Exploring student ideas on change of basis in quantum mechanics

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A common task when problem solving in quantum mechanics, including in a spins-first curriculum, involves changing the basis of a given state. Our research in undergraduate quantum mechanics courses at three institutions explores student thinking about basis, basis expansion coefficients, and change of basis in the context of spin-½ systems. Our investigation is based on conceptual and computational written questions as well as student reasoning interviews. We identify student ideas about whether and how changing basis affects the state, examine how students perceive notation as indicative of choice of basis, explore students' interpretations of the structure and meaning of a basis expansion, and identify the range of methods students employ when changing basis. For instance, we find a recurring idea that changing basis alters the physical system, and observe that some students chose to relabel the ket representing a quantum state vector after changing basis. Together, these results paint a broad, qualitative picture of a variety of ways that students grapple with basis and change of basis, with potential implications for instruction.

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I. INTRODUCTION

Basis is a fundamental concept in quantum mechanics, and converting between bases is a routine task when solving quantum mechanical problems. States and observables are typically expressed in terms of a single set of basis vectors. Physicists often choose to represent quantum states in terms of a basis that corresponds to an observable (measurable quantity) of interest. Depending on the physical system, the bases used may be continuous or discrete, but a basis is always present when writing the state of a quantum system. In this paper, we explore student ideas related to basis, methods of changing basis, and coefficients in a basis expansion within the context of a spins-first quantum mechanics course and with a focus on Dirac notation.

Although all representations of a quantum state provide equal information in the abstract sense, the choice of basis determines which information is easily accessible. Changing basis involves performing a calculation that converts between such representations. It neither affects the mathematical identity of the represented state nor does it alter the physical system that the state vector describes.

As an example, in spin-½ systems, there are two possible outcomes for measuring the spin angular momentum (**S**) along a given axis: $+\hbar/2$ or $-\hbar/2$. Any such two-state system can be described using two orthonormal basis states, corresponding to each of the possible outcomes. Consider the following two expressions for the same quantum mechanical state, $|\psi\rangle$:

$$|\psi\rangle = \frac{2}{\sqrt{5}}|+\rangle + \frac{1}{\sqrt{5}}|-\rangle, \tag{1}$$

$$|\psi\rangle = \frac{3}{\sqrt{10}}|+\rangle_x + \frac{1}{\sqrt{10}}|-\rangle_x, \tag{2}$$

where $|\pm\rangle$ are the eigenstates of the S_z operator, often referred to as the z basis or S_z basis, and $|\pm\rangle_x$ are the eigenstates of the S_x operator, often referred to as the x basis or S_x basis. The coefficients in Eq. (1) are probability amplitudes related to measuring a particle to have spin $+\hbar/2$ or $-\hbar/2$ along the z axis, corresponding to the eigenstates $|+\rangle$ and $|-\rangle$, respectively. Equation (2) displays different probability amplitudes because the values are related to measurements of the S_x operator corresponding with spin along the x axis. It is convention to assume that the kets $|\pm\rangle$ are the eigenstates of S_z unless there is a subscript indicating a different direction.

Although Eqs. (1) and (2) are written in different bases, both represent the same quantum mechanical state. Any advantage of one expression over the other depends on the particular information that is desired or the calculation that is being performed. Converting between bases can be useful, e.g., for determining measurement probabilities for different operators or for determining how states evolve in time.

In recent years, there has been a growing body of research on students' difficulties with quantum mechanical concepts [1–10] and on the resources students use when working with those concepts [11–13]. Further research has focused on the mathematical components of quantum mechanics, including student understanding of Dirac and linear-algebraic representations [14–16], calculation and sense making of expectation values [1,7,17–19], boundary conditions when solving for energy eigenstates [20], and the process of normalization [21]. Serbin *et al.* have investigated student and expert use of language related to basis in quantum mechanics [22], but little other research has specifically attended to student understanding of basis and change of basis in quantum mechanics.

There are multiple calculations that can be used to convert between bases. When working with orthonormal bases as is typically done in quantum mechanics, the method that often requires the fewest lines of algebra uses inner products of the given state with the new basis vectors to determine the coefficients in the new basis. For example, the coefficient $3/\sqrt{10}$ in Eq. (2) is given by the inner product of the $|+\rangle_x$ basis state with $|\psi\rangle$: ${}_x\langle+|\psi\rangle$. This calculation (with a similar one for the other coefficient, $1/\sqrt{10}$) can be used to derive Eq. (2) given Eq. (1); i.e., to change $|\psi\rangle$ from being expressed in the S_z basis to being expressed in the S_x basis.

Previous research has explored how students connect the inner product with probabilities in a wave functions context [9]. Wan and colleagues found that students identified the coefficient associated with expansion in the energy basis, but did not know to solve for it using an inner product. This is similar to recognizing $2/\sqrt{5}$ as a coefficient in Eq. (1), but not recognizing how to represent it as an inner product (e.g., $\langle+|\psi\rangle$). Note that although the two situations are analogous, they are not the same: the context of continuous wave functions includes additional concepts beyond those necessary for spins, such as connecting the inner product to an integral expression. As such, it is worth investigating this topic in the context of spins specifically.

In a separate study, Wan and colleagues noted that "many students do not recognize the measurable effects of relative phases" because they do not consider observables associated with bases other than the one provided [23]. Basis is also relevant in other studies of student reasoning about relative phases in spin-½ states [24]. Other work exploring students' methods for computing expectation values found that students may not think to change basis, even if doing so is required to solve the problem [17]. The present study complements prior work with a comprehensive examination of student ideas about basis and change of basis specifically in the discrete (spin-½) context.

