Students' difficulties with the Dirac delta function in quantum mechanics

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In the context of quantum mechanics, students are often asked to use delta functions to solve problems. Here, we investigate three typical problem-solving processes using delta functions: a delta function potential well problem, a position space delta function problem, and a momentum space delta function problem. We studied students' solutions in written exams and conducted think-aloud interviews. We use the activation, construction, execution, and reflection framework for our analysis. We find that students encountered various difficulties in solving quantum mechanics problems using delta functions. Common challenges included difficulty with establishing expressions of delta functions for position eigenstates, difficulty with expressing the orthonormality of eigenfunctions with continuous spectra using delta functions, difficulty with boundary conditions for delta function potentials, and difficulty with calculating integrals involved delta functions. In particular, students rarely used effective reflective methods to gain confidence in their solutions. We commonly observed this challenge for students in all questions we investigated. In addition, we compare the similarities and differences in the use of delta functions in electrostatics and in quantum mechanics, and discuss possible explanations for the reasoning mechanisms that cause these difficulties. Finally, we discuss the potential pedagogical implications of our findings.

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I. INTRODUCTION

The Dirac delta function is a frequently used mathematical tool in quantum mechanics. Students will encounter these delta function related knowledge topics in the context of quantum mechanics courses in Chinese universities. (i) The state of a particle in a potential field has two basic states, the bound and scattering states. A particle in a delta function potential is a typical case in quantum mechanics. (ii) Position is a fundamental operator in quantum mechanics. Its eigenfunction is a delta function that represents the position measurement of the particle. (iii) There are two types of operators in quantum mechanics, with discrete eigenvalues and with continuous eigenvalues. For an observable with continuous spectra, its eigenfunctions satisfy the Dirac orthonormality, which is represented by a delta function. These three are important cases of the wide applications of delta functions in quantum mechanics. Therefore, the use of delta functions is a key element to understand and master the basic content of quantum mechanics.

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The study of students' difficulties that arise in their learning of quantum mechanics is an active area of physics education research [1–36]. Most of the literature has focused on the difficulties that arise in students' understanding of the basic concepts and formulas of quantum mechanics. For example, Singh *et al.* investigated students' understanding of basic concepts such as the wave functions, the bound and scattering states, the measurement results and probabilities of different physical observables for a particle in a one-dimensional potential [2,5,6,9]. The results showed that students experience various difficulties in reasoning these basic concepts.

In the context of quantum mechanics, students are often asked to combine abstract physical concepts with complex mathematical calculations to solve problems. Although there are several reports on the use of differential equation methods in quantum mechanics [32,33], the use of mathematical tools in problem-solving in quantum physics has not been fully investigated. We are not aware of any studies that have specifically explored how students relate quantum mechanics concepts to delta functions in the problem-solving process. In addition, Wilcox and Pollock studied students' difficulties in using delta functions in the electrostatic content [37]. However, the use of a mathematical technique in solving a problem is highly dependent on the physical content associated with that problem [32,33,37–40]. New phenomena can be expected from studying how students use the delta function in the context of quantum mechanics.

In this work, we investigate students' problem-solving abilities when using delta functions in the context of quantum mechanics. In Sec. II, we provide an overview of the literature related to students' difficulties in using the delta function. In this paper, we use the activation, construction, execution, reflection (ACER) framework to analyze the investigation data. This analytic framework was specifically developed to analyze how students use mathematical tools in solving physics problems [38]. In Sec. III, we describe the details of the operationalized ACER framework used for the analysis. In Sec. IV, we describe the data sources and the details of the interview design conducted in the study. Students in the School of Physics at the University of Science and Technology of China (USTC) have used delta functions several times in their quantum mechanics courses. Here, we focus on three typical cases of delta functions: (i) the delta function potential well problem, (ii) the position space delta function problem, (iii) the momentum space delta function problem. They provide an important context for the use of delta functions in quantum mechanics courses. All three types of problems require students to build expressions of the delta function and to calculate integrals involving the delta function to obtain the corresponding states and their properties. The study presented here can provide a sample of students' difficulties in understanding and using delta functions to solve problems in the context of quantum physics.

Then, in Secs. V, VI, and VII, we present findings and analyses of the difficulties students encountered in solving the delta function potential well problem, the position space delta function problem, and the momentum space delta function problem, respectively. Although the mathematical method of the delta function is general, how this method is used in solving physics problems highly depends on the specific physics context. Therefore, when students use the delta function to solve problems in the new context, some difficulties can persist, and some new difficulties can emerge. Finally, in Sec. VIII, we discuss in detail the similarities and differences between our results and those of previous studies on delta functions and related concepts. We also briefly discuss the instruction implications of this work, as well as future work.

II. REVIEW OF THE LITERATURE AND RESEARCH QUESTIONS

A. Student difficulties with delta functions

Most research in the field of quantum physics has focused on the difficulties students have in understanding basic quantum concepts. For example, it has been found that students have various difficulties in interpreting quantum interference phenomena [17–19], in determining the time evolution of wave functions [9,12,21,26], in obtaining measurement results and probabilities of physical observables [6,7,11], and in representing quantum systems using different notations such as the Dirac notation, the algebraic wave function notation, and the matrix notation [13,22,25,30].

In the Ref. [34], Muller *et al.* investigated students' difficulties in understanding the concepts of quantum mechanics. Students had a number of common concepts and misconceptions about quantum objects such as photons and atoms, including the distinction between classical and quantum objects (e.g., the distinction between electron orbitals and electron cloud images), determinism and nondeterminism (e.g., the interpretation of the double-slit interference phenomenon), and the uncertainty relation (e.g., whether quantum objects possess both position and momentum). Then the authors developed a new research-based quantum mechanics course in which students discover from the beginning how quantum phenomena deviate from classical experience. The evaluation results of the course show that most students acquire appropriate concepts of quantum mechanics.

In the Ref. [35], Bitzenbauer *et al.* also investigated the phenomenon of students' conceptual understanding of quantum mechanics. (i) Some students seemed to stick to their preconceptions dominated by the classical physics and describe the quantum world as a small-scale classical world. (ii) Some students knew the significant differences between quantum and classical physics and describe the quantum world through effects or aspects that do not exist in classical world. (iii) In particular, some students knew the importance of quantum physics for technologies and describe the quantum world as the world of technology. Therefore, the authors suggested that raising students' awareness of the importance of modern quantum technologies can contribute to their study of quantum physics.

In the article [36], Uhden *et al.* developed a new model for analyzing mathematical reasoning in physics. In this model, mathematical reasoning is divided into different levels: for example, mathematization (from physical models to mathematical expressions), interpretation (from mathematical expressions to physical models), and pure mathematics (technical mathematical operation). In this sense, this model is similar to some elements of the ACER framework. The authors demonstrated the applicability of this model in the analysis of physical-mathematical reasoning processes by using an example from classical mechanics. This model can illustrate different reasoning ways and locate possible reasoning difficulties. Thus, it can be used as a tool to analyze students' thinking and help teachers to design and evaluate more applicable teaching approaches.

In particular, Singh *et al.* explored students' difficulties with position measurement [2,5,6,9], a topic related to the delta function. They developed a conceptual survey of quantum physics that consists of a series of multiple-choice questions to probe students' understanding and mastery of various concepts. After a position measurement, the particle is in the position eigenstate, which is a delta function. Research has found that students had several sets of conceptual difficulties with position measurement. (i) Students had difficulty in distinguishing between the stationary state and the position eigenstate. For example, many students claimed that if a particle is in a position eigenstate at initial time, i.e., has a definite position value, the particle will stay in that position at all times. Some other students claimed that the wave function becomes a delta function after an energy measurement and drew a peak at a certain position value. (ii) Students had difficulty in realizing that the system state will change after the position measurement. For example, a particle is initially in a superposition of the ground state and the first excited state. After a position measurement the system wave function collapses to a delta function, which is a superposition of many energy eigenfunctions. However, many students claimed that the particle is still in the initial state, i.e., the superposition of only the ground state and the first excited state.

In addition, Wilcox and Pollock investigated the students' difficulties with the delta function in the context of electrostatics [37]. The charge distribution can be expressed using the delta function. The researchers used traditional exam questions, standard conceptual assessments and interviews and utilized the ACER framework to analyze students' thought processes. They found that students had common challenges with the delta function. (i) During the activation stage, students had difficulty in spontaneously invoking the delta function. (ii) During the construction stage, students had difficulty in translating the description of the charge distribution into a mathematical expression of the delta function and had difficulty in recognizing that the delta function could have units. (iii) During the execution stage, students had difficulty in integrating three-dimensional or non-Cartesian delta function expressions.

In summary, there have been a number of investigations [1–36] of students' difficulties in reasoning about the basic concepts and fundamental formulas of quantum mechanics. These studies addressed many specific concepts and contexts, and students encountered similar difficulties in learning these fundamental concepts. These difficulties were often due to overgeneralization of concepts in one context to another context where they are not directly applicable (e.g., difficulties in distinguishing between stationary states and other operator eigenstates). In other words, reasoning difficulties in distinguishing between closely related concepts are common.

B. Research questions

These previous studies have focused on students' difficulties in understanding concepts of quantum physics. However, there is little research on students' problem-solving skills in the context of quantum mechanics [32,33,41–43]. Learning quantum mechanics is challenging, not only because the concepts of quantum physics are very different from those of classical physics, but also because students have tremendous difficulty in relating the concepts of quantum physics to complex mathematical calculations. Delta functions are an important mathematical method that appears repeatedly in quantum mechanics courses. Therefore, more work needs to be done to investigate student difficulties with using delta functions in the context of quantum mechanics. These investigations can provide additional insight into how students understand quantum physics concepts and use the corresponding mathematical methods.

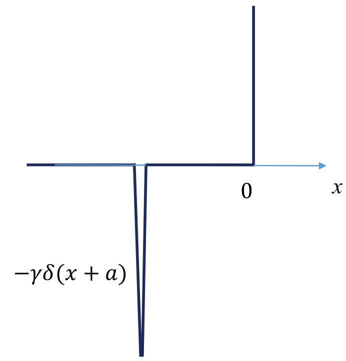

FIG. 1. Problem 1: A particle in a delta function potential well.

On the other hand, as far as we know, there is only one paper on students' difficulties with using delta functions in electrostatics content [37]. Although delta functions are a standard mathematical technique, how to translate the physical content into delta function expressions and how to calculate the results of delta functions and interpret their physical meaning are highly dependent on the corresponding physical context. Therefore, it would be interesting to study students' thought processes in using delta functions in different physics contexts and to compare them.

Determining the energy levels of a particle in delta potentials, using position eigenstates to determine measurements of a particle, and using orthonormality of momentum eigenstates to determine the properties of a particle are among the topics that upper-division students encounter in quantum mechanics. In these cases, the "tool" we refer to is the delta function, and we define "problem solving" as the combination of the mathematical tools of the delta function with the conceptual knowledge of quantum physics to solve the above long and complex physical problems in the context of quantum mechanics. In the present work, we are interested in the following three research questions.

(1) To what extent are students able to use delta function tools in the context of quantum mechanics? That is, to what extent are students able to correctly relate the concepts of quantum physics to delta functions?
(2) What are the common difficulties students have when using delta functions in the problem-solving process? What are the possible causes of these difficulties?

TABLE I. Exam questions to probe student difficulties with delta functions.

Problem 1: Consider a particle with mass m interacting with the potential (Fig. 1): $V(x) = -\gamma\delta(x+a), x < 0$; $V(x) = \infty, x > 0$. Here $\gamma > 0$, $a > 0$. Derive the transcendental equation for the allowed energies of the bound states.

