic; an equation can afford many possible limiting cases, which may yield more or less insight depending on one's purpose. This aspect of LCA—the choice of limits to take—is not addressed in past studies of students' use of LCA, and only cursorily addressed in teaching materials. Polya's texts stand out as providing detailed elaboration of the roles that curiosity and experience can play in choosing limiting cases. For example, he writes, "You may consider a generalization, or particular cases, limiting cases, analogous cases. There is a chance to find something interesting as well as a chance to learn to do research" [17] (p. 192). Polya also describes limiting case analysis as curiosity driven. He often invites the reader to ask questions, e.g., "Is the whole straight line a limiting case of an ellipse? Or that of a hyperbola? Or is some part of the line this and the other that? And so on." (p. 201). This presentation of limiting case analysis as a discovery and curiosity-driven process differs from the more routinized or scripted approaches promoted in some contemporary physics teaching materials and even in our first treatment of LCA in the present study [10].

In the early years of the study, when we asked students to check equations "in the usual three ways" (year 1) and in as many ways (year 2), we noticed that students almost never checked a limiting case for a constant [2]. To better understand how students decided which limits to take, we modified the prompts in years 3 and 4 to ask students which limits they would check first, and why, and which limits they would be unlikely to check at all, and why. We imagine that this question serves to make the students aware of the variation in the problem, and suggests to the students a certain "randomness" in how to proceed, and further suggests that understanding is valued compared to efficiency, thus hopefully, facilitating the development of adaptive expertise. With this prompt, we are inviting them to create a set of instructions…to realize that they have to make their own script.

As an example of how this plays out, consider how an interviewee responded when asked how she decided to choose which variables to check in this formula for the tension when two charged balls are suspended symmetrically as shown (see Fig. 3):

> *Some of it is kinda just a guess…I think in some problems, it's a little more clear, like in the wording …I checked the mass because the mass was a fairly easy one to check, although I forgot that there was a mass right here as well (revisits mass check and notes a problem) that I didn't think about when I was initially doing this…also at least in my mind, I already kind of had an idea of what the…should happen if the mass gets larger…*

Note that, in addition to alluding to the flexible application of knowledge, she also states that she chooses based on what seems "easy" to check, conveying that her criteria for easiness include her own evaluation of her intuition about the mass dependence. We will have more to say about intuition in the next section.

In written responses to these questions about what limits they would check and how they decided, many students reported that of all possible "symbols" in an equation, they would be least likely to check a constant such as the Coulomb constant (k) or magnetic constant (μ_0), as we expected. Students' explanations for why they would refrain from checking limiting cases for constants ranged from the tautological ("Since it is a constant"), to utilitarian ("μ_0 is a constant (also g) so taking the limiting case does not make sense or give you any information"), to contextual ("constant. this is the amount of resistant encounter when forming a B-field in a classical vacuum, unless we were trying to prove μ_0 or refute it's [sic] value, there wouldn't be a reason to change it" and "these are constants (or very nearly constant) and would not give a great deal of physical meaning about the problem if changed as this can be assumed to be in a lab setting"). The aversion to these kinds of checks seems to us to be related to Bing and Redish's suggestion that students' perceptions of "valid uses of math in physics class tend to align with physical reality," leading to their hesitancy to differentiate with respect to universal constants [63] (pp. 010105–5). From the perspective of developing students' adaptive expertise, the utilitarian and contextual explanation are the most appealing in that they leave open possibility for innovation, whereas the tautological arguments seem to close the door on any possibility of future learning from limiting cases involving constants.

For any given equation, students did not all conduct the same limiting case checks; rather, there was quite a bit of variation in which limits students would check first, second, and so on. For example, during cohort 3, students were presented with a diagram of a cross-sectional view of a copper wire surrounded by an aluminum casing. The particle at point P is stated to be moving into the page at a constant speed V_0 parallel to the wire at a fixed distance D above the wire. The prompt included multiple parts (see the Appendix). Here, we focus on layer 3 where students are provided a possible formula for the mass of a particle traveling in this way and asked which limiting or special case checks would help them determine if the formula is sensible (see Fig. 4).

Figure 5 shows which symbols students from cohort 3 planned to check. Of the eight symbols offered, six were selected by at least one student for the first check (V_0, D, g, J_1, J_0, and R). A similar variation was seen in other prompts. From the perspective of adaptive expertise, the variation is important evidence that, collectively, students have not internalized a single procedure for the choice of limiting cases; rather, each student is having to figure out how to apply their knowledge and experience to select among a number of viable options.

While there was variation in the selection, there is also some consistency across responses. For example, as mentioned previously, students were least likely overall to

$$T = mg\sqrt{1 + \left(\frac{kq^2}{4mg(L+b)^2}\right)^{2/3}}$$

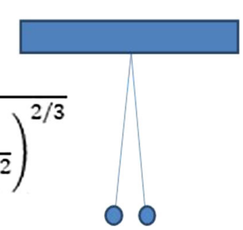


FIG. 3. Proposed tension in symmetric suspended charged balls problem.

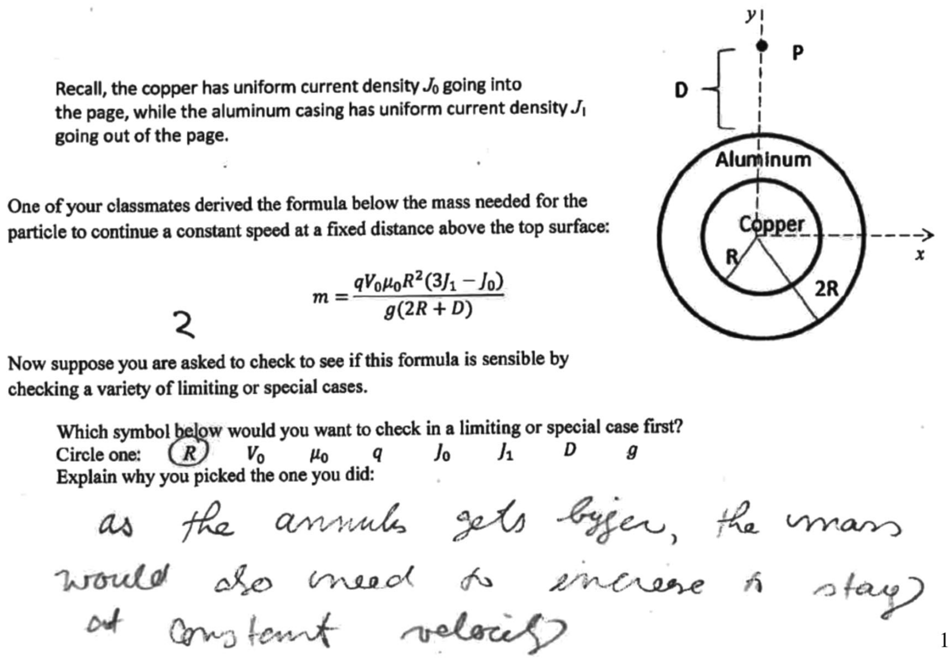


FIG. 4. Example prompt from the electricity and magnetism class asking students to describe their choice of limiting cases. Student selects R because *as the annulus gets bigger, the mass would also need to increase to stay at constant velocity* (Images of student work are modified to improve readability).

check constants, with the exception of one student. Symbols representing distance (such as D, representing the distance of the particle from the wire in Fig. 4) were selected most often across many prompts. Almost all students referred to checking the "large" limit of D, e.g., the particle far from the wire. This check seemed to be preferred because students had a clear sense of what the sensible result should be. One student wrote,

When D gets large

→ force obviously decreases

→ requiring ~~mo~~ less mass for attracted particle to balance.