In the mathematics education community, research has focused on student understanding of two properties associated with a general basis in linear algebra [25]: span and linear independence [26–30]. In quantum mechanics, basis states are typically further restricted to being orthogonal and normalized. Because physicists can exploit orthonormality, the methods for changing basis in physics classes are different from the more universal methods needed for general bases in linear algebra. Mathematics education research on basis has a broader scope and does not account for the specific use of basis in quantum mechanics.

Given the significant role that basis plays in a spins-first course, as well as the importance of changing basis when solving problems in quantum mechanics more generally, we extended prior research by administering several surveys, conducting interviews, and examining three years of students responses on quizzes and exams. Applying a phenomenographical perspective [31–33], we address the following research questions in a spin-½ context:

RQ1: How do students interpret a change of basis with respect to both the physical nature and the representation of the state?

RQ2: What methods do students use when performing a change of basis for a quantum state?

RQ3: How do students interpret coefficients in a basis expansion and/or relate them to inner products?

A state may be represented in a number of ways (e.g., with Dirac notation, with matrix notation, or graphically). Although this study focuses primarily on Dirac notation, we do discuss examples where other representations are relevant.

Using a variety of data sources, we identified the ways students change basis and the ideas related to different facets of understanding basis. This qualitative analysis will help inform the development of curricular materials [34].

II. METHODOLOGY

Data were collected over three years from three public universities. Universities A and B are large, Hispanic-serving, primarily undergraduate institutions with between 25 and 55 students enrolled in upper-division quantum mechanics in a given year. University C is an R1, Ph.D. granting institution with 60+ students enrolled in quantum mechanics each semester.

The quantum mechanics courses from which data were collected used a spins-first instructional paradigm following McIntyre's "Quantum Mechanics" textbook [37]. Each course was taught by one of us, who used similar interactive instructional materials including multiple choice concept questions and tutorials to emphasize conceptual understanding of the material [35]. We met weekly to discuss the curriculum and shared select quiz and exam questions. Two of the courses were paced similarly and only covered material through the infinite and finite square well. Additional topics covered at University C included the free particle, orbital angular momentum, and the hydrogen atom. All courses met in person.

To address the three research questions, we analyzed a variety of data sources, including interviews, surveys, and for-credit assessments. Tables I, II, and III show an abbreviated list of questions that were given to students for the purposes of this study. A standard change of basis question (Table II) was given at different points over the three year span on exams at the three universities. An isomorphic version of the question was given to students in interviews conducted at Universities A and B—we refer to these as the "exploratory interviews." Several ungraded surveys (S1, S2, and S3) were later administered to provide insight into specific ideas related to research questions RQ1 and RQ3. Interviews probing these ideas were also conducted with students at University C—we refer to these as the "follow-up interviews." The surveys and interviews are described in more depth below.

We employed methods of phenomenography to investigate the range of student ideas. A phenomenographical approach is a qualitative approach that aims to identify and categorize the variations among individuals' perceptions and conceptualizations of phenomena [31–33]. Students responding to a question in similar ways may share a similar conceptualization of that phenomena regardless of correctness. In the present study, the categories used in analysis were emergent from the data and used as a description for procedures or lines of reasoning that arose multiple times. Because the majority of the data collected includes interviews and written explanations, we approach the data from a qualitative perspective to identify student ideas related to basis and change of basis. In this work, we catalog both correct and incorrect ideas and procedures, as our goal is to uncover the breadth of ideas that students express.

A. Data and analysis: Interpretations of changing basis (RQ1)

Of the three administered surveys, S1 and S2 included questions designed to probe students' interpretations of a change of basis. In addition to these surveys, both rounds of interviews included elements that targeted RQ1. In the exploratory interviews, which followed a semi-structured format, students were asked to carry out a change of basis, and then asked if the state at the end of calculation (in the new basis) was the same state as the one given in the problem (in the basis of S_z) or whether a new state had been created. The exploratory interviews preceded and informed the design of the surveys.

Survey S1 (Q1–Q3, Table I; Q8, Table III) was given as part of regular online prelecture assignments, which

TABLE I. Survey questions targeting RQ1 regarding students' interpretation of a change of basis. Surveys S1 and S2 were both administered at one institution throughout a quantum mechanics course. Identical versions of surveys S1 and S2 were administered to 6 student interviewees at a second institution. Correct answers to true or false and multiple-choice questions are given in bold.

Survey S1 $N = 29$	Q1	By writing the state $\vert\psi\rangle$ in the x-basis, we've changed the probabilities for measuring along the z-direction. Choose one and *explain your response*: True **False**
	Q2	We can't know the probabilities for measurements along both the z-direction and the x-direction at the same time. Choose one and *explain your response*: True **False**.
	Q3	By representing the state $\vert\psi\rangle$ in the x basis, we've created a new quantum state. Choose one and *explain your response*: True **False**
Survey S2 $N = 27$	Q4	Consider a spin-½ state written in terms of the z-basis vectors: $\vert\phi\rangle = \frac{3}{5}\vert+\rangle - \frac{4}{5}\vert-\rangle$. Suppose we rewrote this state in terms of the x-basis vectors. (The coefficients in all the answers are correct.) How would you choose to label the state now? *Explain your choice.* a. $\vert\phi\rangle = -\frac{1}{5\sqrt{2}}\vert+\rangle_x + \frac{7}{5\sqrt{2}}\vert-\rangle_x$ **(Don't relabel the state)** b. $\vert\phi\rangle_x = -\frac{1}{5\sqrt{2}}\vert+\rangle_x + \frac{7}{5\sqrt{2}}\vert-\rangle_x$ (Add an x subscript on the state) c. $\vert\alpha\rangle = -\frac{1}{5\sqrt{2}}\vert+\rangle_x + \frac{7}{5\sqrt{2}}\vert-\rangle_x$ (Give the state a new name)
	Q5	In the previous question, did rewriting the state in the x basis create a new quantum state? *Explain your choice.* a. Yes b. **No**
	Q6	$\vert+\rangle$ represents the spin-½ state with S_z eigenvalue $+\hbar/2$. What is $\vert+\rangle_y$? *Explain your choice.* a. $\vert+\rangle_y$ **is the state with** S_y **eigenvalue** $+\hbar/2$ b. $\vert+\rangle_y$ represents $\vert+\rangle$ in the y basis. c. BOTH of the above.