Problem 2: Consider a particle with mass m in a one-dimensional infinite square well with boundaries at $x = 0$ and $x = a$ (Fig. 2). The nth energy eigenfunction and eigenvalue for the particle are $\varphi_n(x) = \sqrt{a/2}\sin(n\pi x/a)$ and $E_n = n^2\pi^2\hbar^2/2ma^2$. Perform a measurement of position at $x = a/2$. If energy is measured immediately after the position measurement, calculate the probability of the energy measurement which yields the value of E_n.

Problem 3: For a momentum space wave function $\Phi(p)$, prove the relation $\langle x\rangle = \int_{-\infty}^{+\infty}\Phi^*(p)(i\hbar\frac{d}{dp})\Phi(p)dp$. Hint: Notice that $xe^{ipx/\hbar} = (-i\hbar\frac{d}{dp})e^{ipx/\hbar}$.

(3) What difficulties are similar and what difficulties are different compared to previous studies?

III. THEORETICAL FRAMEWORK

A. Context for research

In our investigation, we use three tasks to probe how students apply their knowledge and tools of delta functions in the context of quantum mechanics. As shown in Table I, in problem 1, students are asked to determine the possible energies of the bound states with a delta function well potential. In problem 2, students are asked to determine the probability of a particular energy measurement outcome after position measurement. In problem 3, students are asked to derive an expression for the expectation value in momentum representation.

B. The ACER framework

In upper-level physics courses, students' problem-solving processes are often long and complex. Students can make a variety of errors at each step of the problem-solving process. In order to extract useful information from complex data, we use the activation, construction, execution, reflection (ACER) framework to guide our investigation of students' solutions [38]. The ACER framework was developed based on a resources theory of the nature of learning, which divides the problem-solving process into four stages.

(i) Activation stage: activate the related mathematical resources.

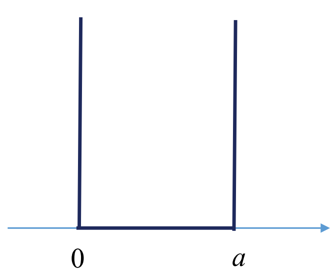

FIG. 2. Problem 2: A particle in an infinite square well.

(ii) Construction stage: construct the corresponding equations for the physics problem.
(iii) Execution stage: execute the calculations to the equations.
(iv) Reflection stage: reflect the final answers.

It is clear that how mathematical tools are used depends on the corresponding physical context. Therefore, the ACER framework is operationalized for specific physics problems. Below, we list the operationalized framework for each specific question, and a more detailed description can be found in Appendices A, B, and C.

First, the researchers selected a typical problem related to the delta function in quantum physics. One researcher worked through the problem and documented all elements of a complete solution. Then these elements are discussed with several other researchers to reach a consensus that all important elements of the solution have been identified. These problem-specific elements can be organized into the four components of the ACER framework, which appear consistently in the use of complex mathematical tools (e.g., delta functions) to solve various content-rich problems (e.g., quantum physics problems). These four components are activation of the tool, construction of the model, execution of the mathematics, and reflection on the results.

Students were asked to solve the same problems in the exams and interviews. The researcher identified each key element of the ACER framework that appears in the student's solution. Each element was then coded. This coding helped ensure that the ACER framework developed by the researcher did not miss important but unanticipated elements of the student solution. By classifying each of the student's key actions into one of the four components of the ACER framework, the student's problem-solving process could be analyzed.

C. The operationalized framework for the delta function well problem

(i) **Activation of the relevant tools:** The activation stage identifies the appropriate mathematical tools to solve the problem. In general, activation is influenced by the problem's given prompt. The

numbering of these elements is for labeling purposes only, which indicates that there are multiple options.
- A1: The question text directly provides a delta function expression for the potential.
- A2: The question figure provides a spike shape of the potential and asks for an expression for the potential.

(ii) **Construction of the physical equations:** The construction stage maps the specific physical situations to the corresponding equations. Here, we divide the construction stage into four elements whose numbering indicates the order of the problem-solving process.
- C1: Write down the stationary Schrödinger equation in different regions based on the shape of the delta function potential.
- C2: Determine the range of energy constants based on the shape of the delta function potential.
- C3: Apply an integral around the delta function potential to determine the boundary condition for the wave function.
- C4: Write down the boundary conditions at $x = 0$ and infinity.

(iii) **Execution of the mathematical calculations:** The execution stage performs mathematical calculations to the equations built in the construction stage. Here, we divide the execution stage into three elements, whose numbering also indicates the order of the problem-solving process.
- E1: Obtain the general solution to the stationary Schrödinger equation (i.e., an ordinary differential equation).
- E2: Calculate the integral containing the delta function.
- E3: Organize the condition satisfied by the energy constant.

(iv) **Reflection on the solutions:** The reflection stage refers to the use of steps to check whether the results are consistent with physics-based expectations in order to gain confidence in the results. Here, we divide the reflection stage into several elements. They are numbered for labeling purposes only, which indicates that there are multiple methods of checking.
- R1: Check the units of the expressions.
- R2: Check that the wave function satisfies the boundary conditions.
- R3: Check that the limit behavior of the wave function is consistent with expectation.

D. The operationalized framework for the position space delta function problem

(i) **Activation of the relevant tools:** The numbering of these elements are for labeling only.
- A1: The question text directly provides a delta function expression for the position eigenfunction.
- A2: The problem involves position measurements and asks for corresponding position eigenfunctions.

(ii) **Construction of the physical equations:** The numbering of the elements indicates the order of the problem-solving process.
- C1: Relate the position eigenvalue to the argument of the delta function and write down the corresponding expression for the position eigenfunction after position measurement.
- C2: Expand the delta function expression of the position eigenfunction into a superposition of energy eigenfunctions.
- C3: Apply an integral of the delta function to express the probability amplitude.
- C4: Use the probability amplitude to express the probability.

(iii) **Execution of the mathematical calculations:**
- E1: Calculate the integral containing the delta function.

(iv) **Reflection on the solutions:** The numbering of these elements is for labeling purposes only.
- R1: Check the units of the expressions.
- R2: Check that the result is consistent with the expectation based on the form of the wave function.

E. The operationalized framework for the momentum space delta function problem

(i) **Activation of the relevant tools:** The numbering of these elements are for labeling only.
- A1: The question text directly provides a delta function expression for the orthonormality of momentum eigenstates.
- A2: The problem involves the momentum representation and uses language associated with the delta functions (e.g., representation transformation, Fourier transform).
- A3: The problem involves the momentum representation and asks for expressions of the orthonormality of momentum eigenstates.

(ii) **Construction of the physical equations:** The numbering of the elements indicates the order of the problem-solving process.
- C1: Express the expectation value in terms of the wave function in position space.
- C2: Relate the wave function in momentum space to the wave function in position space.
- C3: Use the delta function to express the orthonormality of the momentum eigenstates.

(iii) **Execution of the mathematical calculations:** The numbering of the elements indicates the order of the problem-solving process.
- E1: Organize the integral expression.

- E2: Calculate the integral containing the delta function.

(iv) **Reflection on the solutions:**
- R1: Check the units of the expressions.

In the following sections, we apply the coding scheme to study the students' efforts to solve these problems.

IV. RESEARCH METHOD

A. Data sources

We collected data from the quantum mechanics course at USTC. This course covers chapters 1–12 of Zeng's book [44] (or chapters 1–10 of Griffiths' book [45]) for one semester, with 4 weekly sessions of 50 min each. Upper-division students in the school of physics take this course with a typical class size of 60–100 students. Both quantitative and qualitative methods were used in this study, with two different sources of data: students' solutions to traditional midterm exams and "think-aloud" interviews on the problem-solving process. For the quantitative data, we investigated the students' written solutions to the exams and identified their common difficulties. Then, for the qualitative data, we collected and analyzed students' responses in the interviews to gain more insight into the possible causes of these reasoning difficulties.

B. Written exams

We all taught the quantum mechanics course at USTC. We designed three exam problems using delta functions through several discussions: the delta function potential well problem (i.e., problem 1 in Table I), the delta function problem in position space (i.e., problem 2 in Table I), and the delta function problem in momentum space (i.e., problem 3 in Table I). On each exam, students were asked to solve one of the problems using the delta function.

In total, we collected data from six sets of exams over six years. For problem 1, there were two sets of exam data with a total of $N_t = 466$ students. For problem 2, there are two sets of exam data with a total of $N_t = 506$ students. For problem 3, there were two sets of exam data with a total of $N_t = 394$ students.

We then used the ACER framework to analyze the students' written solutions. Their solutions were coded according to the elements that appeared in the ACER framework. Finally, we organized the results to identify the students' problem-solving ideas and to identify the difficulties that arose.

C. Design the interviews

To further understand students' difficulties with problem-solving and to uncover underlying reasoning mechanisms, we conducted think-aloud interviews [46]. We collected data from six sets of interviews, each of which was scheduled after the midterm exam. For problem 1, there were two sets of interviews with a total of $N_t = 30$ students. For problem 2, there were two sets of interview data with a total of $N_t = 24$ students. For problem 3, there were two sets of interview data with a total of $N_t = 20$ students.

The students interviewed were volunteers willing to participate in the study. The total number of interviews was smaller than the total number of students who took the midterm exams. Most of the interviewed students scored between 50 and 80 on the exams and they made a variety of mistakes in solving the exam questions. We chose them to participate in the interviews because we wanted to investigate more deeply how they made these mistakes.

The interviews were scheduled to take place within a week of the exams. Because of the short interval, the students who participated in the interviews still clearly remembered how they had solved the same problem during the exams. That is, the solutions they gave in the interviews were very similar to the solutions they gave in the exams. Therefore, if they made a certain mistake in the exams, they would make the same mistake in the interviews. In this way, we can study why they made that mistake.

The interviews were conducted individually outside the classroom. We noticed that the students became tired if the interviews were too long. Therefore, each interview lasted about an hour or so. At the beginning of each interview, we explicitly told the students, "This is just a think-aloud interview. We want you to explain your thoughts aloud as you solve the problem. You do not be nervous because this is not a test and we will not grade you for your performance. This interview is simply because we are interested in your problem-solving process and ideas." All of the students interviewed did not agree to video or audio recording because this approach would make them feel watched and give them an uncomfortable feeling. Therefore, in this study, the results of all interviews were transcribed verbatim. For the sake of consistency, all interviews were conducted by the first author of this paper.

The interviews were designed using a semi-structured think-aloud protocol. Surveys of students' written solutions on exams can provide information about the pattern and frequency of students' difficulties. However, exam data provides limited insight into students' thinking about problem-solving and into the reasons for their difficulties. Therefore, we asked the students interviewed to solve the same problems as on the midterm exams and asked them to articulate their thought processes.

After the students expressed their thoughts as clearly as possible, if they had not mentioned them, we asked a list of questions about their thought processes. Our list of questions were designed based on the elements of the ACER framework.

(i) For the activation element, we asked students what prompted them to use the relevant knowledge resources.

(ii) For the construction element, we asked students how they interpreted each expression they wrote down. For example, "What does this equation represent?", "Can you give an explanation for this expression?", and "What do you call this quantity in the expression?".

(iii) For the execution element, we asked students how they performed the corresponding mathematical calculations. For example, from the exam data, we found that students had difficulty in calculating the integrals of the delta function. We would like to know if these errors were due to simple calculation errors or were caused by some other mechanisms. So, we asked students, "How did you get this integral result?"

(iv) For the reflection element, it was difficult to determine from the exam data whether students checked their solutions. Therefore, in the interviews, when students finished solving the problem, we asked them if and how they checked their solutions.