Students seemed to value when it was "obvious" what *should* happen in the case of a particular limit. It is our sense that obvious often corresponded to familiar

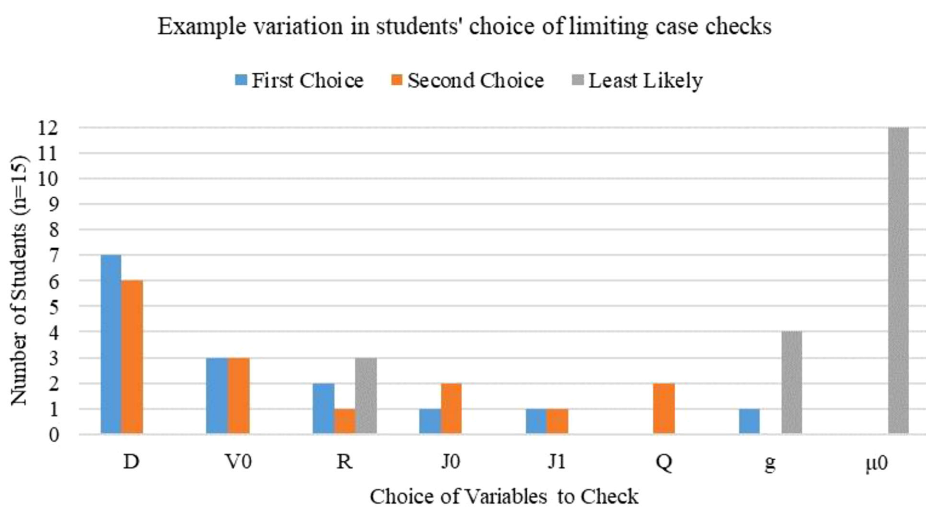


FIG. 5. Summary of student responses to prompts related to Fig. 4, asking which symbols they would check first and second, and which they would be least likely to check.

quantities—distances, masses, and velocities. The preference for familiar quantities indicates students' attention to efficiency; the prompt asks students if the formula is sensible; to accomplish that goal expediently, they look for limiting cases that could immediately reveal a fundamental flaw in the formula or that corresponds to a scenario for which the outcome seems especially apparent. The prompts did not readily reveal if students were willing to sacrifice this efficiency in order to develop knowledge that might be useful for the future. However, students did show a preference for "simple" limits that yield information they already know to be valuable, as one student wrote about checking the initial velocity:

> *Checking the limiting cases of the initial velocity involves a simple limit but gives you valuable information. Of course you know that if $Vo \to 0$, m will be zero because there is no initial velocity to launch mass m and therefore won't travel parallel to the wire. So if you get a different answer you know there is something wrong. If $Vo \to infinity$, then $m \to infinity$ because if the initial velocity is very fast the particles [stet] mass must be heavy in order to travel at a fixed distance at a constant speed.*

This response (and others also) indicates metacognitive activity; in selecting which checks to conduct, students are considering whether or not they are equipped with the physical or mathematical intuition needed to evaluate the

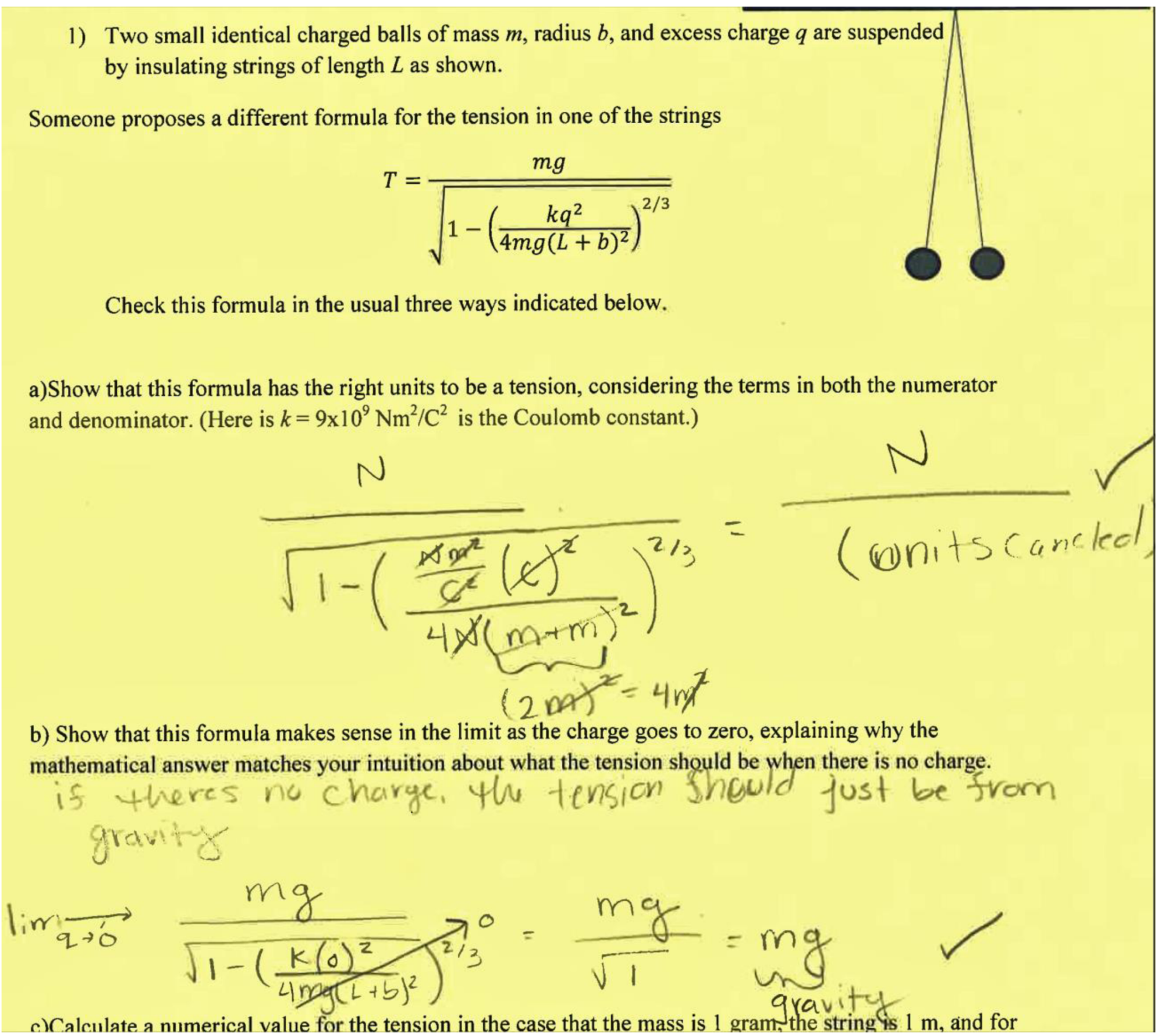


FIG. 6. Student responses to the first LCA prompt provided during year 2 of the study. The prompts provide explicit instructions about which checks to conduct.

results, an observation which we elaborate on in the next section.