received participation credit only. The survey was completed by $N = 23$ students at University B after relevant course instruction. Survey S2 (Q4–Q6, Table I) was given in the same class at the end of the semester as an optional survey ($N = 21$).

Survey S1 was also given at University C in a set of interviews carried out with 3 pairs of student volunteers ($N = 6$) who were currently enrolled in the quantum mechanics course. These follow-up interviews followed a semistructured format. First, interviewees complete survey S1 working alone. Then the interviewed pairs completed an instructional activity related to basis and change of basis within the context of spin-½ systems, with limited guidance from the interviewer. Finally, the interviewed students also completed survey S2 on their own several weeks after their interview.

Questions Q1–Q3 are true or false questions given to probe ways that students might think a state has been physically altered by changing basis. These questions are presented in Table I. Q1 asks whether changing basis affects measurement probabilities. Q2 asks whether measurement probabilities (not outcomes) are simultaneously knowable for multiple directions. Q3 asks whether changing basis creates a new state. These questions are primarily targeted towards students' interpretation of change of basis with respect to the physical nature of the state.

Questions Q4–Q6 probe student understanding of notation as it relates to basis (also in Table I). Q4 asks whether a ket should be relabeled after changing basis. Relabeling a ket $|\phi\rangle$ means either adding a subscript (e.g., $|\phi\rangle_x$) or giving it a new name (e.g., $|\alpha\rangle$). Q5, like Q3, asks whether a new state has been created by the change of basis. Q5 was asked again to complement Q4, allowing the survey to distinguish students' use of notation for states in different bases from their interpretation of the process and consequences of change of basis. Q6 asks students to interpret the subscript on $|+\rangle_y$; we asked Q6 to probe student understanding of subscripts as they relate to basis to complement Q4 answer (b), which suggests applying a subscript to a ket after changing basis. These questions are primarily targeted towards students' interpretation of change of basis with respect to the representation of the state.

When coding students' responses to questions Q1–Q3, we identified specific *reasoning elements*. A reasoning element is an idea or line of argumentation a student uses to justify or explain their answer; for example, the "conflation of change of basis with measurement" element arose in students' explanations for why they expected measurement probabilities to differ after a change of basis was applied to a state. Reasoning elements are similar to *resources* [38], but are not necessarily as primitive or fundamental. In line with our phenomenographical approach [31–33], the reasoning elements we present generalize multiple students' statements, and we present all reasoning elements regardless of correctness.

We also identified *broad ideas*, which we define as possible answers to RQ1: according to students, what effect (if any) does changing basis have on the physical nature and/or the representation of a state? *A priori*, one might anticipate at least two broad ideas—changing basis *does* or *does not* have or result in a physical effect—as well as, possibly, others related to ways in which changing basis affects or is reflected in the *representation* of the state. We found that three distinct broad ideas emerged from student responses, outlined below (Sec. III A). Each reasoning element is consistent with only one broad idea, but a single broad idea may have multiple consistent reasoning elements. For example, we label statements that changing basis alters the physical system as belonging to a single *broad idea* category, but students employed a variety of reasoning elements to justify answers consistent with this idea.

The broad ideas and reasoning elements were originally identified by the first author, and then refined in discussion with five of the authors. These codes, which were emergent from the survey S1 data, were then applied to the analysis of related questions from survey S2. We also analyzed discussions from both the exploratory and follow-up interviews in light of these codes; in particular, interviews afforded us a closer look into student thinking related to the broad ideas and reasoning elements. Although we had *a priori* notions for how to categorize these data, using emergent coding allowed us to identify unexpected distinctions in students' ideas. Results from this analysis are discussed in Sec. III A.

B. Data and analysis: Methods for changing basis (RQ2)

To determine which methods students use when changing basis, question Q7 was given in interviews and on for-credit assessments at all three institutions. Shown in Table II, Q7 asks students to write a state in terms of a new set of basis states. In addition to being given on exams,

TABLE II. Interview and exam question targeting RQ2 regarding methods students use when changing basis. Versions of this question were administered to students at each of the three institutions on several for-credit quizzes or exams in multiple semesters. Different versions omitted the hint and/or changed the numerical coefficients in the definitions of $|\pm\rangle_n$.

Interviews and assessments $N = 237$	Q7	Consider a spin-½ particle prepared in the state $|\psi\rangle$, written in the z basis as $|\psi\rangle = \frac{1}{\sqrt{2}}|+\rangle + i\frac{1}{\sqrt{2}}|-\rangle$ Write the state in the n basis assuming $|+\rangle_n = \frac{1}{\sqrt{3}}|+\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|-\rangle$ and $|-\rangle_n = \frac{\sqrt{2}}{\sqrt{3}}|+\rangle - i\frac{1}{\sqrt{3}}|-\rangle$. *Hint: Solve for the values of a and b such that* $|\psi\rangle = a|+\rangle_n + b|-\rangle_n$

a version of Q7 was part of a set of interviews across Universities A and B in the second year of data collection (the exploratory interviews, $N = 22$). These think-aloud interviews were conducted with paid student volunteers who were enrolled in the quantum mechanics course. The interviews followed a semistructured protocol, allowing room to probe students' conceptual understanding of basis. To address RQ2, we focus on how students went about changing basis in the interviews.