Thus, these interview questions explicitly address all aspects of the ACER framework. In addition, based on a particular student's responses, we designed additional questions on the spot to explore his or her thought processes in more depth. In some interviews, we also asked students broader questions, such as what, in their opinion, are the main difficulties in learning quantum mechanics. We did not interrupt students as they elaborated their ideas and answered questions, because we wanted them to elaborate as clearly as they could. If students were quiet for a long time, we would remind them to continue talking.

V. FINDINGS ON STUDENTS' DIFFICULTIES WHEN ADDRESSING A DELTA FUNCTION WELL PROBLEM

In this section, we use the ACER framework to identify and classify students' difficulties in solving the bound states of a particle in a delta function well (problem 1 in Table I).

A. Activation of the tools

In the exams and interviews, all students were able to write expressions related to the delta function. But this does not suggest that students do not have difficulties in motivating the corresponding mathematical knowledge and tools, because the question explicitly provides a delta function potential well.

B. Construction of the equations

Step C1: In step C1, students are required to establish the time-independent Schrödinger equation containing the delta function potential well. There were $N = 466$ students who took the exams, and among them, $N = 456$ students completed this step correctly. In the remaining $N = 10$ solutions, the students inappropriately established the time-dependent Schrödinger equation but could not reduce it to the time-independent Schrödinger equation.

In the interviews, one student wrote down the time-dependent Schrödinger equation. Then he spent a long time trying to reduce it to the time-independent Schrödinger equation, however he failed. When he explained why he did this, he said, "The time-dependent Schrödinger equation is the fundamental equation in quantum mechanics, which I have already written down. I remember that I need to use the separation of variables method to deal with the equation. But this process is too complicated, especially since the present equation contains a delta function. I really can't do it." The interview results imply that some students were so focused on the time-dependent Schrödinger equation that they did not directly establish the time-independent Schrödinger equation when solving stationary state problems.

Step C2: In step C2, students need to determine the range of values of the energy constant E. There were $N = 456$ students who established time-independent Schrödinger equation, and among them, $N = 400$ students correctly commented that the bound states should be $E < 0$. In the remaining $N = 56$ solutions, students incorrectly discussed the case of $E > 0$.

In the interviews, four students incorrectly believed that the energy constant could have the case of $E > 0$. Their explanations can be divided into two categories.

Two students wrote down the equation $E > 0$. One of them explained, "I remember that I solved a similar problem when I was doing homework. We usually need to consider bound and scattering states. Here we are considering the bound state, whose energy should be less than the potential energy at infinity $E < V(\infty) = \infty$. Thus we have $E > 0$." The other student gave a similar statement. Actually, the correct criteria is that when the energy is less than the potential energy at both plus and minus infinity, the particle is in a bound state. For the present problem, the potential energies at plus and minus infinity are different: $V(+\infty) = +\infty$, $V(-\infty) = 0$. Thus the specific potential leads to the equation $E < V(-\infty) = 0$. The interview results suggest that some students do not realize that to form a bound state, the wave function must be bound on both sides.

Another two student also discussed the case of $E > 0$. One of them explained, "This is a delta function well, which is equivalent to an infinitely deep well. So the particle can be bound in this potential well regardless of the energy $E > 0$ and $E < 0$. Thus we should discuss both cases $E > 0$ and $E < 0$." The other student gave a similar statement. The interview results indicate that some students have an incorrect belief that a delta function well can confine particles of all energy ranges.

Step C3: In step C3, students are required to construct an integral containing the delta function $\int_{-a-\epsilon}^{-a+\epsilon} \delta(x+a)\varphi(x)dx$, which leads to the boundary condition satisfied by the wave function. There were $N = 384$ students who progressed to this step, and among them, $N = 294$ students successfully completed this step. In the remaining $N = 90$ solutions, the common errors included: not building an integral expression for the delta function; or setting an incorrect range of integration.

The interviews provided insight into students' difficulties. Their responses can be divided into two types.

In the interviews, three students had never established the integral of the delta function. We asked them if they knew the integral property of the delta function and why they did not use this property. One student wrote down the integral formula of the delta function $\int_{-\infty}^{+\infty} \delta(x)dx = 1$ and explained, "This integral formula shows that the delta function is a narrow spike, whose integral over the whole space is 1. But I believe that I do not need to use the integral of the delta function here because I has already obtained the wave functions on both sides of the delta function well." The other two students gave similar answers. In fact, because the derivative of the wave function is discontinuous at the point where the potential is infinite, we need the integral of the delta function to determine the boundary condition satisfied by the wave function. The interview results suggest that some students do not understand that the integral of the delta function is to establish the corresponding boundary condition.

Seven interviewed students wrote down the integral expression for the delta function but set the incorrect range of integration: $\int_{-\infty}^{+\infty} \delta(x+a)\varphi(x)dx$. One of them explained, "I learned about the delta function in my mathematical course, and it is a very special function. Usually we need to use its integral formula as $\int_{-\infty}^{+\infty} \delta(x)dx = 1$. This formula shows that although the delta function is infinite at a point, its integral over the entire space is 1. So in all specific cases, we should establish the integral over the full space." The other six students gave similar answers. In fact, in the quantum mechanics context that includes a delta function potential, it is necessary to build the integration around the point where the delta function is located, rather than over the whole space. In this way, one can establish the boundary condition satisfied by the wave function at this point. We believe that many students are used to the general integral formula for the delta function, i.e., the range of integration from minus infinity to plus infinity, which may discourage them from building the integrals over a finite region.

Step C4: In step C3 students have established the boundary condition at $x = -a$, so in step C4 they need to establish other boundary conditions. There were $N = 211$ students who progressed to this step, and among them, $N = 195$ students correctly set up all other boundary conditions. In the remaining $N = 16$ solutions, students either lacked the boundary condition for the wave function at infinity or incorrectly believed that the derivative of the wave function is always continuous.

In the interviews, one student made a mistake in constructing the boundary condition at $x = 0$. When we asked him about the general rule of boundary conditions in quantum mechanics, he replied, "The wave function and its derivative should be continuous at the boundary." Then we asked him if the derivatives of the wave function is continuous at $x = 0$, where there is an infinitely high potential. He hesitated for a moment and then replied, "I made a mistake earlier. At this point, the derivative of the wave function should be discontinuous." In quantum mechanics, the wave function is always continuous, and the derivative of the wave function is also continuous except at the point where the potential is infinite. We believe that some students do not grasp the standard boundary conditions for the wave function in quantum mechanics.

C. Execution of the calculations

Step E1: In step E1, students are required to solve the ordinary differential equation established in step C2. There were $N = 400$ students who established the correct ordinary differential equation, and among them, $N = 384$ students obtained the correct general solution to this equation. In the remaining $N = 16$ solutions, the common errors included providing incorrect forms of the exponential function (e.g., obtaining $e^{\pm ikx}$ instead of $e^{\pm kx}$); or dropping or adding a constant factor.

In the interviews, all students wrote down the general solutions to the ordinary differential equation directly without any derivation process. When they were asked how they provided these general solutions, a typical response was, "It is an ordinary differential equation with constant coefficients. I remember that its general solution is an exponential function." The interview results imply that many students do not actually solve ordinary differential equations, but rather memorize their general solutions.

Step E2: In step E2, students are asked to calculate the integral expression of the delta function built in step C3. There were $N = 294$ students who built the correct integral expression, and among them, $N = 211$ students obtained the correct integral result, $\int_{-a-\epsilon}^{-a+\epsilon} \delta(x+a)\varphi(x)dx = \varphi(-a)$. In the remaining $N = 83$ solutions, the common errors included incorrectly assuming that the integral result is 1 or $\varphi(0)$; or stopping at this step.

In the interviews, eight students made mistakes in this step. Their interview results can be divided into three types.

Four participants incorrectly wrote down the integral result as $\int_{-a-\epsilon}^{-a+\epsilon} \delta(x-a)\varphi(x)dx = 1$. One of them explained, "The delta function is an infinite function, and its integral result is 1." We then asked him if all integrals of the delta function are 1, and he answered very clearly, "Of course, whenever an integral expression contains a delta function, the result of that integral is 1. This is a special property of delta functions." The other three students gave similar responses. The interview results suggest that some students are used to the integral result $\int_{-\infty}^{+\infty} \delta(x)dx = 1$ and inappropriately generalize this integral property of the delta function to all integral cases.

Two participants incorrectly wrote down the integral result as $\int_{-a-\epsilon}^{-a+\epsilon} \delta(x-a)\varphi(x)dx = \varphi(0)$. We then asked them how they obtained this result, and one of them replied, "I remember learning in class that the integral of a delta function and any function is the value of that function taken at the origin." The other student gave a similar answer. In fact, the example often given in class is a delta function at the origin that leads to the integral result of $\int_{-\infty}^{+\infty} \delta(x)\varphi(x)dx = \varphi(0)$. We suspect that some students are used to the integral result of this particular example and incorrectly generalize it to all cases. Obviously, the delta function here is at the point $-a$, not at the origin.

Two interviewed students stopped at this step without progressing further. When we asked them why they did not proceed, one of them replied, "I remember the result of the integration of the delta function over the full space, $\int_{-\infty}^{+\infty} \delta(x)f(x)dx = f(0)$. However, the domain of integration here is $[-a-\epsilon, -a+\epsilon]$. I don't know how to calculate the integral over such a finite region. So there is no way for me to go on." The other students gave a similar response. In fact, the integral does not need to go from minus infinity to plus infinity. What is important is that the region of integration includes the point where the delta function is located, so a finite region would do. The interview results suggest that some students are used to the result of integral over the full space, which may discourage them from understanding and calculating the integral over a finite region.

Step E3: In step E3, students need to simplify the equations built in steps C3, C4, and E2 to get the final answer. Since this step is mostly simple algebraic calculations, we find that students were usually able to successfully pass this step without making mistakes on exams and interviews.

D. Reflection on the solutions

For problem 1, one of the effective ways to check and interpret the solution is to check the units of the expressions (element R1). In this problem, the unit of the delta function is L^{-1}, where L is the unit of length. Thus the unit of the constant γ in the expression of the delta function potential is ML^3T^{-2}, where M and T are the units of mass and time, respectively. Among the 30 students interviewed, only 5 students explicitly commented on the units of the delta function and the constant. However, they all incorrectly assumed that the delta function itself must be unitless. For example, one of the students replied, "The delta function is definitely a unitless quantity. It is infinite at this point, and an infinite physical quantity is impossible." This result indicates that our students have difficulty in understanding and applying the unit of the delta function.

For problem 1, looking at the boundary conditions of the wave function can facilitate the interpretation of the physical meaning of the delta function potential (element R2). An effective way to do this is to draw the wave function in the vicinity of the delta function potential. It is clear that the wave function has a kink at the point of the delta function potential. With this graph, it is helpful to understand the role of the delta function potential in determining the boundary conditions of the wave function at this point. In the interviews, 15 students explicitly commented that they checked the boundary conditions of the wave function. However, they only checked the boundary conditions in a purely mathematical, rote way: look at the expressions for the boundary conditions and see if they can find an error. They did not draw the corresponding diagram of the wave function or interpret the behavior of the wave functions to gain a deeper physical understanding. One student tried to draw the graph of the wave function when specifically prompted by the interviewer. Then, he replied excitedly, "This is the first time I have used the method of drawing a wave function to understand the boundary conditions satisfied by the wave function. I had never thought of this graphical approach before." For experts, the mathematical expressions of the boundary conditions and their physical meaning are closely related. However, this result suggests that this relationship may still be developing for our students.