B. Aspects of intuition in students' limiting case analysis

Limiting case analysis involves *application* of one's physical and mathematical intuitions [64] and simultaneously provides a means for *developing and refining* intuitions [17,18]. Because these processes cannot be meaningfully reduced to routines or scripts in the context of professional physics, we model LCA as characteristically adaptive. LCA requires flexible application of intuitions to solve problems, as well as the willingness to sacrifice immediate efficiency to explore and refine intuitions. Here we address some aspects of how students respond when asked questions about their "intuition" while conducting limiting case analysis.

1. Attempts to elicit student elaboration about intuitions

Note that no systematic attempt was made to define intuition for the students other than to use the phrase in the contexts shown below. Early in the semester for each cohort, the instructor indicated what he had in mind with explicit direction such as "show that this formula matches your intuition in the limit as the charge goes to zero…" Students often responded to this explicit instruction satisfactorily (in the instructor's view), such as in the response shown below in Fig. 6, where the student writes "if there is no charge, the tension should just be from gravity" and uses mathematical notation to indicate that the formula produces that result in the limit as q goes to zero.

Likewise, in other cohorts, we have similar examples where the students write out their expectations in brief, such

> *As q goes to zero…$T = mg$ makes sense because it is a free fall problem if no charge is included*

> *When there is no charge the balls do not repel each other and hang only by the string keeping them from the ground. Thus tension should just be mg. When q is 0 the denominator is 1 and mg is the tension.*

These responses provide some insight into students' physical and mathematical intuitions. For example, students identify causal relationships between variables (e.g., charge and force) and describe physical relationships between material components of the problem (e.g., strings and ground). They also point out when a limiting case yields a familiar equation (e.g., free fall problem).

These responses are to be contrasted with much more terse responses that we collected if the phrase about explaining why it matches your intuition was not explicitly included, at least at the beginning. When using open-ended prompts such as that indicated above in Fig. 1 and in Fig. 7—"check to see that this formula is sensible in as many ways as you can think of, explaining your thinking clearly."—we saw little in the way of student commentary

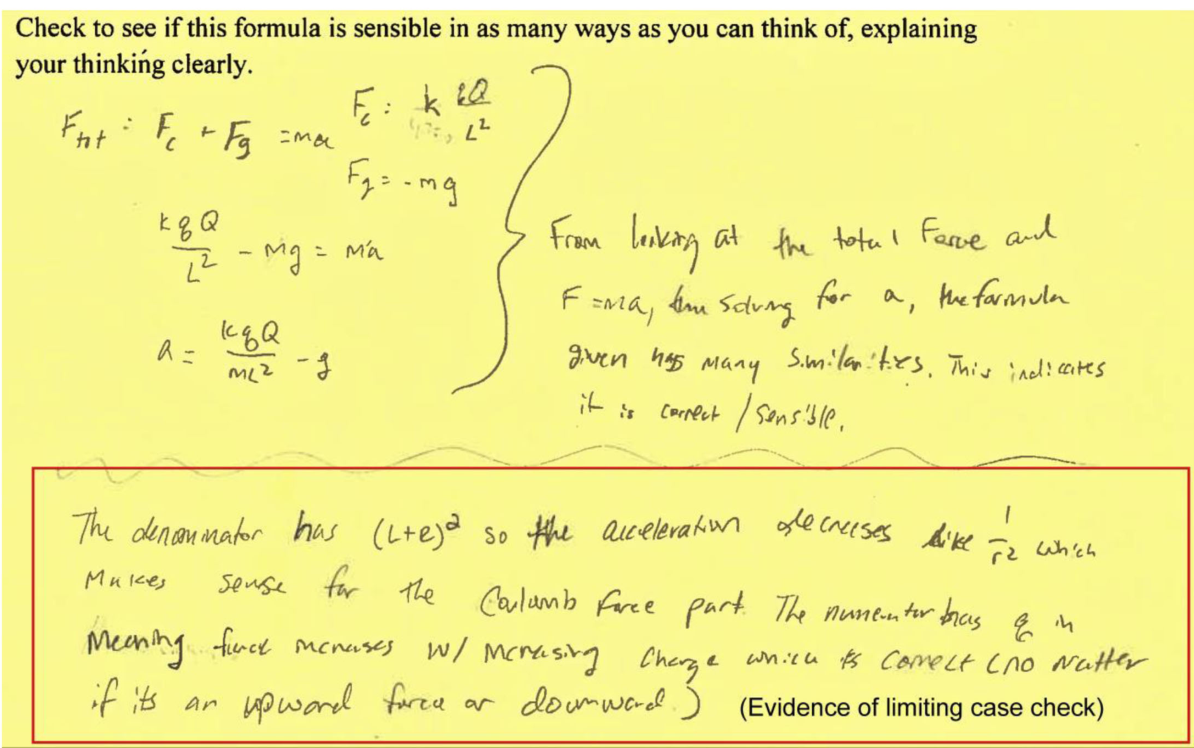


FIG. 7. More elaborate responses eventually forthcoming even with vague prompt, once earlier specific prompts were used.

reflecting intuition. Students provided meager commentary, simply asserting that the formulas "make sense" or behave "as expected" without indicating what they mean, as in the student response below.

If L increased, the tension would decrease as expected. If m increased, the tension would increase, as expected If q increased, the tension would increase, also as expected.

Typically, the instructor would mark this kind of response as inadequate. However, we note that this type of response is very similar to the verbiage that students encounter in physics textbooks. Griffiths' [6] widely used text for upper-level undergraduate electricity and magnetism courses uses limiting cases as a sort of pedagogical tool, to introduce new formulas. In example 2.1, to derive the equation for the electric field at some distance from the midpoint of an infinite straight wire of uniform charge, students are shown a solution to calculate the field at a distance from a line segment 2L (at this point, students do not know where the derivation is headed). Then, students are shown that in the limit $L \to \infty$, an equation for a wire of infinite length is obtained. No explanation is provided for why such an equation is useful, but the importance is tacitly suggested in a series of practice problems that follow the derivation. This approach to LCA happens repeatedly in the text, and there are a few aspects of this use we wish to highlight. First, the result is often treated as a forgone, trustworthy conclusion; equations derived by taking limits of other equations are generally not accompanied by dimensional analysis or numerical value checks to confirm sensibility. Second, it is rarely explained to the student how the choice to take a particular limit (e.g., $L \to \infty$, $z \gg L$) and not another (e.g., $L \to 0$; $\varepsilon_0 \to \infty$) is made. Finally, the interpretation of the limiting case as "making sense" is often abbreviated; Example 2.1 shows that in the limit far from the line of charge, the equation reduces to that of a point charge. This result is labeled as making sense, and it is taken for granted that students will be able to rationalize why a line should behave like a point from far away (nor are students explicitly invited to challenge this "intuitive" result.) In contrast, we explicitly encouraged students to question whether they thought the results of their limiting case check were intuitive or sensible and to explain their reasoning. By doing so, we hoped to get a better sense of the considerations that go into students' determination of whether a result makes sense, and simultaneously, we hoped students would begin to view themselves as arbiters of reason, rather than defer to instructors about whether a result makes sense [7].