Beyond the interviews, separate variations of question Q7 were given in different semesters for a total of 215 written responses from the three universities ($N = 237$ including interviews). With the exception of one administration ($N = 37$), all variations included complex coefficients but used different numerical values for the probability amplitudes in the given state and basis states.

Analysis of the procedural change of basis question (Q7) produced categories describing the different approaches students used to calculate a change of basis. For example, student responses that used inner products to calculate coefficients were labeled as using the "projection" method, while student responses that used algebraic manipulation and substitution were considered a separate category. The application of these categories required minimal interpretation beyond parsing students' handwritten work. All of the responses were coded by two authors, and the list of categories were agreed upon by all authors. Results from this analysis are discussed in Sec. III B

C. Data and analysis: Coefficients and inner products (RQ3)

Survey questions were also administered to investigate students' interpretations of basis expansion coefficients and the ways they relate to inner products. Relevant questions are shown in Table III. Survey S1, described above, also included Q8. In question Q8, students were given a state and asked what information is provided by the coefficients in the expansion of a state in a basis.

Survey S3 (Q9 and Q10) was given in class ($N = 24$) after all relevant instruction. Question Q9 asked students to describe the meaning of an inner product written in Dirac notation, ${}_y\langle +|\psi\rangle$. Question Q10 probed whether students could generate the generic expression for the expansion of a state in a basis; e.g., $|\psi\rangle = a|+\rangle_n + b|-\rangle_n$ in the n basis. When previously asked to change basis (Q7), the question text included this expression in a hint.

When analyzing student responses to Q8–Q10, we focused on two steps that are involved when using projection inner products as a method for changing basis: recognizing that a state can be written as an superposition of other basis states; and connecting basis expansion coefficients with inner products. We identified the different ways students addressed these concepts in their responses, using a coding strategy similar to our strategy for RQ1: categories emerged from the data and were refined in discussion among us to increase confidence in our interpretation of students' written statements.

III. RESULTS

In this section, we present a variety of student ideas on basis and change of basis in the context of a spin-½ system. First, addressing RQ1, we describe the ideas that students express when they are asked to interpret the meaning of a change of basis and its implications on the state (Sec. III A). Second, addressing RQ2, we identify the procedural methods that students use when they are asked to represent a state in a new basis (Sec. III B). Finally, addressing RQ3, we discuss student understanding of the structure of a basis expansion and students' interpretation of the coefficients, of inner products, and of the relationship between the two (Sec. III C).

A. Interpretations of changing basis (RQ1)

To explore student understanding of how changing basis affects the quantum state, interviews and written questions were designed to target specific conceptual and notational aspects of basis and changing basis. Three broad ideas arose in student responses to these survey questions, as well as in dialogue in interviews. These ideas are summarized in Table IV and described in detail below.

A basis change is akin to expressing a vector in a different coordinate system. As such, it has no effect on the quantum state, and does not require relabeling the associated ket (quantum state vector). This is broad idea (i): the idea that changing basis does *not* affect the physical system

TABLE III. Survey questions targeting RQ3 regarding students' interpretations of coefficients in a basis expansion. Survey S1 also included questions presented in Table I. Like surveys S1 and S2, survey S3 was administered at one institution in the quantum mechanics course.

Survey S1 $N = 29$	Q8	Consider a spin-½ electron prepared in the state $|\psi\rangle = \frac{1}{\sqrt{3}}|+\rangle + \frac{\sqrt{2}}{\sqrt{3}}|-\rangle$ What do the coefficients in this expression (the $\frac{1}{\sqrt{3}}$ and the $\frac{\sqrt{2}}{\sqrt{3}}$) tell you about the state?
Survey S3 $N = 24$	Q9	Consider a quantum state $|\psi\rangle$. What is the meaning of ${}_y\langle +|\psi\rangle$? Describe what the symbols are as well as any physical meanings you associate with the expression as a whole.
	Q10	How can we represent a general state $|\psi\rangle$ in the S_x basis? Write an expression for any unknown variables you use.

TABLE IV. Students expressed a range of reasoning elements on different questions probing their understanding of change of basis and its affect on both the physical system and the quantum state vector (ket) describing that system. Each reasoning element is consistent with one of three broad ideas, which represent possible interpretations of the effect of change of basis on a quantum state. Different questions (from Table I) invoked different reasoning elements. The table also gives examples of answers that were consistent with the broad idea.

Broad idea	Consistent expressed reasoning elements	Answers given
(i) Changing basis does not affect the physical system or the state vector (Sec. III A 1)	Choice of basis is strictly a means of representing a state vector. Change of basis is strictly a reversible calculation.	Stating that changing basis does not change measurement probabilities, and choosing to retain a ket's original label when representing it in a new basis. For example, answering "False" on Q1 or (a) on Q4.
(ii) Changing basis alters the physical system (Sec. III A 2)	Changing basis is the same thing as making a measurement. One cannot know probabilities for spin measurements along different directions at the same time (misinterpretation of the uncertainty principle). $\lvert\psi\rangle$ is associated with just one pair of spin-up/down probabilities.	Stating that changing basis changes measurement probabilities. For example, answering "True" on Q1.
(iii) Changing basis can involve relabeling the state vector (i.e., the ket) (Sec. III A 3)	Notation (e.g., a subscript on a ket) indicates choice of basis (by convention). It is useful to indicate choice of basis with notation (even if convention does not require it). A ket or state vector is mathematically distinct when represented in a new basis (even if it describes an unchanged physical system).	Choosing to relabel a ket when representing it in another basis (e.g., $\lvert\psi\rangle \to \lvert\alpha\rangle$), or choosing to add a subscript to a ket when representing it in another basis (e.g., $\lvert\psi\rangle \to \lvert\psi\rangle_x$). For example, answering (b) or (c) on Q4.

or the state vector. In other words, changing basis does not affect the state.