In addition to commenting on units and boundary conditions, we would like students to recognize some limiting behavior and to ensure that this limiting behavior is consistent with physical expectations (element R3). An effective way to do this is to check the form of the wave function at infinity depending on whether the particle is a bound or scattering state. None of the students in the interviews explicitly commented on this point. One of the students used a wave function in the form of a plane wave e^{ikx}. He was confused when we pointed out that this expression was wrong. We then went on to suggest that the particle in problem 1 is a bound state and therefore the wave function should tend to zero at infinity. He then recognized his error and stated, "I usually just memorize the solution to the Schrödinger equation for exams. I have never understood the wave functions of bound and scattering states in this light."

E. Overview of students' performance

In total, $N_t = 466$ students took the exams and attempted to solve the delta function potential problem. In Fig. 3, we show the Sankey diagram of the students' solutions. On the left, the source represents the total number of students who took the exams. On the right, a set of destinations shows the various errors that occurred at each step. The width of the flow represents the number of solutions that made errors.

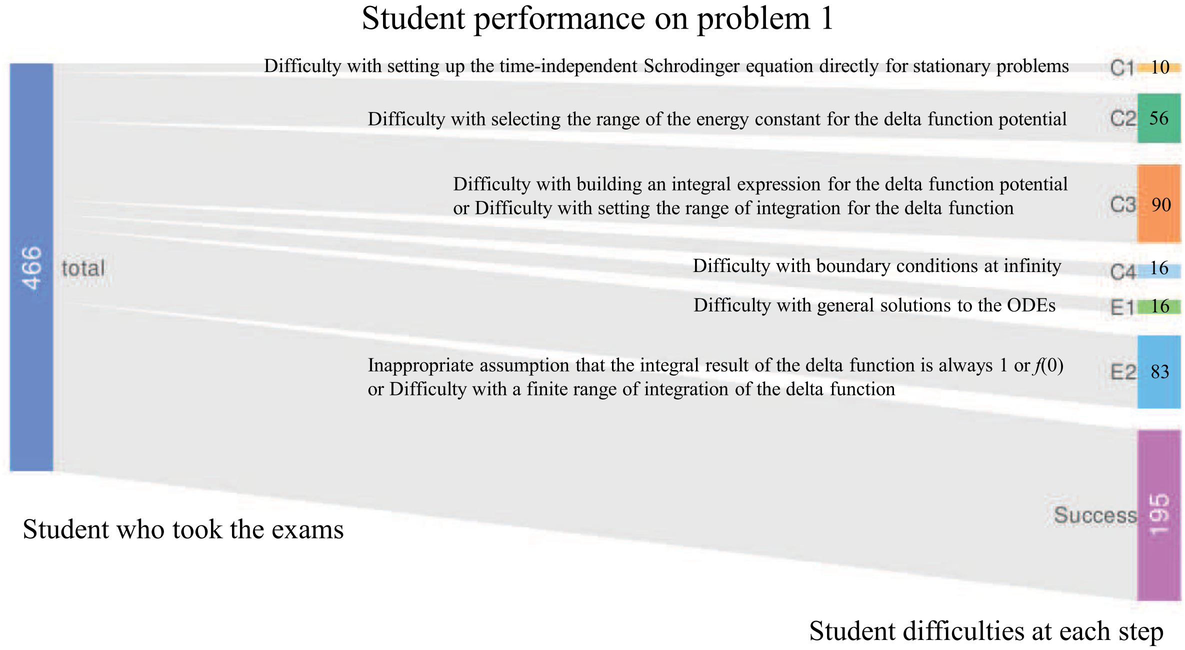

FIG. 3. Sankey diagram depicting the pathways of students' solutions as they progressed through the delta function well problem. The diagram shows the flow from the beginning to the different types of solutions that have difficulties at a particular step, where the width of the arrows are proportional to the number of solutions in each flow.

Finally, $N = 195$ students (approximately 42% of the total) successfully passed these steps and obtained the correct answers. As shown in Fig. 3, students tend to make mistakes in three areas: difficulty with selecting the range of the energy constant for the delta function potential (C2), difficulty with building an integral expression for the delta function potential (C3), and difficulty with calculating the integral of a delta function (E2). Therefore, these three areas constitute the main difficulties for students to solve the problem.

VI. FINDINGS ON STUDENTS' DIFFICULTIES WHEN ADDRESSING A POSITION SPACE DELTA FUNCTION PROBLEM

In this section, we use the ACER framework to classify and identify students' difficulties in solving the position space delta function problem (problem 2 in Table I).

A. Activation of the tools

The question asks for an expression of the wave function after the position measurement, activating delta functions as the appropriate mathematical tool. There were $N = 506$ students who took the exams, of which $N = 18$ students did not know how to proceed.

In the interviews, three students expressed confusion about being asked to provide the wave function after the position measurement. When we asked them why they did not continue, a typical answer was "Measuring the position of the particle, the state of the particle will become the position eigenstate. But I don't remember the expression for the position eigenstate. I vaguely remember that it seems to be an exponential function e^{ikx}, but I am not sure." The other two students gave similar responses. The interview results indicate that some students do not come up with the delta function as a mathematical description of this process and they tend to incorrectly assume that the position eigenfunction is a plane wave. That is, they have difficulty recognizing the delta function as the appropriate mathematical tool, even though they know that the wave function collapses to a position eigenfunction after position measurement.

B. Construction of the equations

Step C1: In step C1, students need to know that the position measurement will collapse the system to the position eigenstate. In addition, they need to know that the position eigenfunction is a delta function and relate the position eigenvalue to the argument of the delta function. There were $N = 488$ students who started this step, and among them, $N = 406$ students wrote down the correct expression of the delta function for the wave function after the position measurement $\Psi(x) = A\delta(x - \frac{a}{2})$. In the remaining $N = 82$ solutions, the common errors included not building an expression of the delta function at all; or giving an incorrect expression of the delta function.

The interviews provided additional insight into students' difficulties in constructing expressions of delta functions. Their responses can be divided into three types.

Four interviewed students wrote down an expression as $|\Psi(x)|^2$. One of them explained, "Measuring the position of a particle is measuring the probability of the particle appearing at that point." The other three students gave similar responses. The interview results indicate that some students are unaware of a fundamental concept in quantum mechanics: a measurement of a physical observable collapses the state of the system to the eigenstate of the physical observable. This results in their difficulties in distinguishing between the position eigenfunction and the probability of position measurement.

Two interviewees gave an incorrect expression for the delta function as $\Psi(x) = \delta(x)$. One of them explained, "After the measurement, the state of the system becomes a position eigenstate. The position eigenstate is a delta function of this form." The other student gave similar responses. In fact, measured at x_0, the position eigenfunction is $\delta(x - x_0)$ with the eigenvalue x_0. We believe that some students do not understand the meaning of the position eigenvalue and therefore incorrectly generalize the form of the commonly used delta function $\delta(x)$ to all cases.

Two participants wrote down an expression for the delta function as $\Psi(x) = \delta(x - x_0)$, but did not explicitly give a specific value for x_0. When we asked them why they did not give a specific value for x_0, one student replied, "Measuring the position of a particle, we do not know exactly where it is, so we cannot write a specific value for x_0. In fact, x_0 can take any value, representing the possibility that the particle can appear in the whole space." The other student gave a similar answer. The interview results imply that some students did not distinguish between the two basic concepts: the eigenvalue and the probability of the position measurement.

Step C2: In step C2, students are required to express the wave function established in step C1 as a superposition of the energy eigenfunctions. There were $N = 406$ students who gave the correct wave function after the position measurement, and among them, $N = 385$ students rewrote this wave function in a linear combination form of the energy eigenfunctions. In the remaining $N = 21$ solutions, a common error was not establishing the superposition form of the energy eigenfunctions.

In the interviews, one student wrote down an expression as $|\Psi(x)|^2$ at this step. He explained, "The result of a measurement is the probability of finding a particle at a certain point. The square of the absolute value of the wave function $|\Psi(x)|^2$ represents this probability." The interview results suggest that some students confuse position measurements with energy measurements.

Step C3: In step C3, students are required to express the probability amplitude as an integral of the wave function and each energy eigenfunction. There were $N = 385$ students who progressed to this step, and among them, $N = 341$ students correctly wrote the expression for the probability amplitude $c_n = \int_0^a \varphi_n(x)\delta(x - \frac{a}{2})dx$. In the remaining $N = 44$ solutions, a common error was the failure to establish the integral expression for the probability amplitude.

Three interviewed students wrote down a correct inner product in Dirac notation for the probability and then obtained an incorrect result as $c_n = \langle\varphi_n|\Psi\rangle = \varphi_n(x)\delta(x - \frac{a}{2})$. When we asked them why there was no integral part, one of them replied, "To calculate the probability of a wave function, we should only calculate the probability of it being at a certain point. Thus we do not need to include the integral part. Only when we calculate the probability of a region, we need to include the integral part." The other two student gave a similar response. The interview results show that some students incorrectly assume that an expression for the probability amplitude of measuring an observable does not involve the integral, which may lead to difficulties in converting abstract Dirac notations into concrete integral expressions in position representation.

Step C4: In step C4, students are required to express the probability as the absolute square of the probability amplitude. There were $N = 291$ students who proceeded to this step, and among them, $N = 285$ students correctly wrote down the expression for probability. In the remaining $N = 6$ solutions, a common error was to treat the probability amplitude directly as the probability.

During the interviews, one student first wrote down an expression for probability as c_n, then he erased it and wrote down another expression as $|c_n|^2$. He explained, "Probability is the absolute square of the probability amplitude. These two quantities are very similar, so I sometimes confuse them. Fortunately, here I have written the correct expression for it." We suspect that some students tend to confuse the two concepts of probability and probability amplitude.

C. Execution of the calculations

Step E1: In step E1, students need to calculate the integral expression established in step C3. There were $N = 341$ students who built the correct integral containing the delta function, and among them, $N = 291$ students obtained the correct integral result, $\int_0^a \sin(n\pi x/a)\delta(x - a/2)dx = \sin(n\pi/2)$. In the remaining $N = 50$ solutions, the common errors included: incorrectly calculating the integral as 1; or stoping at this step.

The interviews provided insight into the students' difficulties in calculating the integrals of delta functions. The results of the student interviews can be divided into two categories.

Five students wrote down the integral as $\int_{-\infty}^{+\infty} \sin(n\pi x/a)\delta(x - a/2)dx = 1$. One of them explained, "An integral that contains a delta function should result in 1, which is a fundamental property of the delta function." The other four students gave similar responses. The interview results suggest that some students overgeneralize the basic integral formula of the delta function $\int_{-\infty}^{+\infty}\delta(x)dx = 1$ to cases where it does not apply.

Another two interviewees stopped at this step. One of them explained, "I remember that all integrals on the delta functions are from minus infinity to plus infinity. But here the integral is from 0 to a. I think it should result in 1, but I am not sure, so I did not continue writing." Another student gave a similar response. The interview results indicate that some students do not understand the key to the integral property of the delta function: the integration interval need not be the full space, but need only include the point where the delta function is located.

D. Reflection on the solutions

For problem 2, a powerful tool is to check the units of the expressions (element R1). Among the interviewed students, only 4 students explicitly commented on the unit of the delta function, and all of them believed that the delta function is unitless. Similar to the findings regarding problem 1, this result suggests that our students' belief that the delta function is unitless is widespread and persistent. This may lead to difficulties in their understanding of the physical meaning of the delta function in specific contexts.