In our study, the students only began to elaborate on their intuitions after being given an explicit prompt such as that shown above in Fig. 6, part 1b) early on; afterward, we then found that much less explicit prompts are needed to draw out richer student commentary about intuition in later iterations. Here is just one example of several from day 8 layer 2 cohort 1; note that the prompt is again more generic, and the student responds with "The denominator has $(L+R)^2$ so the acceleration decreases like $1/r^2$ which makes sense for the Coulomb force…"

2. LCA and student intuition via the lens of adaptive expertise

We argue that because LCA is nonalgorithmic in many respects, it has the potential to be more valuable pedagogically than other means of checking answers. This corresponds to the flexible application of knowledge in response to "random" variations in conditions that Hatano and Inagaki emphasize. Students engaging in LCA are afforded opportunities to query themselves about their own knowledge of physics in order to decide which quantity to vary in their imagined limit taking. Affording students the time and space to discern whether their own choices of limiting cases make sense to them sends a signal that there is value in understanding even at the expense of efficiency. The act of making these choices, we believe, is productive for students as they are developing as physicists. For example, we see evidence of a more nuanced understanding of "infinity" among students in these cohorts as they respond to various prompts over the semester. Early in the semester, when asked "what happens to the electric field due to a charged rod, say, as you get farther away?," we see that students almost universally respond with some version of "the field goes to zero," but as the semester progresses the students learn to ask, "How exactly does it go to zero, as $1/r^2$ like a point charge, or as $1/r$ like a line of charge, or as $1/r^2$ like a dipole?

As a further example of this, consider student 3's response on day 19 of the class. The second layer (page) of the prompt provides an expression for the mass required such that the particle maintains a constant speed at a fixed distance above the wire as shown in Fig. 4 and asks students to explain how they would go about checking if the expression is sensible. Student 3 listed D as the first symbol to check, providing a clear articulation in layer 3: "confirming that distance behavior is a $1/D$ relation confirms that the proportionality of mass to the magnetic field is kept ($ma = qv \times b$); B at $D \to \infty$ decreases at rate $1/D$ for a wire."

Here student 3 shows clear evidence that they are picking this limiting case to check because of their confidence in their knowledge that the magnetic field strength should go to zero as $1/D$ since this cylinder can be treated as a wire in that limit. One might raise the question whether this qualifies as a physical intuition or rather something else such as "memorized facts about electrostatics." For our purposes, it is useful to see that the student is comparing the expression at hand to their storehouse of knowledge to see if it is sensible. Furthermore, student 3 notes that checking J_0 will be useful for "confirming that a high current density

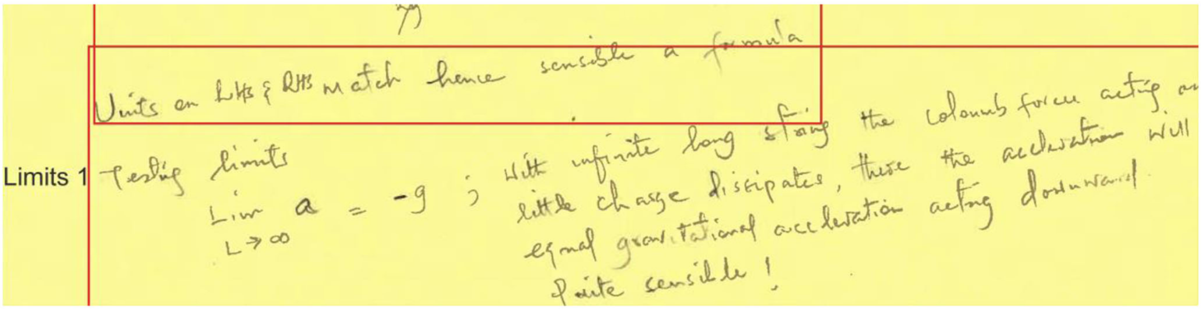


FIG. 8. Student shows some enthusiasm for LCA when results go as expected.

requires a higher mass, as the increased repulsive force has to be canceled by a stronger gravitational force." The student then conducts a couple of different limiting case checks, most of which hold up to the student's expectations. However, one special case involving the current densities (the limit as J_0 goes to infinity with $J_0 > 1$) yields an "impossible" result of negative mass and the student interprets that this means that the equation is indicating that the only way to maintain the velocity of the particle, in this case, is to have the charge on the particle to be negative. This indicates an admirable level of flexibility on the part of the student, in our view, in how to interpret the equation so that it aligns with their intuition. In other words, the student notes that the equation can still give a relevant mass for the problem if $J_0 > J_1$, but only if one ascribes the opposite sign to the particle's charge. This willingness to seek out meaning (and this ability to find meaning) in the expression, even when moving beyond the original specifications of the scenario, seems consistent with the idea of adaptive expertise, as well.

We also note here that a certain level of enthusiasm for performing LCA is also evident, such as in the example shown above (Fig. 8) where the student writes, after noting that the limit of the acceleration of this sphere approaches "$-g$" as L goes to infinity and explains how this can be understood from physical considerations "Quite sensible!" This aligns with the idea that adaptive expertise has a playful aspect to it. However, we also see evidence that students are sometimes more focused on "performing school" when engaging in LCA. In the example below (Fig. 9), a student first checks the limit as $L \to \infty$ and concludes, "this matches their intuition since the electrostatic force would become negligible as the small sphere moved away." Then the student turns to the limit as $R \to 0$ and indicates that the expression goes to $1/R^2$ like a point charge, matching their intuition (and overlooking the R^3 dependence in the numerator). The student may have been so eager to see the familiar point charge result that they gloss over the complications presented by the R^3 factor in the numerator.

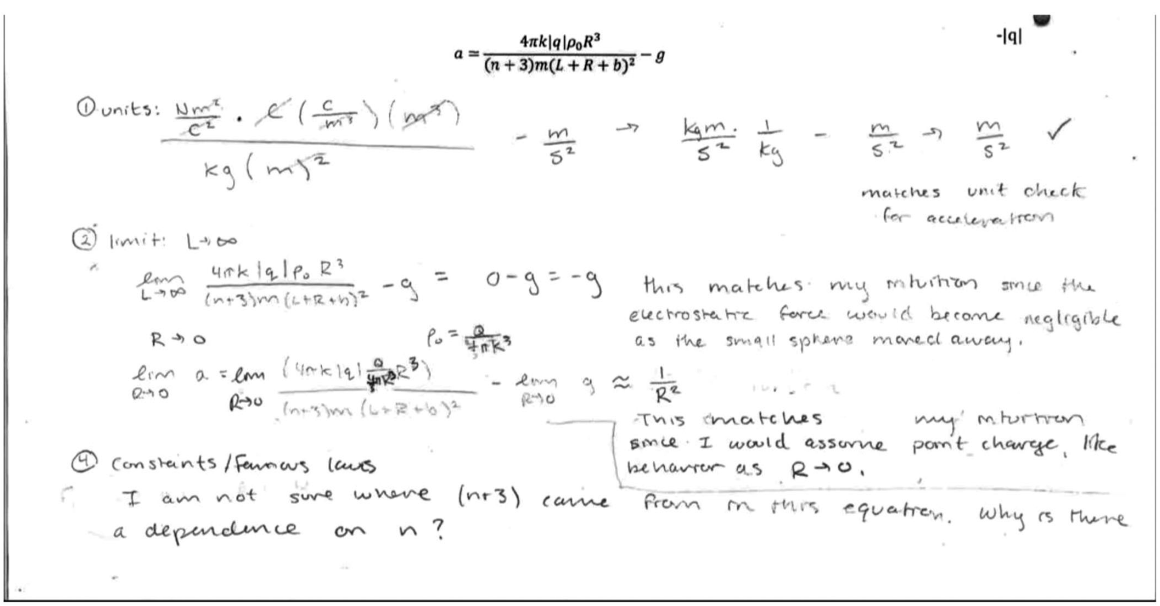


FIG. 9. Student recognizes $1/R^2$ dependence is to be expected and overlooks other R dependence.