Broad idea (ii) is the idea that changing basis has a physical effect, such as modifying measurement probabilities or outcomes. If we ask "Does changing basis affect the state?," a student holding broad idea (ii) might answer affirmatively. However, we must disambiguate the word "state," as students may use it to mean the state vector (the ket), or the physical system. Broad idea (2) specifically refers to students thinking that the physical system is changed as a result of changing basis.

A related question is, "Does changing basis involve relabeling the state vector (i.e., the ket)?" Broad idea (iii) says yes: it is the idea that a state vector can or should be labeled differently when represented in a different basis. We saw no instances where students simultaneously argued that the physical system had been altered by a change of basis but that the notation for the state vector should remain unchanged. However, we did encounter cases where students opted to relabel the state vector following a change of basis, but nonetheless argued that the physical system was *unaltered*. Therefore, broad idea (iii) emerged as a distinct category from broad idea (ii). As such, students opting to relabel after a change of basis certainly places their response under broad idea (iii), but is insufficient evidence determine whether or not they also hold broad idea (ii).

In quantum mechanics, there is (up to global phase) a one-to-one correspondence between the configuration of a physical system and the quantum state vector describing that configuration. When the distinction between physical system and quantum state vector is relevant to distinguishing student ideas in the discussion below, we use the terms "physical system" and "quantum state vector" (or "ket") explicitly. Otherwise, when we use the term state, we refer to *both* the physical system it represents and the corresponding state vector. When students say state, we cannot always be certain if they mean one or both of these things. For the purposes of this analysis, whenever possible, we infer what students mean by the word state from context.

Students' responses to the various questions included a range of reasoning elements consistent with one of the three broad ideas, as summarized in Table IV. Although individual responses largely fell under one of these three ideas, different responses from the same student sometimes aligned with multiple ideas, indicating that students' ideas about the meaning of basis and change of basis were fluid, and that different questions could draw out different reasoning elements for students.

In the following three subsections, we elaborate on the three broad ideas and provide relevant examples from student work. The survey questions referred to in this section can be found in Table I.

1. Idea that changing basis does not affect the physical system or the state vector

Correct answers to questions Q1–Q5 are consistent with this broad idea. Students answering these questions correctly provided explanations along two lines: that basis is just a means of representing a state, and that change of basis is a reversible calculation.

First, some students argued that changing basis did not alter the physical system nor the state vector because a basis is just a means of representing the same state vector. In response to Q5, one student said

> *It isn't creating a different state, it's writing the same state in a different representation.*

Some students drew an explicit analogy with coordinate systems, as in this response to Q5:

> *Our choice of basis does not change the state, just like our coordinate system does not change where something physically is.*

Another student made a similar analogy but instead with unit conversion, saying in response to Q1,

> *To me, I see doing a change in basis similar to doing unit conversions in a way. "Going from mass to moles does not change our initial mass."*

Whereas the first line of reasoning centers on the meaning of basis, the second focuses on the idea that change of basis is only a calculation—in particular, one that can be undone. One student presented this reversibility argument in response to Q3 as follows:

> *It is the same quantum state...converting back and forth will not change the original state.*

The argument that change of basis is a reversible calculation arose in multiple students' correct explanations for these five questions.

Students in both rounds of interviews cited both reversibility and the idea that a basis is a choice of representation as reasons why changing basis does not alter the state. No other lines of reasoning identified as completely correct were evident in our sample.

2. Idea that changing basis alters the physical system

Questions Q1, Q2, and Q3 targeted student thinking around how changing basis might affect the physical system—an answer of true to any of those question suggests student reasoning that is consistent with the broad idea that changing basis alters the physical system, and students' explanations illuminate reasons why they may hold this belief. These findings confirm what we observed in the exploratory interviews, in which students made statements consistent with broad idea (2). We encountered three reasoning elements in explanations to answers consistent with the broad idea that changing basis has a physical effect.

The first is the notion that a state is associated with a specific basis and thus tied to a specific pair of (physical) spin-up or spin-down probabilities. In reality, a single state encodes all possible pairs of probabilities at once—that is, probabilities for spin measurements along any axis. Among the reasons given in favor of True for Q1 and Q3 was the response that "the probabilities" would be different after the change of basis. By probabilities, students appear to mean the pair of probabilities for measuring spin-up or spin-down along a specific direction. For example, in an explanation for Q1, one student wrote

> *If you change the orientation, you will get different probabilities.*

While this statement would be true if the student were discussing changing the orientation of a *measurement*, they gave it as an explanation for why changing *basis* from the z basis to the x basis would alter the probabilities for measurement along the z direction. This line of reasoning arose in responses to both Q1 and Q3. Similar reasoning arose in the exploratory interviews: some students argued that the probabilities for "spin-up" and "spin-down" had changed with the change of basis. Again, students seemed to associate the state with just one pair of spin-up or spin-down probabilities.