Another effective reflection method is to check that the results are consistent with the expectations based on the physical situation (element R2). In particular, one can draw the position eigenfunction, which is a delta function centered at $a/2$. Then, one can draw the energy eigenfunction $[\sin(\frac{n\pi x}{a})]$, which is a function of odd (even) parity centered at $a/2$ for the case where n is odd (even). With this graph, it can be expected that the final result should be different for odd and even cases. However, none of the students in the interviews commented this relatively subtle argument. One student's answer contained an error that the result was the same for both odd and even cases. After our prompting, he tried to draw the wave functions and realized his mistake. He excitedly replied, "Only in classical mechanics, such as problems with a particle moving, I have used graphical method to help solve the problem. In quantum mechanics, I have never used this kind of graphical method to help solve a problem. This is quite enlightening. It is really great!" Another two students still drew qualitatively incorrect wave functions after promoting. For example, one student drew two axes with the horizontal and vertical axes labeled x and Ψ, respectively, and then drew a point on the x-axis to represent the position eigenfunction. He explained, "After the position measurement, the system collapses to a point in space." Another student drew only one x axis and drew the energy eigenfunction as a peak on the x axis. He explained, "After the energy measurement, the state collapses to a specific state, so it should be represented by the shape of a delta function at a certain point." A graphical representation of the wave function can help students gain a deeper understanding of the physical situation. However, this result shows that our students rarely use this reflection method and have various difficulties in graphing wave functions.

E. Overview of students' performance

In total, $N_t = 506$ students took the exams and attempted to solve the position space delta function problem. In Fig. 4, we show the Sankey diagram of the

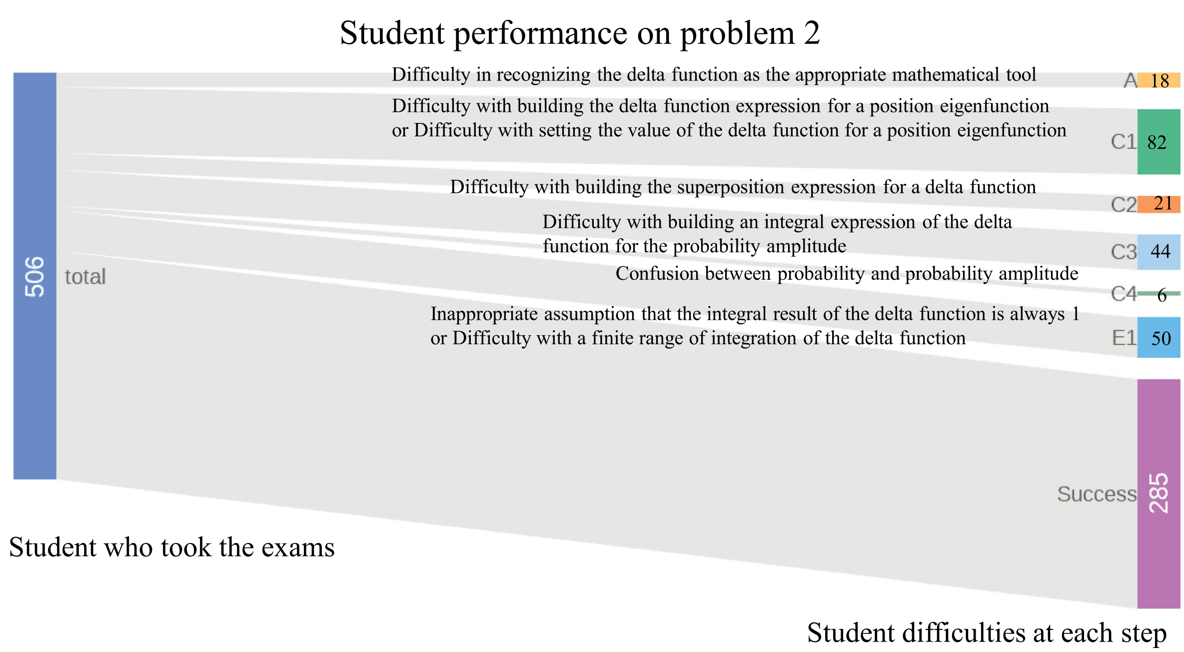

FIG. 4. Sankey diagram depicting the pathways of students' solutions as they progressed through the position space delta function problem.

students' problem-solving process on this problem. The types of errors on each solution step are listed on the right side of the graph, and the width of the flow in the graph represents the number of solutions that have certain errors. Finally, $N = 285$ students (about 56% of the total) successfully passed these solution steps and obtained the correct answers. We found that students tend to make mistakes in three areas: difficulty with building the delta function expression for a position eigenfunction and difficulty with setting the argument of the delta function for a position eigenfunction (C1), difficulty with building an integral expression of the delta function for the probability amplitude (C3), and difficulty with calculating the integral of a delta function (E1). These three areas therefore constitute the main difficulties for students in solving the problem.

VII. FINDINGS ON STUDENTS' DIFFICULTIES WHEN ADDRESSING THE MOMENTUM SPACE DELTA FUNCTION PROBLEM

In this section, we use the ACER framework to identify and classify students' difficulties in solving the momentum space delta function problem (problem 3 in Table I).

A. Activation of the tools

The question asks for an expression of the orthonormality of momentum eigenfunctions, activating delta functions as the appropriate mathematical tool. There were $N = 340$ students progressing to this step, of which $N = 23$ students stopped the exams at this step.

In the interviews, four students lost their way at this step. One of them claimed, "This integral is long and has many terms. I have been thinking about it for a long time and I don't know how to proceed." Then we prompted that there are two momentum eigenfunctions in the expression and the orthonormality of the momentum eigenfunctions can be used. He thought for a while and replied, "I remember the expressions for the orthonormality of energy eigenfunctions, because I often deal with this kind of topic. But I really don't remember the expression for the orthonormality of momentum eigenfunctions, because I have hardly ever dealt with such a topic before." The other three students gave similar responses. The interview results suggest that some students had little training on the orthonormality of the eigenfunctions with continuous spectra. Students knew the concept of the orthonormality of momentum eigenfunctions, but did not come up with the delta function as a mathematical description of this.

B. Construction of the equations

Step C1: In step C1, students are required to build the integral expression for the expectation value of position operator $\hat{x}$. There were $N = 394$ students trying to solve this problem, and among them, $N = 384$ built the correct integral expression. In the remaining $N = 10$ solutions, students omitted or added some terms to the integral expression.

In the interviews, one student wrote down an expression as $\langle x \rangle = \int \Psi^*(x)\Psi(x)dx$, but left out the most critical term (i.e., the position operator $\hat{x}$). He explained, "For position, its expectation value is the absolute square of the wave function." The interview results indicate that some students did not distinguish between the expectation value of position and the probability of measuring position.

Step C2: In step C2, students are required to build an expression for the wave function in the momentum space. There were $N = 384$ students who proceeded to this step, and among them, $N = 340$ set up the correct momentum space wave function $\Phi(p) = 1/\sqrt{2\pi\hbar} \int_{-\infty}^{+\infty} e^{-ipx/\hbar}\Psi(x)dx$. In the remaining $N = 44$ solutions, the common errors included not creating an expression for the momentum space wave function; or missing some terms (e.g., the exponential factor) in that expression.

In the interviews, one student stopped at this step. When we asked him if he knew the momentum space wave function, he hesitated for a moment and then replied, "I know the position space wave function, but I have no idea what a momentum space wave function is. So I really can't carry on."

Another three students wrote down an incorrect expression, for example, $\Phi(p) = \int_{-\infty}^{+\infty} \Psi(x)dx$. One of them explained, "Usually, in class and in homework, we use wave functions in position space and rarely use wave functions in momentum space. I probably remember that there seems to be an integral between the wave function in momentum space $\Phi(p)$ and the wave function in position space $\Psi(x)$. So I write it in this form." The other two students gave similar responses.

On the other hand, in the interviews, a student explicitly commented, "The concepts of wave functions in momentum space and position space, a superposition of momentum eigenstates, the probability amplitude for measuring momentum, the orthonormality of momentum eigenfunctions, and the completeness of momentum eigenfunctions are all related together. We can use one formula to derive another." Then, he correctly and fluently used the delta function expression for the orthonormality to connect the wave function in momentum space $\Phi(p)$ with the probability amplitude for measuring momentum $\langle p|\Psi\rangle$, as

$$\begin{aligned}\langle p|\Psi\rangle &= \int \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\Psi(x)dx \\ &= \int \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\left[\int \frac{1}{\sqrt{2\pi\hbar}}\Phi(p')e^{ip'x/\hbar}dp'\right]dx \\ &= \int \Phi(p')\left[\frac{1}{2\pi\hbar}\int e^{-ipx/\hbar}e^{ip'x/\hbar}dx\right]dp' \\ &= \int \Phi(p')\delta(p'-p)dp' = \Phi(p).\end{aligned}$$

In contrast, several other students claimed that they knew about wave functions in momentum space or the orthonormality of momentum eigenstates, while they did not comment at all on the connection between these concepts. Moreover, when we asked them to derive the above equations, they encountered various difficulties and did not proceed. Expertise about delta functions in momentum space contains different nodes that represent different knowledge pieces in this particular knowledge domain. The interview result suggests that students who appropriately connect different nodes of knowledge about momentum space often lead to a functional understanding of these knowledge. In contrast, students who simply memorize each knowledge piece have only local consistency in their understanding of these knowledge pieces, but lack global consistency. This may lead to difficulties in their reasoning about momentum space, such as difficulties in understanding and using the delta function and its closely related concepts in momentum space.

Step C3: In step C3, students are required to utilize the orthonormality of momentum eigenfunctions. There were $N = 340$ students who proceeded to this step, and among them, $N = 236$ students correctly wrote down the delta function expression for the orthonormality, $1/(2\pi\hbar)\int_{-\infty}^{+\infty} e^{-ipx/\hbar}e^{ip'x/\hbar}dx = \delta(p' - p)$. In the remaining $N = 81$ solutions, students either did not build an delta function expression or built incorrect forms of the delta function.

In the interviews, eight students made various mistakes in this step. Their answers can be divided into two types.

Three students wrote down an expression as $\int e^{-ipx}e^{ip'x}dx = 1$. One of them explained, “The integral of two momentum eigenfunctions results in 1. This is the orthonormality that is generally satisfied by the eigenfunctions.” We then asked him if he could give an example to illustrate. He replied, “For example, a particle in an infinite square potential well has energy eigenfunctions as $\varphi_n(x)$. Then the orthonormality of any two energy eigenfunctions can be written as $\int \varphi_m^*(x)\varphi_n(x)dx = \delta_{mn} = 1$ for $m = n$.” Two other students gave similar answers. In fact, the eigenfunctions with continuous spectra have the orthonormality as the delta function, while the eigenfunctions with discrete spectra have the orthonormality as the Kronecker delta δ_{mn}. The Kronecker delta reduces to 1 only if the two eigenfunctions are the same. The interview results suggest that some students confuse the orthonormality of discrete spectra and continuous spectra.

Another five students wrote down an expression as $\int e^{-ipx}e^{ip'x}dx = \delta(x - x')$. One of them explained, “I remember that the momentum eigenfunction is very special and its orthonormality is expressed as a delta function.” We then asked him to write the orthonormality of the position eigenfunctions, and he still wrote a similar expression as $\int \varphi(x)\varphi(x')dx = \delta(x - x')$. We asked him why the two expressions have the same result. He thought for a moment and replied, “Both momentum eigenfunctions and position eigenfunctions satisfy the same orthonormality as the delta function.” The interview results show that some students did not distinguish between the orthonormality of momentum eigenfunctions and position eigenfunctions.