3. A rich example wherein LCA and intuition interact

As previously discussed, one of the reasons that students provided for why they checked certain "symbols" was because they had an idea of what to expect or an *intuition* about what would be sensible. However, students did not always articulate an intuition about what limiting behavior would "make sense," and some openly professed difficulty formulating an intuition at all. Here we provide a contrasting example to those above, an example that conveys some of the nuance and breadth of issues about how intuition figures into students' limiting case analysis.

Recall that in layer 2 of the prompt, the student is provided with an expression that gives the mass necessary to travel at constant speed at a fixed distance above the wire as shown in Fig. 4 and asks students to explain how they would go about checking if the expression is sensible. Student 8 lists the checks in the usual order (units, limits, and values). For the limiting case check, the student writes, "In this scenario, I would check the limit as $V_0 \to 0$, $V_0 \to \infty$ to see how the mass behaves with an increasing/decreasing velocity (all other things held constant)." In contrast to student 3 discussed earlier, student 8 does not offer any sort of expectation of what behavior would be sensible for either of these limiting cases. On layer 3 of the prompt, students are asked which limiting cases they would check first, second, or not at all, and why. Student 8 selects D first because "provided all other factors stay the same, changing the distance could be a good indicator of how the mass will change." Notice that student 8 does not describe what result might be sensible or how mass should change with D. Instead, the student seems to present limiting case analysis as a way to determine what the functional relationships might be. The student picks R as the second limit to check, saying that this check might provide a "good indication of how the mass may change" but without actually suggesting what functional relationship might be sensible. It is difficult to tell from this response what, if any, physical intuition student 8 activates while planning a limiting case check.

However, in contrast to the earlier layers of the prompt, student 8's actual execution of the checks on layer 4 does reveal more about what they are thinking. The student first finds that as $V_0 \to 0$, $m \to 0$, and writes, "This doesn't match my intuition because just because something isn't moving doesn't mean it will be massless." The student draws a big "X" next to the check to indicate that the result is not sensible. The student also checks $D \to \infty$ and does not think the result makes sense: "This doesn't match my intuition because just because we move away from the wire, doesn't mean it will lose its mass." Though this represents a misinterpretation of the prompt, the student's commentary does show evidence of activating physical intuition, in our view. Student 8 seems to read this limiting case check as a causal story, e.g., $V_0 \to 0$ causes $m \to 0$ (rather than saying that as V_0 goes to zero, one expects that the particle's mass would have to be smaller since there is less magnetic force to "levitate" the particle.). The physical intuition is activated in assessing the reasonableness of these causal stories. Student 8's everyday intuition is that an object's mass does not change as the velocity or distance changes. We suspect that student 8 did not activate any sort of physical intuition until the check was actually conducted when the student was trying to determine whether the result was "sensible." In this problem, the student treated limiting case analysis as a process of discovering potential causal relationships and then pausing to evaluate whether those relationships make sense. Student 8's approach to LCA is an important counterexample to Burkholder *et al.*'s claim that "in order to think about limiting cases, students need to have a sophisticated enough mental model of the problem to have certain expectations for the results they will find" [64] (pp. 020134–4). In fact, we suggest, a better indicator of adaptive expertise is when a student willingly checked a limiting case because the results were not immediately obvious to them and they wanted to see what they could learn, even if it turned out the limit was difficult to interpret or not particularly insightful, a sort of willingness to go down "rabbit holes." This perspective helps us see merit in student 8's approach, which we might have otherwise overlooked.

C. Examples of students learning from LCA

So far, we have discussed indications that students implemented LCA in ways suggestive of adaptive expertise, for example in recognizing the opportunity for choice in response to variation in problem scenarios, as well as the flexible application of knowledge. We have also noted where our pedagogical intervention fell short, for example, by reinforcing students' attention to efficiency, potentially at the cost of their developing useful knowledge for the future. In our final year of the project, we designed a problem specifically intended to help us see whether students in cohort 4 could learn new physics from performing LCA. The initial context is the problem featured at the beginning of the paper, wherein a small charge is suspended by a string beneath a large sphere of charge. The string is cut and the students are asked about the subsequent motion of the small charge.

In this version of the problem, the radial distribution of charge in the large sphere is given by $\rho = \rho_0 (r/R)^n$, that is, it is directly proportional to r^n, with the charge density ρ_0 being the same constant no matter which value of n is considered. The students are then asked to "Think about the motion of the ball after the string is cut; what ideas do you have about how this motion depends upon the value of "n"? (see the Appendix for the full version of this question). In response to this question, student 1 initially predicts that the electrostatic attraction will increase if n is increased:

e) Someone proposes that the acceleration of the little sphere just after the string is cut is (upward is taken to be the positive direction);

$$a = -\frac{4\pi k q \rho_0 R^3}{(n+3)m(L+R)^2} - g$$

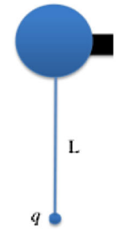


FIG. 10. Proposed formula for the acceleration of a small charged sphere.

As n increases, that greater charge density will be closer to the surface of the sphere than the center so more charge will be closer to the ball. The ball will experience greater electrostatic force as n increases. It's [sic] acceleration downward from the force of gravity will be more offset by the greater electrostatic attraction.

After the students have had a chance to write about their initial ideas regarding how the motion of the small ball after the string has been cut depends upon n, they are then presented with a proposed equation for the acceleration of the small sphere (see Fig. 10). The students are asked to examine this formula in various limits to see whether it matches up to their own earlier predictions.

Student 1 recognizes explicitly that this formula conveys that the Coulomb interaction (and thus the upward acceleration of the little sphere) weakens with increasing n (opposite of what they had predicted); furthermore, they present some reasoning about why their earlier idea was misguided, acknowledging that some of the charges on the big sphere will be closer to the small charge, but some will be further as n gets larger, making it difficult to extract the full effect, saying

This formula is more involved version of what I thought on the other page. Only difference is I didn't figure out what Fc would be. The Coulomb's force term [Fc] decreases as n increases or as charge becomes more distributed on outer area. I had hard time figuring out what could happen as n changes because of how ball is positioned relative to sphere. As more charge comes closer, charge on other side of sphere is getting further away but the r^2 term makes that hard to picture w/o math.