The second reasoning element that arose was misapplication of the uncertainty principle. Q2 asked whether it is possible to know the probabilities for measurements along multiple axes simultaneously. Because these probabilities are directly encoded in basis expansion coefficients, responses arguing that knowing this information simultaneously is physically impossible (answers of true) are consistent with the idea that the state vector representing a physical system is tied to a single basis, and that therefore change of basis alters the physical system. Some students who answered Q2 in this way alluded to the uncertainty principle, incorrectly extending it to apply to measurement probabilities as well as measurement outcomes. For example, one student said

> *We can only be certain of the probabilities for measurements in only one direction at a time.*

This interpretation of the uncertainty principle was also consistent with other students' justification for answering true for Q1:

> *By changing the z basis to x basis we...now only can calculate the outcome in the x direction.*

If this interpretation of the uncertainty principle were accurate—meaning that probabilities for incompatible measurements could not be known simultaneously—it would imply that changing basis creates and destroys information and would suggest that change of basis is (or is associated with) a physical process.

The third reasoning element we observed demonstrated a conflation between the process of change of basis and the act of making a measurement. Unlike change of basis, measurement is a physical procedure that does alter probabilities. For example, measuring spin along the x direction provides a definite value for that observable, but in general invalidates any previously known probabilities for, say, the z component of spin.

If a student conflates measurement with change of basis, they may misapply a valid understanding of measurement to draw invalid conclusions about the consequences of a changing basis. Some responses appeared consistent with this conflation. For example, some students referred to the effect of measurement to justify their answer for Q2. One student said

> *The act of measuring in one direction will alter the system such that the other direction would be random.*

Arguments about measurement arose in responses to Q1 and Q3 as well. One student responded to Q1 with the argument that,

> *…measuring in another basis will alter the probability of the output that comes out in the z direction. I think this is because measuring a state alters it.*

The same student made a similar argument in answer to Q3:

> *By measuring it in that axis we have an altered quantum state…*

Responses like these suggest that some students conflate change of basis with the physical act of making a measurement. The same conflation was made by some students in the follow-up interviews. In a debrief at the end of their interview, one student named distinguishing basis-change from measurement as one of the key things they had learned during the interview. (The follow-up interviews included an instructional activity, which was completed after interviewees took survey S1.)

3. Idea that changing basis can involve relabeling the state vector (i.e., the ket)

In both interview and written contexts, students often elected to relabel a ket when representing it in a different basis. Options (b) $|\phi\rangle_x$ and (c) $|\alpha\rangle$ for Q4, which asked how best to label a ket after a change of basis, exemplify this idea (see Fig. 1). We consider these choices incorrect because relabeling is not necessary and has the potential to lead to confusion (for example, relabeling could suggest that the Dirac notation expressions for the state in each basis cannot be related by an equals sign, whereas they are in fact equal).

> **Q4.** Consider a spin-1/2 state written in terms of the z-basis vectors: $|\phi\rangle = \frac{3}{5}|+\rangle - \frac{4}{5}|-\rangle$.
>
> Suppose we rewrote this state in terms of the x-basis vectors. (The coefficients in all the answers are correct.) How would you choose to label the state now?
>
> **a.** $|\phi\rangle = -\frac{1}{5\sqrt{2}}|+\rangle_x + \frac{7}{5\sqrt{2}}|-\rangle_x$ **(Don't relabel)**
>
> b. $|\phi\rangle_x = -\frac{1}{5\sqrt{2}}|+\rangle_x + \frac{7}{5\sqrt{2}}|-\rangle_x$ (Add an x-subscript)
>
> c. $|\alpha\rangle = -\frac{1}{5\sqrt{2}}|+\rangle_x + \frac{7}{5\sqrt{2}}|-\rangle_x$ (Give it a new name)
>
> *Explain your choice.*

FIG. 1. Q4 from Table I, repeated here for convenience. When considering whether changing basis involves relabeling the state vector, students may either (a) not relabel; (b) relabel by adding a subscript to the ket; (c) relabel by using a new symbol inside the ket. The correct answer is (a).

Students' justifications for relabeling a ket after changing basis appeared to fall in two categories: one argument was that the notation for a ket plays a role in indicating choice of basis; the other was that a ket represented in a different basis is somehow a mathematically distinct object from the ket represented in the original basis.

The first argument that students gave in favor of relabeling a state vector after changing basis was that the notation for a ket either does, or should, indicate the basis used to represent the state. This reasoning element arose both in responses to Q4 and in the interviews. Students who choose to relabel a ket after changing basis may understand that neither the physical system nor the ket itself have changed, but they may still consider relabeling to be notationally useful.

Subscripts—specifically subscripts on the angle bracket of the ket, such as $|+\rangle_x$—arose as the notation of choice for some students who opted to relabel a ket after changing basis. Some did not see the addition of a subscript as equivalent to giving the ket a new name, and some saw subscripts as playing a special role in indicating choice of basis. This explanation for choosing $|\phi\rangle_x$ as the answer to Q4 demonstrates both of these ideas:

> *We can write states in different basis if we indicate it with a subscript on the state. The state itself doesn't change in a different basis so there's no need to give it a new name [referring to* $|\alpha\rangle$*].*

Q6 was designed to probe students' interpretation of subscripts on a ket. Some students opted for choice (b) alone, which argued that $|+\rangle_y$ is $|+\rangle$ in the y basis. Other students argued that both (a) and (b) were true, with choice (a) correctly defining $|+\rangle_y$ as an eigenstate of S_y. Choices (a) and (b) are mutually inconsistent if you also recognize that $|+\rangle$ is *not* also an eigenstate of S_y. However, both choices are consistent if one believes that a subscript on a ket indicates the basis used to represent the state.