C. Execution of the calculations

Step E1: In step E1, students are required to perform some transformations on the expression established in step C2, such as $x = -i\hbar\frac{d}{dp}$. Because this equation was explicitly provided in the text of the problem, students did not make mistakes in this step either in the exams or in the interviews.

Step E2: In step E2, students are required to calculate the integral containing the delta function built in step C3. There were $N = 236$ students who set up the correct integral expression, and among them, $N = 194$ students obtained the correct result $\int_{-\infty}^{+\infty}\delta(p' - p)\Phi(p')dp' = \Phi(p)$. In the remaining $N = 42$ solutions, the students obtained incorrect integral results as 1 or $\Phi(0)$.

In the interviews, five students made mistakes in this step. Their answers can be divided into two types.

Three interviewees calculated the integral as $\int_{-\infty}^{+\infty}\delta(p' - p)\Phi(p')dp = 1$. One of them explained, “The integral containing the delta function is always 1, which is the most important property of the delta function.” The other two students gave similar responses. The interview results show that some students inappropriately generalize the particular integral of the delta function $\int_{-\infty}^{+\infty}\delta(x)dx = 1$ to all integral cases.

Another two participants calculated the integral as $\int_{-\infty}^{+\infty}\delta(p - p')\Phi(p)dp = \Phi(0)$. One of them explained, “I remember that the result of the integral of the delta function and an arbitrary function is the value of that function taken at the origin point. This is a general formula.” The other student gave a similar response. We believe that some students incorrectly generalize the special integral formula for the delta function $\int_{-\infty}^{+\infty}\delta(x)f(x)dx = f(0)$ to all integral situations.

D. Reflection on the solutions

For problem 3, checking the units of the expressions is an effective reflection tool (element R1). However, none of the students in the interviews explicitly commented on the units of the expressions. Five of the students wrote down an incorrect expression for the orthonormality as $\int_{-\infty}^{+\infty} e^{-ipx/\hbar}e^{ip'x/\hbar}dx = \delta(p' - p)$, where they left out a constant factor $1/(2\pi\hbar)$. One student was confused when we suggested that the units on the left and right sides of this expression did not agree. We then asked him to comment on the unit of the integral on the left side and the unit of the delta function on the right side. He correctly thought that the unit of the integral is L, but incorrectly thought that the delta function is unitless. Thus he could not reconcile the two ideas: that the units of the expression are consistent and that the delta function should be unitless. We then prompted that the unit of the integral is L, while the unit of the delta function is $M^{-1}L^{-1}T$, where L, M, and T are the units of length, mass, and time, respectively. So in order to make the units of the left and right sides consistent, the left side should have a constant factor $1/(2\pi\hbar)$. He replied, "I used to memorize these formulas by rote, so it was easy to miss some constant factors. This is the first time I understand these formulas from the perspective of units, which is interesting. It gives me a refreshing feeling." Another two students gave similar responses. This result suggests the potential value of this reflection in capturing errors and facilitating interpretation, while also suggests that our students rarely use such reflection spontaneously.

E. Overview of students' performance

In total, $N_t = 394$ students took the exams and attempted to solve the momentum space delta function problem. In Fig. 5, we show the Sankey diagram of the students' problem-solving performance on this problem. The flow from left to right indicates the types of errors students made at each step, while the width of the flow indicates the number of solutions that made a certain error. Finally, $N = 194$ students (about 49% of the total) obtained the correct answers. As shown in the figure, students were prone to make mistakes in three areas: difficulty with building the expression for the wave function in momentum space (C2), difficulty with building the delta function expression for the orthonormality (C3), and difficulty with calculating the integral of a delta function (E2). These three areas are the main difficulties students have in solving the problem.

VIII. DISCUSSIONS AND CONCLUSIONS

A. Findings regarding the delta function problems

It is commonly assumed that if students learn mathematical methods in a given physical context, they are able to transfer their knowledge and skills from one physical context to another. However, this phenomenon was not observed in our study.

In this work, we investigated students' difficulties by analyzing their exam answers and conducting think-aloud interviews. Then, we analyzed the data according to the ACER framework. We found that students made a variety of errors when using the tools of delta functions to solve the problems in the context of quantum mechanics. For the three different quantum physics problems, we summarize in Table II, the primary difficulties and possible causes that students exhibited at each stage of the problem-solving process. Here, primary difficulties refer to the errors made by multiple students (typically > 20 students).

B. Comparison of delta functions in different application contexts

The above sections present the results of our study on students' difficulties with delta functions in three contexts. It is interesting to compare students' difficulties in different contexts to see what are the similarities and differences.

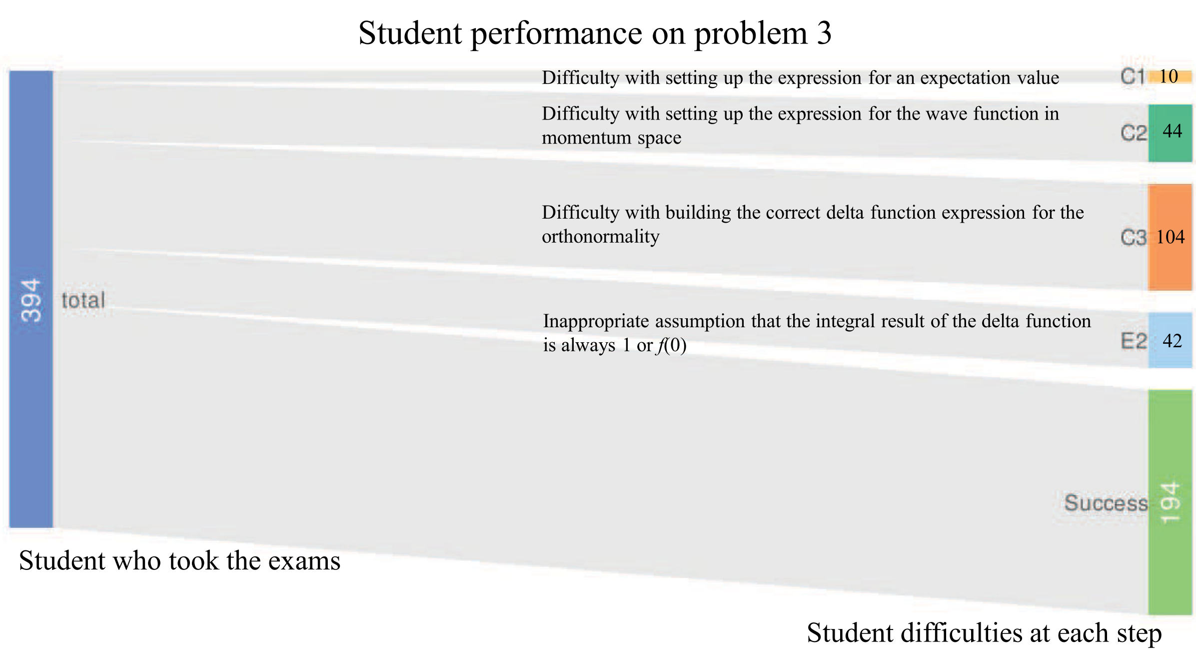

FIG. 5. Sankey diagram depicting the pathways of students' solutions as they progressed through the momentum space delta function problem.

TABLE II. A summary of student difficulties with delta functions in quantum mechanics.

Stages	Student difficulties	Possible reasons	Evidence in previous studies
Activation	Students know that the wave function collapses to a position eigenfunction after position measurement, but have difficulty recognizing the delta function as the appropriate mathematical tool	Incorrect belief that the position eigenfunction is represented by a plane wave form	Novel
	Students know the concept of the orthonormality of momentum eigenfunctions, but have difficulty recognizing the delta function as the appropriate mathematical tool	Unfamiliar with the orthonormality of eigenfunctions with continuous spectra	Novel
Construction	Difficulty with selecting the range of the energy constant for the delta function potential	Incorrect belief that a delta function potential can confine particles of all energy ranges	Novel
	Difficulty with building an integral expression for the delta function potential	Difficulty in realizing that the delta function potential must determine the boundary condition of the wave function	Novel
	Difficulty with setting the range of integration for the delta function	Incorrect belief that the range of integration of a delta function should be infinite	Novel
	Difficulty with building the delta function expression for a position eigenfunction	Confusion between the position eigenfunction and the probability of position measurement	Discussed in [9]
	Difficulty with setting the argument of the delta function for a position eigenfunction	(i) Difficulty in realizing the meaning of the position eigenvalue (ii) Confusion between the position eigenvalue and the probability of position measurement	Novel
	Difficulty with building the superposition expression for a delta function	Confusion between position eigenstates and energy eigenstates	Discussed in [9]
	Difficulty with building an integral expression of the delta function for the probability amplitude	Incorrect assumption that the expression for probability amplitude does not involve an integral part	Discussed in [9]
	Difficulty with building the expression for the wave function in momentum space	Difficulty in linking related concepts of wave functions in momentum space	Novel
	Difficulty with building the delta function expression for the orthonormality	(i) Confusion between the orthonormality of discrete spectra and continuous spectra (ii) Confusion between the orthonormality of momentum eigenfunctions and position eigenfunctions	Novel
Execution	Difficulty with calculating the integral of a delta function	Overgeneralization of the special integral formula $\int \delta(x)dx = 1$ or $\int_{-\infty}^{+\infty} \delta(x)f(x)dx = f(0)$ to all integral cases	Partially discussed in [37]
	Difficulty with the integral of a delta function for the case of finite integration range	Incorrect assumption that the integration range of a delta function should be infinite	Novel
Reflection	Difficulty with using effective reflection methods such as checking units	Incorrect belief that the delta function is unitless	Discussed in [37]
	Difficulty with using effective reflection methods such as checking boundary conditions	Difficult in graphing a wave function in the situation of the delta function potential	Novel
	Difficulty with using effective reflection methods such as checking limits	(i) Difficult in graphing a delta function form for the position eigenfunction (ii) Incorrectly graphing a delta function form for the energy eigenfunction	Novel

(i) Differences in the activation stage: In solving the delta function potential well problem, the problem text explicitly suggests an expression for the delta function. It effectively shortens the activation process but also provides little information about the activation process.

In solving the position space delta function problem, students needed to activate the resources related to the delta functions. However, we found that students knew that a position measurement collapses the particle wave function to a position eigenstate, but did not come up with the delta function as a mathematical description of this.

Similarly, in solving the momentum space delta function problem, students also needed to activate the resources related to the delta functions. However, we found that although students knew that the momentum eigenstates should satisfy the orthonormality, they did not come up with the delta function as a corresponding mathematical description for this.

(ii) Differences in the construction stage: In solving these problems, our students made various mistakes in the construction process. In the case of the delta function potential well, some students did not set up the integral of the delta function potential. There are two reasons for this difficulty: (i) students do not realize that the delta function potential can determine the boundary condition of the wave function at the point where the potential is infinite. (ii) students are used to examples that the range of integration of the delta function is infinite, which discourages them from establishing a finite range of integration of the delta function.

In the case of position space delta function, some students either did not build the delta function or built the incorrect expression of the delta function. This may be due to their difficulties with some basic concepts: (i) they do not recognize that the position measurement would collapse the state to the position eigenfunction, (ii) or they do not distinguish between the position eigenvalue and the probability of position measurement.

In the case of the momentum space delta function, some students also did not correctly establish the delta function expressions for the orthonormality of the momentum eigenfunctions. This may be due to the following two reasons: (i) students confuse the orthonormality of discrete spectra and continuous spectra, (ii) or they confuse the orthonormality of position eigenfunctions and momentum eigenfunctions.