We claim that this excerpt provides evidence that by performing LCA (on a proposed but unsubstantiated equation) Student 1 has achieved more insight into this charge distribution and how it should affect the acceleration. For us, this is reminiscent of Zietsman and Clement idea [18] that by performing extreme case analysis, the student's attention is focused on the nature of the charge distribution in more detail, recognizing a feature that went unnoticed before and strengthening the student's overall grasp of the situation. At the end of the exercise, when asked directly whether new physics was learned, student 1 writes, "I think you can learn new physics by examining the expressions as long as you know the assumptions the expression is based on. Sometimes an expression looks valid but does not hold for all cases or situations and could teach you wrong physics." The student explicitly conveys that LCA could teach you new physics in a specific regime, or could possibly be a problem if the solution does not apply in that regime, but emphasizes that it is important to consider what assumptions have been made. Note that the student is admirably cautious about concluding too much here; also note that they are willing to reconsider their earlier claims about the dependence even though there is no assurance given that the provided equation should be trusted. We also see this example as providing an indicator of adaptive expertise in that the student shows willingness to abandon the original idea as decisive (that the larger value of n would mean that the charges would be closer on average to the ball) revealing a flexibility in the student's application of knowledge.

As another example of a student learning about physics from LCA, we present the following, also from cohort 4 but later in the semester (day 20; see the Appendix for the full set of questions). The students are asked about how the magnetic field outside a long fat wire depends upon the distribution of current inside the wire, with the parameter n once again playing the role of the power of the radial current distribution within the wire (that is, we can write $J(s) = J_R(s/R)^n$ *for* $s < R$, and $J(s) =$ zero otherwise, where J_R is the same constant independent of n). In this case, there is a charged particle at P (see Fig. 11) traveling (anti-)parallel to the wire at distance D above the upper surface of the wire (influenced only by the wire's magnetic field and the gravitational field). The particle is traveling at precisely the speed "v" necessary so that it maintains a straight-line trajectory at distance D above the surface of the wire. Here again, we are especially interested in what the students think about how this speed depends upon the power n.

In response to the query, "Think about the speed of the small particle; what ideas do you have about how the speed of the particle depends upon the value of n?" a student writes, *So the question is which value of n produces the greatest total current, a greater value of n should produce a greater total current which would have a stronger B-field, which would cause a lower velocity.*

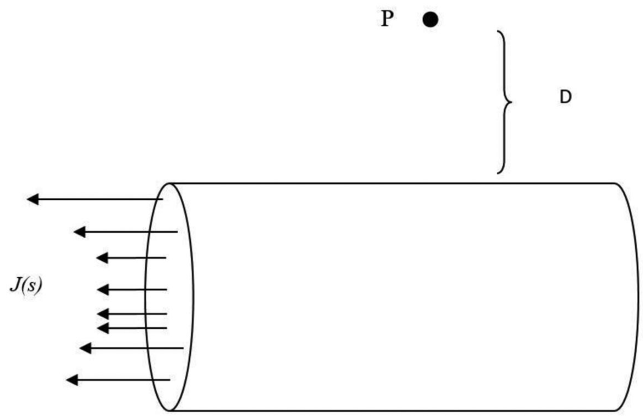


FIG. 11. Particle at point P is traveling in the direction opposite the current and remains parallel to the cylinder with a nonuniform current distribution.

In this exercise, after the students have expressed their initial views about how v might depend upon n, the students are asked to work in groups to arrive at an equation relating these two parameters, starting from "famous laws of physics." Most groups, including the group with student 5, apply Newton's second law and Ampere's law to arrive at a solution for v in terms of n and other parameters such as

$$v = \frac{mg(R + D)(n + 2)}{q\mu_0 J_R R^2}.$$

The student then notes that the equation that they have derived predicts the opposite of what their initial ideas were:

Prompt: What does this formula say about how the speed depends upon the value of n?

Student: "*as n increases v increases*"

Prompt: Does this match your ideas from earlier (see other side)? Explain.

Student: *"no, I said they would be inversely related"*

Prompt: Have any new ideas about how the speed depends upon the charge q or the parameter n emerged from when you first thought about the problem to now, after you have solved the problem in detail? Explain.

Student: "*Yes, I understand better how n plays a role into it and how q does as well*"

Note that while the student does not explain in detail what it is that they understand better, it seems clear that the student believes that they have learned something from the equation (and presumably from the derivation that they themselves performed in the group work). This is the only example we saw where students explicitly examined a bit of physics that they developed and appeared to have come away with a more expert perspective on some aspect of the related physics, having learned something from doing LCA, but we do feel that this example captures in some sense the ambitions expressed by the cited educators in teaching LCA explicitly. Because the student has adopted a new conceptual relationship between the parameter n and the speed v, we maintain that this scenario suggests adaptive expertise is at play in this context.

In our discussion of our findings throughout this section we have referred to Figs. 4, 10, and 11, but these figures are only abbreviated versions of the full prompts. For the full version of the prompts corresponding to Fig. 4, see Figs. 12, 13, 14, and 15 in the Appendix. To see the full version of the prompts for Fig. 10, see Figs. 16 and 17 in the Appendix; Finally, the full version of the prompts represented by Fig. 11 can be seen in Figs. 18, 19, 20, and 21 in the Appendix.

VII. DISCUSSION AND RESEARCH IMPLICATIONS

This study employed a design-based research methodology to study students' use of limiting case analysis in an upper-level electricity and magnetism course. From a research perspective, our orientation to understanding variation in students' use of limiting case analysis led to important new discoveries regarding how students select cases for analysis, and how they attribute meaning to the results of their limiting case checks. We also observed the benefits of our research for teaching practice. Because we were able immediately to assess students' responses to our intervention, we could quickly make adjustments informed by systematic analysis of our data. Some discoveries that we made during the study were not related to limiting case analysis but were very important for students' understanding of electricity and magnetism. For example, students' responses to one of the prompts about charge density led the instructor to realize that students were struggling with the concept of nonuniform density. The continuous feedback between research and practice led us to constantly ask: What is the pedagogical value of students completing this task? What is the research value of students completing this task?

As our data collection consisted primarily of individual students' responses, we were limited in what kinds of information we could glean about students' perceptions of limiting case analysis. If we had asked students to discuss limiting case analysis with each other in class or observed their study groups, we may have gathered more information about how they perceived this tool as part of a physics community resource or norm. Stephens and Clement [47] observed that during physics class discussions, students' spontaneous use of extreme case reasoning was always accompanied by a "depictive gesture." Because our data collection focused on students' written responses, we were unable to observe other types of evidence, such as gestures, which might have increased the instances of limiting case analysis observed or revealed more about the substance of students' LCA. In addition, most of our data collection prompts occurred during the "high stakes" situation of quizzes and tests. We did not look at how students take up limiting case analysis in more relaxed settings such as on homework assignments or in study groups.

The results of our study might be confined to our unique instructional context. We worked in small classes (less than 20 students, usually about 10–12 students), with high instructor-student interaction. We are not sure if the ways that limiting case analysis was encouraged in this class would "work" in a larger or different type of class, although the instructor uses the phrase "check your answers in the usual three ways" when he teaches larger classes as well. We also note that there and, in fact, even in the small setting described here, there was evidence of routinized enactment of the checks. Whether we should be hopeful or discouraged by routine enactment is unclear, but we want to acknowledge it occurred.

An important shift in our conversation occurred during this study, from "Can students do LCA?" to "What do students do and learn from LCA?" We have shown that in our local context, students will conduct LCA, and further, they will enact LCA in a variety of ways that are consistent with representations of LCA in the professional physics literature. Our initial efforts to encourage LCA may have inhibited other kinds of useful checks that are part of students' adaptive expertise, an important lesson for future work. However, we also discovered through our prompts that students exhibited adaptive expertise in how they made choices about which limiting cases to check. Asking students to provide the rationale for which checks they would conduct (or not conduct) and why was critical to our deepening understanding of how students enact LCA. We recommend additional work to support students' informed choice in LCA, as choice is an important component of adaptive expertise and a thoughtful research design might allow for refinement of the theory of adaptive expertise itself.