The idea that the subscript on a state vector denotes basis arose in the interviews as well. After concluding that a ket need not have a subscript added after changing basis, one student also concluded that the x subscript on $|+\rangle_x$ was optional, while another suggested that S_x and S_y were the same matrices expressed in different bases. Note that the notation for a ket does not indicate basis choice, but there are other contexts where subscripts are used to indicate basis, such as in matrix representations [39].

The second argument students used for relabeling a state after changing basis was not that it was simply useful (or required) notation, but that the state vector written in the new basis was somehow distinct from the original state vector. Here we refer to the "state vector" as a distinct concept from the physical system it describes. Hence this line of reasoning belongs here, under broad idea (iii), as opposed to under broad idea (ii), because either we could not assume from context that the student believed the physical system had been altered, or alternatively, the student made it clear they did *not* believe anything had changed physically.

Students making the argument that the state vector somehow changed appeared to consider the choice of basis as part of the identity of the state vector. Interviews illuminated this reasoning element. Comparing the vector (temporarily labeled $|k\rangle$) resulting from a change of basis to the original vector (labeled $|u\rangle$), one student argued that

> *They are the same vector when they are both expressed in the same basis...$|k\rangle$ in the i basis is just $|u\rangle$.*

This student identified being "expressed in the same basis" as part of what it means to be "the same vector." Another made a similar claim:

> *It's still different components even though it's the same vector. It isn't $|u\rangle$ [the original vector] because $|u\rangle$ is in a different basis*

This student argued that a vector written in two different bases can no longer be identified by the same symbol ($|u\rangle$) because of the different axes and values used to describe it, even if they both describe the same unaltered physical system. They focused on the difference in representation rather than the fact that both versions represented the same physical system.

Broad idea (iii)—that changing basis can involve relabeling the state vector—can be distinct from broad idea (ii)—that the physical system has been altered. As an example, one student who opted to give the ket a new label in Q4 nonetheless argued that it described the original quantum state in response to Q5 (*not* consistent with broad idea (2)):

> *It's the same state, but it is just represented in a different coordinate system. (it looks different but its the same.)*

This student argued convincingly that the physical system has not been altered by the change of basis. Their response is consistent with broad idea (iii), because they elected to give the ket a new label [choice (c)], but *in*consistent with broad idea (ii), because they argued the state was the same. For this student, the change in representation (the fact that the state vector "looks different") was sufficient for them to choose to relabel the ket in Q4. (Because the student distinguished the state from its representation, we believe they used the word state to refer to the "physical system.")

B. Methods for changing basis (RQ2)

Analysis of question Q7 from interviews and several years of quiz and exam data from three different institutions have shown that students use a variety of approaches to change the basis of a state in the spin-½ context. It is possible to correctly change basis using various methods, including projection (i.e., inner products), algebraic manipulation, or substitution. All three of these methods arose in the data.

Often, the most mathematically efficient method is to use orthogonal projection. This method produces the coefficients $a_\pm$ for the state in the new basis, $|\psi\rangle = a_+|+\rangle_n + a_-|-\rangle_n$, by calculating the inner products $a_\pm = {}_n\langle\pm|\psi\rangle$ (Fig. 2). More directly, the new representation can be found by operating with the identity $\mathbb{1} = \Sigma_i |a_i\rangle\langle a_i|$ on the state $|\psi\rangle$, where $|a_i\rangle$ are the eigenstates in the desired basis, but this first step is often skipped.

The remaining two methods are more algebraically intensive and were more subject to error for students. Both methods involve algebraic manipulation and substitution but are distinguished by which step is first in the process. The algebraic manipulation method involves treating the new basis states as a system of equations in order to solve for $|+\rangle$ and $|-\rangle$ in terms of $|+\rangle_n$ and $|-\rangle_n$. Finding these representations allowed students to insert them into the given state and then combine like terms.

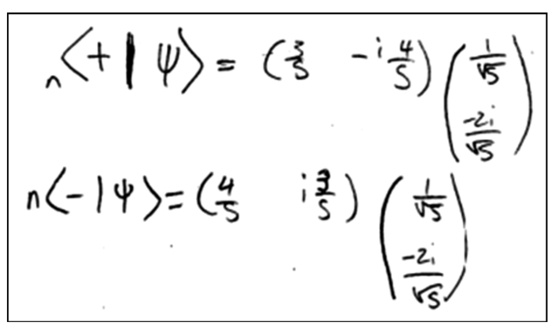

FIG. 2. Student work showing the start of the projection method of finding a change of basis. The student began by writing the inner products and expanding into matrix notation.

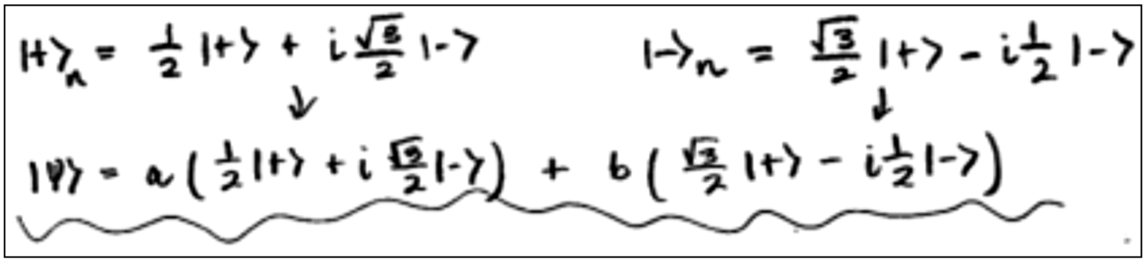

FIG. 3. Student work showing an alternative correct method for finding the representation of a state in a different basis. In this method, the student substitutes the new basis states into the generic representation. Subsequent combining of like terms allows the student to compare the expression to the original state and solve for a and b.