(iii) Similarities in the execution stage: In all three problems, students were required to calculate the integral of the delta function. We found that students commonly made the same mistakes: incorrectly calculating the integral results as 1 or $f(0)$. This is due to the fact that students overgeneralize the special integral formulas $\int_{-\infty}^{+\infty}\delta(x)dx=1$ or $\int_{-\infty}^{+\infty}\delta(x)f(x)dx=f(0)$ to all integrals of delta functions, where they are not directly applicable.

(iv) Similarities in the reflection stage: Reflection refers to the use of appropriate approaches to check whether the solution is consistent with physics-based expectations in order to gain confidence in the solution. For the problems covered in this study, the following reflection methods can be used: checking the units, or confirming that the boundary conditions are satisfied, or checking the limiting behavior of the solution.

While our students are able to perform meaningful reflection when explicitly prompted, few are able to execute these reflections spontaneously. They often use purely mathematical, rote procedures without a deeper understanding of the physical picture and physical meaning of the problem. Such an approach is highly inefficient.

On the one hand, students rarely comment spontaneously on the units of the expression including a delta function. And they all believe that delta functions are unitless, a misconception that surprisingly persists whether the delta function is in position space or in momentum space.

On the other hand, in addition to commenting on the units of a given expression, we would like our students to recognize the physical interpretation of this expression and to ensure that the limit behavior and physical interpretation are consistent. For experts, the units, the limit behavior and its physical meaning of a delta function are closely linked; however, the results of the study suggest that this relationship may still be developing for our students.

In summary, in the construction stage, students need to construct expressions for the delta function. They need to translate the physical situation into mathematical expressions (e.g., establish the delta function expression for the corresponding position eigenfunction after a position measurement) or explain the physical meaning of the mathematical expressions (e.g., recognize that the argument of the delta function corresponds to the position eigenvalue). The study results indicate that the students do not develop a functional understanding of the concepts relevant for solving these problems (e.g., difficulty in distinguishing between the eigenvalue and the probability of position measurement). The differences between the difficulties encountered by students in different contexts are understandable, as the construction stage of the problem-solving process is highly dependent on the specific physical context of the problem. In the execution stage, the mathematical computing errors made by the students on these three problems are similar because the integral calculation of the delta function is content independent.

C. Comparison with previous studies on delta functions in the context of electrostatics

As mentioned in the literature overview, there is only one study on students' difficulties using the delta function in the context of electrostatics [37]. It is important to compare our study with previous studies to see what pre-existing difficulties are persistent and what difficulties are emerging.

(i) Similarities and differences in the activation stage: The previous study [37] has found that students have difficulty in recognizing the delta function as the appropriate mathematical tool for charge distribution even they can provide a correct physical interpretation of it. Similarly, we found this type of difficulty in more aspects of quantum physics. On one hand, students know that the wave function collapses to a position eigenfunction after position measurement, but have difficulty recognizing the delta function as the appropriate mathematical tool. On the other hand, students know the concept of the orthonormality of momentum eigenfunctions, but have difficulty recognizing the delta function as the appropriate mathematical tool.

(ii) Similarities and differences in the construction stage: In the study of electrostatics [37], the shape of the charge distribution is related to the number of delta functions. The researchers found that students sketch the charge distribution on the three-dimensional Cartesian axes as a very narrow spike. This is a one-dimensional graphical representation of the delta function at $x = 0$ as an infinitely high and thin distribution, which is commonly used when the delta function is first defined. However, students attempt to apply this one-dimensional representation of the delta function to a three-dimensional representation, leading to difficulties in relating the shape of the charge distribution to the coordinate system and the number of delta functions.

In the present study of quantum mechanics, we focus on the case of one-dimensional delta functions. During the interviews, we found that students mostly understand the definition of the delta function and they are able to draw a schematic representation of the delta function: a spike at a certain position. Unlike the case of electrostatics, it is a correct visualization to relate this one-dimensional representation of the delta function as a very narrow spike to the mathematical expression of the delta function. In order to probe how students use high-dimensional delta functions and possible difficulties, in future studies we will design questions related to high-dimensional delta functions in the context of quantum mechanics.

The previous study [37] has found that students have difficulty in relating the values of the parameters of the delta function to the locations of the charge distribution. This type of difficulty also appeared in our study, for example, students made errors in determining the eigenvalues of the position eigenfunctions. When the delta function is used to express the charge distribution, the parameter value indicates the charge position. When the delta function is used to express the position eigenstate, the parameter value represents the position of the particle after a position measurement. Thus, students' difficulties with the physical meaning of the parameter values of the delta function persist in different physics contexts.

The previous study [37] has also found that students have difficulty in building integrals for regions of the charge distribution. In fact, we found that this type of difficulty arise in more contexts of quantum mechanics: for the delta function potential, students do not know why they need to build an integral to construct the corresponding boundary condition for the wave function. For the measurement probability, students have difficulty in translating the abstract Dirac notation into the concrete integral expression, especially when this integral involves the delta function. A combination of the previous study and our study indicates that although students mostly understand the integral nature of delta functions, it is challenging for them to construct the integrals of delta functions from scratch for physical situations. Taken together, these studies suggest that although the delta function methods are general, students have considerable challenges in translating specific physical concepts into these delta function expressions.

(iii) Similarities and differences in the execution stage: The previous study [37] has found that students have difficulty in calculating the integrals of delta functions. Further, the study also found that this difficulty is mainly due to the fact that students recognize that the delta function can pick out the value of a variable, but they always apply incorrectly, e.g., they incorrectly believe $\int_{-\infty}^{+\infty} \delta(x - x')dx = x'$.

In our study, we also observed that students have various difficulties in calculating the integrals of delta functions. However, the reasons for these difficulties are somewhat different from previous findings. On the one hand, students tend to incorrectly generalize the integral formulas $\int_{-\infty}^{+\infty} \delta(x)dx = 1$ or $\int_{-\infty}^{+\infty} \delta(x)f(x)dx = f(0)$ to all integral cases where these integral formulas are not applicable. On the other hand, students are used to the case that the limits of integration of the delta function are from negative infinity to positive infinity, leading their difficulties with the case of the finite region of integration.

(iv) Similarities and differences in the reflection stage: The previous study has found that students rarely use effective checks, and in particular have difficulty in checking the units of delta functions and constants. A similar difficulty was found in our study: students believed that the delta function is unitless. Thus, students' difficulties with the units of delta functions persist in both electrostatics and quantum mechanics contexts.

In summary, as shown in Table II, the many difficulties in using the delta function seem to persist in the students' reasoning for solving electrostatics problems (as documented in the previous study) and for solving quantum physics problems (as documented in this paper). In addition, some new difficulties in constructing and computing delta functions are identified in our study.

D. Comparison with previous studies on position measurement

To the best of our knowledge, there are no studies specifically on the use of delta functions in the context of quantum mechanics. In addition, there is some work that deals with the topic of position measurement [2,5,6,9]. Position is an important physical operator whose eigenfunction is a delta function, and its eigenvalue is the value of the parameter in the delta function.

Previous studies [2,5,6,9] have found that students do not realize that the state of the system would become the eigenstate of the observable after the measurement. This type of difficulty also appeared in our study. For example, some students do not recognize that the state of the system would collapse to the position eigenstate after the position measurement, which discourages them to built the delta function expression for the position eigenstate.

Previous studies [2,5,6,9] also found that students have difficulty in distinguishing between the probability of measuring position and the outcome of position measurement. This type of difficulty manifested itself in numerous ways in our study. On the one hand, some students confuse the absolute square of the wave function $|\Psi|^2$ with the resulted state of position measurement, which leads to their difficulties in writing down the delta function expression for the position eigenstate. On the other hand, some students do not understand the physical meaning of the values in the delta function $\delta(x - x_0)$ and incorrectly believe that their values cannot be determined due to probability.

Previous studies [2,5,6,9] have found that students confuse between positional eigenstates and energy eigenstates. This type of difficulty also appeared in our study. For example, the students' confusion between position measurement and energy measurement results in their difficulties in writing the position eigenstate as a superposition form of the energy eigenstates.

Moreover, in our study, many difficulties arose in establishing the integral of a certain delta function and calculating the result of this integral. However, quantum mechanics concept investigations [2,5,6,9] usually consists of multiple-choice questions that do not ask students to perform calculations, so these difficulties have not been observed in previous studies.

In summary, as shown in Table II, synthesizing previous related studies and our study, students have some conceptual difficulties with position measurements. These difficulties with conceptual understanding affect how they translate these concepts into mathematical expressions of delta functions.

E. Implications for instruction

Our investigation of students' common difficulties can provide several implications for instruction on delta functions in quantum mechanics.

First, students know the image of the delta function and they can draw a spike to explain the delta function. However, they have difficulty in translating the relevant quantum physics concepts into expressions of the delta function. In particular, for the delta function expression of position eigenstate, they often confuse the following concepts: the result of position measurement, the probability of position measurement, and the position eigenfunction after position measurement. We suggest that for the position operator, students should be asked to do more conceptual test questions, which provide an opportunity for them to try to distinguish various quantum concepts related to the position operator.

Second, students have difficulty in establishing expressions of the delta function for the orthonormality of the eigenfunctions with continuous spectra. In the traditional homework at USTC, students are often exposed to eigenfunctions with discrete spectra, such as the energy eigenfunctions in an infinite square potential well, and they are able to express the orthonormality of these energy eigenfunctions. However, they incorrectly assume that this simple orthonormality can be generalized to the case of all eigenfunctions. We suggest that more exercises on the eigenfunctions with continuous spectra can be assigned and students can try to solve these problems using delta functions.

Third, students have difficulty in creating integral expressions for the delta function potential. In fact, students often incorrectly believe that the wave function and its derivative are always continuous. We suggest that the correct boundary conditions should be emphasized in teaching: although the wave function is always continuous, its derivative is discontinuous at points where the potential is infinite. It would also be more enlightening to draw a graph of the wave function at these singular points for a specific example. This would allow the students to understand why the delta potential can determine the boundary condition for the derivative of the wave function.

Fourth, students make various mistakes when they perform the integral calculations concerning delta functions. In particular, students incorrectly generalize the integral formulas of the delta function (e.g., $\int_{-\infty}^{+\infty} \delta(x)dx = 1$) to all situations where they are not applicable. Since students often encounter these typical examples, they will remember them and use pattern matching to deal with the problems they encounter on the exams. We suggest that more complex integral calculations of delta functions can be assigned for homework, so that students can be asked to use the basic properties of delta functions instead of memorizing simple examples.

In the calculations, students incorrectly assume that the integration of the delta function must be from negative to positive infinity. We suggest that this context should be emphasized in teaching: all that matters is that the region of integration includes the point where the delta function is located.

The ability to meaningfully reflect on or interpret the solutions is a defining characteristic of physicists. Checking the solution and constructing a meaningful interpretation are as important as a practice that gets solutions. Yet these are the areas where our students struggle most when manipulating delta functions. These methods of reflection are not sufficiently emphasized in our current physics courses. Questions and activities should be designed to develop students' reflective skills.

In general, there are several effective ways to reflect on the solutions in quantum mechanics: checking whether the units of the expressions are consistent, checking whether the wave function satisfies the boundary conditions, and checking whether the limiting behavior of the results is consistent with expectations.

Checking the units of an expression is one of the most common methods of reflection. Exercises can be assigned while actively prompting students to comment on the units of the expression and the corresponding physical interpretation, and to ensure that the units and physical interpretation are consistent. This will develop the habit of students reflecting on their solutions by checking the units. Specifically for problems related to delta functions, students can start by analyzing the integral expression of a one-dimensional delta function $\int_{-\infty}^{+\infty} \delta(x)dx = 1$ to determine the units of the delta function.