VIII. IMPLICATIONS FOR TEACHING PHYSICS

Upon reflecting upon this study, we feel we have gained some insights about teaching; chief among them is the sense that LCA is important for practicing and developing physicists alike and deserves more attention in class. Like Warren, Hahn, and others we see value in emphasizing LCA in multiple ways, not just telling students to "check their answer" at the end. In this electricity and magnetism class specifically, we see LCA as playing an especially important role in the pedagogy of distance behaviors ("ways that things can go to zero"); see Sec. V B and the discussion of the electric field dependence on a large distance from a long cylinder of charge.

We acknowledge here that the equations upon which our students practiced LCA tended to be quite complicated, and perhaps not ones they might have generated on their own in typical problem-solving settings. While often the framing implies that LCA is to be done at the end of the work, we want to emphasize, as pointed out by Kuo and others, that it should be considered as important to conduct at various points in the problem-solving procedure, not just for checking final answers. Finally, we would add that there is value in promoting even the routinized version of LCA ("check your answer in the three usual ways"), even while providing opportunities to move beyond such scripts to more adaptive expertise.

One thing that we began to appreciate as we moved from asking students to check their answers to asking students which variables they wanted to test first and why is that the latter prompts seemed to move the students to a better starting place pedagogically. The latter prompts seem to elicit responses that had more connection to learning something useful than to performing school. These prompts also helped us see what is confusing or ambiguous to students and also "rules" they are forming (such as "don't check constants"). Attention to students' choices also helped us monitor whether the instructional intervention was becoming too "scripted" for students and not exploratory in Polya's sense; we made specific changes, but in a different context, other changes may be needed.

Prior work has suggested that compared to other forms of answer checking (e.g., units), LCA seems to have the most potential for students' development of problem-solving expertise. Our study suggests that this advantage may be the result of students' invoking their own physical intuition in concert with the mathematical formalism and seeking consistency between the two. We claim that another affordance of LCA is that it often can push students to a better understanding of an aspect of a given problem that they are not so confident about. Examining LCA in the context of trig functions can enhance students' understanding of trig functions because they can leverage their intuition about the situation. We saw evidence of this with regard to nonuniform density, where the student was able to better understand that when the exponent on the density distribution was higher, it resulted in a smaller acceleration of the influenced particle whereas their original view contrasted with that perspective. While we did not investigate the various math skills (algebra, limits, etc.) necessary to accomplish LCA, that also seems like a rich area for future investigation. Furthermore, we see that a very challenging part of our design-based research study was creating conditions under which students felt that they were discovering new physics through LCA. Future research might examine how students come to attribute epistemic value to LCA.

Limiting case analysis is important to practicing physicists. Yet, there is little concrete guidance for physics educators, and a lack of consensus in the research community, about how to help students learn, and learn from, limiting case analysis. This designed-based research study examines how students implement and attribute meaning to limiting case analysis using data largely from a junior-level electricity and magnetism course. Results suggest that limiting case analysis could play a pivotal role in the development of adaptive physics expertise, as it invokes student choice about which variables to examine and the linking of physical intuition with mathematical formalism.

ACKNOWLEDGMENTS

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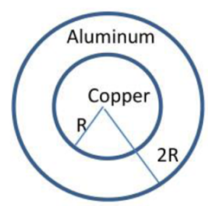
APPENDIX: SELECTED PROMPTS FROM THE STUDY

Here we provide more detailed versions of some of the prompts used in this study, particularly those associated with Figs. 4, 10, and 11. We refer to ‘layers’ within the prompts, by which we mean to indicate that the students were only shown one layer at a given time (a layer might be one or two or even more pages), and students were not able to access responses on a previous layer after viewing and responding to that previous layer.

A very long copper wire of radius R is surrounded by an aluminum casing with inner radius R and outer radius 2R; a cross-sectional view is shown. Assume for now that the copper has uniform current density J_0 going into the page, while the aluminum casing has the same uniform current density J_0 going out of the page. The copper wire has a very thin insulating coating on it to ensure that no current travels radially from the copper to the aluminum.

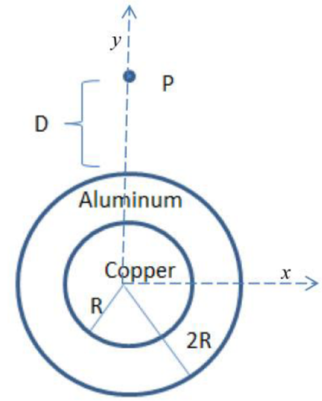


One student, Carla, claims that the magnitude of total current in the copper wire by itself must be more than that of the Aluminum casing even though both have the same magnitude of current density J_0, while Frederick says that he thinks that the total current in the Copper must be equal to that in the Aluminum. Which student, if either, do you agree with, explaining your reasons in detail.

Derive a formula for the current in each metal under the assumptions above, and show that it comports with your comments above.

FIG. 12. Aluminum and copper wire problem referenced in Fig. 4, layer 1 of 5.

Now, suppose, the copper has uniform current density J_0 going into the page, while the aluminum casing has uniform current density J_1 going out of the page. The copper wire has a very thin insulating coating on it to ensure that no current travels radially from the copper to the aluminum.



Suppose further that a particle of mass m is launched with speed V_0 directly into the page at point P as shown.
Which one below is the correct expression for the initial velocity of this particle in the coordinate system shown?

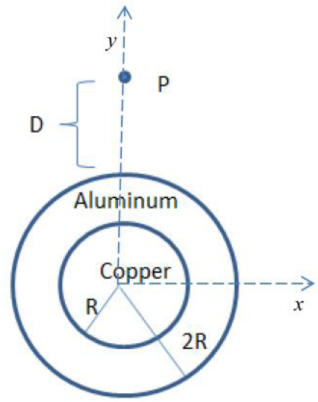
a) $+ V_0\,\hat{\mathbf{x}}$ b) $- V_0\,\hat{\mathbf{x}}$ c) $+ V_0\,\hat{\mathbf{y}}$ d) $- V_0\,\hat{\mathbf{y}}$
e) $+ V_0\,\hat{\mathbf{z}}$ f) $- V_0\,\hat{\mathbf{z}}$ g) none of the above

Now, imagine that one of your classmates derived the formula below for the mass needed for the particle to continue travelling parallel to the wire at constant speed at a fixed distance D from the top surface of the aluminum

$$m = \frac{qV_0\mu_0 R^2(3J_1 - J_0)}{g(2R + D)}$$

What kinds of checks would you first want to do to see if this expression is sensible? List a few different kinds of checks that you think are most important to do, explaining why you chose them, in sentences.

FIG. 13. Aluminum and copper wire problem referenced in Fig. 4, layer 2 of 5.



Recall, the copper has uniform current density J_0 going into the page, while the aluminum casing has uniform current density J_1 going out of the page.

One of your classmates derived the formula below the mass needed for the particle to continue a constant speed at a fixed distance above the top surface:

$$m = \frac{qV_0\mu_0 R^2(3J_1 - J_0)}{g(2R + D)}$$

Now suppose you are asked to check to see if this formula is sensible by checking a variety of limiting or special cases.

Which symbol below would you want to check in a limiting or special case first?
Circle one: R V_0 μ_0 q J_0 J_1 D g
Explain why you picked the one you did:

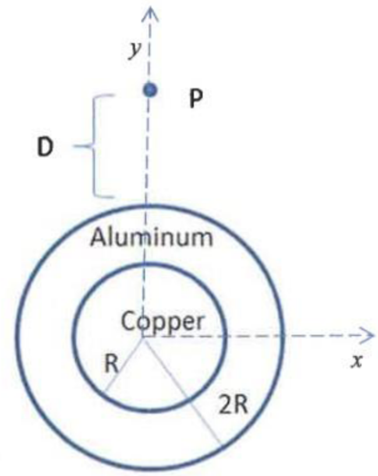
Which one might you want to pick second?
Circle one: R V_0 μ_0 q J_0 J_1 D g
Explain why you picked the one you did:

Which of the symbols would you be ***least*** likely to choose to examine as a limiting case check? Explain.

R V_0 μ_0 q J_0 J_1 D g

FIG. 14. Aluminum and copper wire problem referenced in Fig. 4, layer 3 of 5.

Recall, the copper has uniform current density J_0 going into the page, while the aluminum casing has uniform current density J_1 going out of the page.



One of your classmates derived the formula below the mass needed for the particle to continue a constant speed at a fixed distance above the top surface:

$$m = \frac{qV_0\mu_0R^2(3J_1 - J_0)}{g(2R + D)}$$

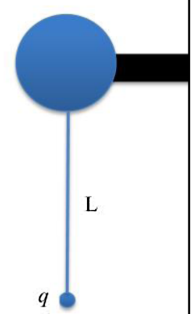
Check to see that this formula is sensible in at least 4 ways, explaining your reasoning.

FIG. 15. Aluminum and copper wire problem referenced in Fig. 4, layer 4 of 5. Layer 5 of this exercise was not relevant to this study and is not included here.

2) (5 pts) A large sphere of radius R, supported near the earth's surface as shown, has **negative** excess charge density $\rho(r)$ that varies as r^n (where n is 0, 1, 2, 3,…) for $0 \leq r < R$, and reaches a value of $\rho_o = -2.0\ \mu C/m^2$ as you get to $r = R$. (so n = 1 means that the density varies linearly with radius, etc.) A non-conducting, uncharged string of length L with a second tiny sphere of negligible radius, mass m, and excess charge q is suspended from the large sphere as shown. Suppose the string is cut gently without otherwise disturbing the setup, and the little ball begins to move.

a) Which value of n corresponds to a solid sphere with charge distributed uniformly throughout? ____________________

b) Think about the motion of the ball after the string is cut; what ideas do you have about how this motion is dependent on the value of q?



c) Think about the motion of the ball after the string is cut; what ideas do you have about how this motion is dependent on the value of m?

d) Think about the motion of the ball after the string is cut; what ideas do you have about how this motion is dependent on the value of n?

FIG. 16. Sphere with nonuniform charge density referenced in Fig. 10, layer 1 of 2.

e) Someone proposes that the acceleration of the little sphere just after the string is cut is (upward is taken to be the positive direction)

$$a = \frac{4\pi k q \rho_0 R^3}{(n+3)m(L+R)^2} - g$$

f) What does this expression predict for the motion in various limiting cases of q? Explain.

g) What does this expression predict for the motion in various limiting cases of m? Explain

h) What does this expression predict for the motion in various limiting cases of n? Explain

i) To what extent were your ideas on the previous page borne out by your analysis on this page? Did anything new come to mind as you were reflecting on the formula?

j) Do you think it is possible to learn new physics by examining an expression such as that given above? …if so, under what conditions? If not, why not?

FIG. 17. Sphere with nonuniform charge density referenced in Fig. 10, layer 2 of 2.

Individually: Suppose you have a large conducting cylinder of radius R that has a variable current density *J(s)* that can be adjusted so that it behaves like s^n for $0 < s < R$ (where n can be any real number greater than -2), reaching the positive constant value J_R as you get to $r = R$. J_R is the same constant for each value of n.

1)Which value of n corresponds to uniform density of current, like regular currents in regular wires?_________

2)Suppose there is a particle (charge q and mass m) travelling above the cylinder through point P at a distance D away from the top surface of the cylinder at just the right speed so that it travels parallel to the cylinder at fixed distance D due to the combined effects of the cylinder's magnetic field and the earth's gravitational field. If q is positive, which way is the particle travelling?____

3)Think about the speed of the small particle; what ideas do you have about how the speed of the particle depends on the value of the charge q of the sphere?

4)Think about the speed of the small particle; what ideas do you have about how the speed of this particle depends on the value of n?

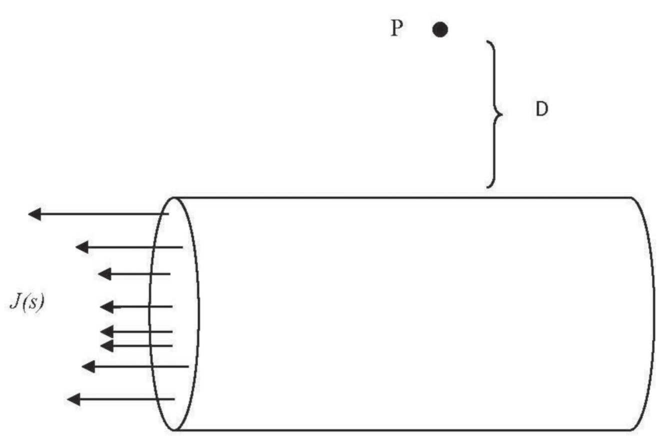


FIG. 18. Magnetic field around a conducting cylinder problem referenced in Fig. 11, layer 1, part A.

Working in groups on whiteboards, show that the total current travelling through the wire can be written

$$I = \frac{2\pi R^2 J_R}{(n+2)}$$

Next, start from a famous law of physics (say which one!) and show that the magnitude of the magnetic field at point P is given by $B_P = \frac{\mu_0 R^2 J_R}{(n+2)(R+D)}$ explaining your reasoning.

FIG. 19. Magnetic field around a conducting cylinder problem referenced in Fig. 11, layer 1, part B.

Finally, start from a famous law of physics (say which one!), and show that the speed v for which the particle will move parallel to the cylinder at fixed distance D from the surface is given by

$$v = \frac{mg(R+D)(n+2)}{\mu_0 q J_R R^2}$$

What assumptions did you make to get this result? Explain.

Identify a place and orientation on earth where one could do this and not worry about the earth's magnetic field messing things up?

FIG. 20. Magnetic field around a conducting cylinder problem referenced in Fig. 11, layer 1, part C.

Write down the formula that your group got for *v* here: *v* =

What does this formula say about how the speed depends upon the value of *q?*

Does this match your ideas from earlier (see opposite side)? Explain

What does this formula say about how the speed depends upon the value of *n?*

Does this match your ideas from earlier (see other side)? Explain .

Have any new ideas about how the speed *v* depends upon the charge *q* or the parameter *n* emerged from when you first thought about the problem to now, after you have solved the problem in detail? Explain.

Did you sense that other students had different initial ideas than you while working in your groups? What were some of these ideas?

FIG. 21. Magnetic field around a conducting cylinder problem referenced in Fig. 11, layer 2.

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