Checking the behavior of the wave function at the boundary or in a particular regime is also an effective method of reflection. Specifically for problems with delta function potentials, students can analyze the wave function in different regions to ensure that the result is consistent with expectations. For example, a boundary with a delta potential of infinity separates two regions with a delta potential of zero, so a reasonable intuition is that the wave functions on both sides should behave similarly, while the wave function should change dramatically at the boundary.

In particular, we can use the Bayesian updating [47,48] to facilitate students' application of reflection. The Bayesian updating activities allow students to test a hypothesis or model using hypothetico-deductive reasoning, and then perform a Bayesian update of their confidence in the hypothesis or model being tested. Previous work [47,48] has found that using Bayesian updating activities in introductory physics courses resulted in significant improvements in students' scores on the epistemic beliefs assessment in the physical sciences (EBAPS).

When solving quantum mechanical problems (e.g., solving the Schrödinger equation for a particle in a potential), students can be asked to reflect on their solutions by following the steps of the Bayesian updating as

Step 1: assume that the solution is correct and give an initial confidence C_i.

Step 2: predict how the result in the solution should change if one of the input physical parameters is changed.

Step 3: redo the calculations to see if the result changes as expected.

Step 4: determine whether this reflective activity confirms or disconfirms the hypothesis.

Step 5: update the confidence in the hypothesis C_f using Bayes' theorem.

$$C_f = \frac{C_i R}{C_i R + 1 - C_i}.$$

Here C_f is the confidence of hypothesis H being true given the newly obtained evidence E, and C_i is the initial confidence of hypothesis H being true before considering the new evidence, and R is a Bayes factor which refers to whether a particular piece of evidence E confirms, or negates, hypothesis H. For example, a confirmatory evidence leads to a large value of R ($R > 1$); a disconfirmatory evidence means a small value of R ($0 < R < 1$).

The ability to learn from reflection on the experimental and theoretical testing of physical hypotheses is central to the practice of physics. These Bayesian updating activities provide students with a consistent structure and clear motivation to reflect on their work, and lead them to independent learning.

In conclusion, we utilized the ACER framework to analyze students' problem-solving processes using delta functions in quantum mechanics. We found that students' understanding of relevant quantum physics concepts affects their performance using the delta function. Compared to previous studies of the use of delta functions in electrostatics, our study showed that some students' difficulties can be perpetuated and new difficulties can arise. The delta functions here are relatively simple and future studies could be extended to more complex delta functions, such as three-dimensional ones. Additional research could provide a broader view of students' problem-solving processes, help identify possible patterns of student reasoning, and allow for the development of appropriate instructional strategies to address students' reasoning difficulties.

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APPENDIX A: OPERATIONALIZATION OF THE ACER FRAMEWORK FOR THE DELTA FUNCTION WELL PROBLEM

In the following we provide the summary of the process to solve the delta function well problem (problem 1 in Table I) according to the ACER framework.

- Step A—In this problem, a particle interacts with a potential well of the delta function form, which can active the tools of delta functions.
- Step C1—Set up the basic equation:

  We can write the time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \gamma\delta(x+a)\right]\varphi(x) = E\varphi(x).$$

- Step C2—Choose the range of the energy constant:

  When the energy E is less than the potential energy at both plus and minus infinity, the particle is in a bound state. Thus we have $E < 0$. Since

$$\delta(x+a) = 0, \quad \text{if } x \neq -a$$
$$= \infty, \quad \text{if } x = -a,$$

we can write the time-independent Schrödinger equation as

$$\frac{d^2\varphi}{dx^2} = k^2\varphi$$

in the region $x < -a$ and in the region $-a < x < 0$. This is an ordinary differential equation (ODE), and $k = \sqrt{-2mE/\hbar^2}$ is real and positive.

- Step E1—Provide the general solutions to the ODE:
  The above ODE has the general solutions as

$$\varphi(x) = Ae^{kx} + Be^{-kx},$$

in the region $x < -a$, and

$$\varphi(x) = Ce^{kx} + De^{-kx}$$

in the region $-a < x < 0$. Here A, B, C, and D are the unknown constants.

- Step C3—Set up the boundary condition at the point $x = -a$:
  We can write the time-independent Schrödinger equation as

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \gamma\delta(x+a)\right]\varphi(x) = E\varphi(x)$$

at the point $x = -a$. Integrating the Schrödinger equation from $-a-\epsilon$ to $-a+\epsilon$, and taking the limit as $\epsilon \to 0$, we thus obtain

$$\lim_{\epsilon\to 0}\int_{-a-\epsilon}^{-a+\epsilon}\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \gamma\delta(x+a)\right]\varphi(x)dx = \lim_{\epsilon\to 0}\int_{-a-\epsilon}^{-a+\epsilon} E\varphi(x)dx.$$

- Step E2—Calculate the integral containing the delta function:
  The integral in the above boundary condition yields

$$\int_{-a-\epsilon}^{-a+\epsilon}\delta(x+a)\varphi(x)dx = \varphi(-a).$$

Then the left side of the above boundary condition is

$$\lim_{\epsilon\to 0}\int_{-a-\epsilon}^{-a+\epsilon}\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \gamma\delta(x+a)\right]\varphi(x)dx = -\frac{\hbar^2}{2m}\left[\frac{d\varphi}{dx}\bigg|_{-a^+} - \frac{d\varphi}{dx}\bigg|_{-a^-}\right] - \gamma\varphi(-a),$$

and the right side of the above boundary condition is

$$\lim_{\epsilon\to 0}\int_{-a-\epsilon}^{-a+\epsilon} E\varphi(x)dx = 0.$$

Thus we obtain

$$\left[\frac{d\varphi}{dx}\bigg|_{-a^+} - \frac{d\varphi}{dx}\bigg|_{-a^-}\right] = -\frac{2m\gamma}{\hbar^2}\varphi(-a).$$

- Step C4—Set up other boundary conditions:
  The second term in the general solution $\varphi(x) = Ae^{kx} + Be^{-kx}$ blows up as $x \to -\infty$, so we are left with $\varphi(x) = Ae^{kx}$. Then we impose boundary conditions: φ continuous at $x = 0$ and $x = -a$,

$$C + D = 0,$$
$$Ae^{-ka} = Ce^{-ka} + De^{ka}.$$

In addition, utilizing the result in Step E2, we have the boundary condition: $\frac{d\varphi}{dx}$ discontinuous at $x = -a$,

$$k(Ce^{-ka} - De^{ka}) - kAe^{-ka} = -\frac{2m\gamma}{\hbar^2}Ae^{-ka}.$$

- Step E3—Determine the energy eigenvalues:
  Using algebraic calculations, we obtain

$$\left(1 - \frac{2}{kL}\right)Ae^{-ka} = 2C\cosh(ka).$$

Here $L = \frac{\hbar^2}{m\gamma}$. This is the equation for the allowed energies of the bound states, where k is a function of the energy E.

- Step R—Use the specific methods to check the solution: e.g., check the units of the expressions or check the boundary conditions.

APPENDIX B: OPERATIONALIZATION OF THE ACER FRAMEWORK FOR THE POSITION SPACE DELTA FUNCTION PROBLEM

Following the ACER framework, a summary of the process to deal with the position space delta function problem (problem 2 in Table I) is shown here.

- Step A—The measurement of position invokes the concept of position eigenfunctions, which can prompt the resources of delta functions.
- Step C1—Write down a delta function expression for the position eigenfunction after a position measurement:

  After the position measurement at the center of the well, the wave function is a delta function as

$$\Psi(x) = A\delta\left(x - \frac{a}{2}\right).$$

- Step C2—Express the wave function as a superposition of the energy eigenfunctions:

  A subsequent energy measurement yields the energy eigenfunctions. Thus we expand the wave function as

$$\Psi(x) = A\delta\left(x - \frac{a}{2}\right) = \sum_n c_n \varphi_n(x),$$

  where the energy eigenfunctions are $\varphi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$.
- Step C3—Set up the equation for the probability amplitude:

  The probability amplitude of a given energy eigenfunction can be expressed as a integral

$$c_n = \int_0^a \varphi_n^*(x)\Psi(x)dx = \int_0^a \varphi_n^*(x)A\delta\left(x - \frac{a}{2}\right)dx.$$

- Step E1—Calculate the integral containing the delta function:

  Applying the property of the delta function, we can calculate the integral

$$c_n = \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)A\delta\left(x - \frac{a}{2}\right)dx$$
$$= A\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{2}\right).$$

- Step C4—Setup the expression for the probability:

  The probability of measuring a given energy value can be expressed as

$$P_n = |c_n|^2.$$

  Thus we can find the probability of measuring a given energy value E_n,

$$P_n = |A|^2 \frac{2}{a} \text{ for } n \text{ odd and } P_n = 0 \quad \text{for } n \text{ even.}$$

- Step R—Apply the specific methods to check the solution: e.g., check the units of the expressions or check that the result is consistent with the expectation based on the form of the wave function.

APPENDIX C: OPERATIONALIZATION OF THE ACER FRAMEWORK FOR THE MOMENTUM SPACE DELTA FUNCTION PROBLEM

According to the ACER framework, a summary of the process to deal with the momentum space delta function problem (problem 3 in Table I) is shown here.

- Step A—The problem involves the momentum representation. The important property in the momentum representation is the orthonormality of the eigenfunctions, which can motivate knowledge related to the delta function.
- Step C1—Express the expectation value in terms of the wave function in position space:

  For the position operator, we have

$$\langle x \rangle = \int_{-\infty}^{+\infty} \Psi^*(x)x\Psi(x)dx,$$

  where $\Psi(x)$ is the wave function in the position space.
- Step C2—Express the wave function in momentum space:

  The wave function in momentum space is a Fourier transform of the wave function in position space,

$$\Phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} e^{-ipx/\hbar}\Psi(x)dx.$$

  Alternatively, its inverse Fourier transform is

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} e^{ipx/\hbar}\Phi(p)dp.$$

- Step E1—Organize the expression of the intergral:

  We substitute it into the expression of expectation value, and we obtain

$$\langle x \rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} e^{-ipx/\hbar}\Phi^*(p)dp\right]x \times \left[\int_{-\infty}^{+\infty} e^{ip'x/\hbar}\Phi(p')dp'\right]dx$$
$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Phi^*(p)\left(i\hbar\frac{d}{dp}\right) \times e^{-ipx/\hbar}e^{ip'x/\hbar}\Phi(p')dp'dpdx.$$

Here we use the hint of the question: $xe^{ipx/\hbar} = (-i\hbar \frac{d}{dp})e^{ipx/\hbar}$.

- Step C3—Apply the orthonormality for momentum eigenfunctions:

  For two momentum eigenfunctions, we can write their orthonormality as

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{-ipx/\hbar}e^{ip'x/\hbar}dx = \delta(p'-p).$$

We substitute it into the expression of expectation value, and we obtain

$$\langle x\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Phi^*(p)\left(i\hbar\frac{d}{dp}\right) \times \delta(p'-p)\Phi(p')dp'dp.$$

- Step E1—Calculate the integral containing the delta function:

  Using the property of the delta function, we can calculate the above intergal as

$$\langle x\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Phi^*(p)\left(i\hbar\frac{d}{dp}\right) \times \delta(p'-p)\Phi(p')dp'dp = \int_{-\infty}^{+\infty}\Phi^*(p)\left(i\hbar\frac{d}{dp}\right)\Phi(p)dp.$$

  This is the final answer.

- Step R—Apply the specific methods to check the solution: e.g., check the units of the expressions.

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