Alternatively, students attempted calculation by substituting the new basis states (e.g., $|+\rangle_n = \frac{1}{\sqrt{3}}|+\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|-\rangle$ and $|-\rangle_n = \frac{\sqrt{2}}{\sqrt{3}}|+\rangle - i\frac{1}{\sqrt{3}}|-\rangle$) directly into the generic representation ($|\psi\rangle = a|+\rangle_n + b|-\rangle_n$) in order to rewrite the state in terms of the basis of S_z (Fig. 3). This step alone leaves the state in terms of unknown variables a and b. Almost all students using this method went on to compare the new coefficients for $|+\rangle$ and $|-\rangle$ with the original state to get a system of equations, which could be solved to find a and b.

Neither the algebraic manipulation nor the substitution methods are generalizable to the context of continuous wave functions. Furthermore, in the context of spins, the amount of work required by these methods meant many students left their answer incomplete. Some got stuck after inserting the definition of the new basis kets into the generic representation. Others identified the system of equations but failed to solve it or attempted to solve it but made errors in the algebra.

The methods discussed so far will all lead to correct answers (when carried out correctly). The remaining "incorrect" methods included some that were only applicable for special cases, such as when there are no complex phase terms. Since Q7 included a complex phase in the relation between the basis states, the use of these methods resulted in incorrect answers for students. Two of the three incorrect methods involved finding the probabilities of measuring $\pm\hbar/2$ along the n direction by calculating $|{}_n\langle\pm|\psi\rangle|^2$ instead of finding the probability amplitudes. Students then either (a) used the results of the probabilities as the values of the coefficients a and b or (b) took the square root of the result to use for the coefficients. Although taking the square root of the probabilities may happen to work in special cases, it does not account for any negative or complex phase associated with the coefficients.

The last incorrect method involved swapping the subscripts on the new basis states (e.g., rewriting $|+\rangle_n = \frac{1}{\sqrt{3}}|+\rangle + \sqrt{\frac{2}{3}}|-\rangle$ as $|+\rangle = \frac{1}{\sqrt{3}}|+\rangle_n + \sqrt{\frac{2}{3}}|-\rangle_n$). Students then substituted these for the $|\pm\rangle$ in the given state and then rearranged the result in terms of $|\pm\rangle_n$. Although this works in some special cases (e.g., for finding $|\pm\rangle$ in the basis of S_x), it does not work in general.

C. Representation of states, coefficients, and inner products (RQ3)

In the preceding sections, we categorized the ways students interpreted the effect of changing basis and the methods they used to do so. Several additional survey questions rounded out the study by probing student thinking about the mathematical structure of a state, and the notation and meaning associated with inner products. We summarize our findings from these questions here.

Q10 asked students to "represent a general state $|\psi\rangle$ in the S_x basis." A correct answer to this question can be broken into two steps. Step one is to recognize that the state $|\psi\rangle$ can be written as a superposition of the S_x basis states,

$$|\psi\rangle = a|+\rangle_x + b|-\rangle_x. \tag{3}$$

Step two is to connect the unknown coefficients (a and b) with inner products between the state and the basis states (${}_x\langle\pm|\psi\rangle$). Some students completed both steps correctly (e.g., Fig. 4), some students only completed step one, and others gave other responses.

Only half of the students who completed step one by writing Eq. (3) also completed step two by providing inner

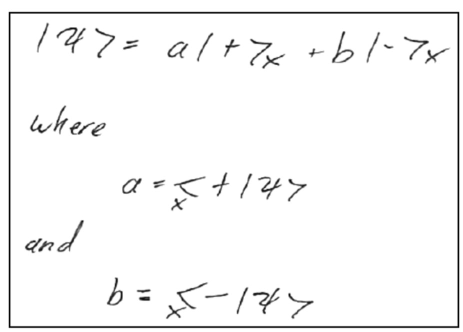

FIG. 4. A student's written response when asked to write the general form of a state in the S_x basis. Here the student uses two unknown variables, a and b, and connects the coefficients to the appropriate inner products.

product expressions for a and b. Students who did not express the variables a and b as inner products either simply noted they were coefficients or invoked the normalization condition. Generating the form of a basis expansion [e.g., Eq. (3)], or at least acknowledging that the current state can be connected to another basis representation, is an important element of changing basis using projection. However, our results show that students' production of a generic basis expansion does not guarantee an explicit connection of coefficients to inner products.

Students also did not connect coefficients to inner products on question Q8, which asked students to discuss the meaning of the coefficients for a state with numerical coefficients (as opposed to the generic basis expansion from Q10). Most students connected the coefficients explicitly to probabilities, with a subset specifying that the coefficients were probability amplitudes that needed to be squared. Other students included that the coefficients showed that the state was normalized. No students spontaneously discussed the mathematical meaning of the coefficients as basis expansion coefficients.

To further probe the connection between the coefficients and projection, students were asked to discuss the meaning of the expression ${}_y\langle + | \psi \rangle$ and the symbols that make it up (Q9, from the same survey as Q10 above). Responses about the expression as a whole included connections to probability or specifically to the coefficient or probability amplitude of the $|+\rangle_y$ state, as shown in the two examples below.

> *This picks out the probability amplitude of measuring spin up in the y direction of the state ψ*
> *I associate the probability of finding $+\hbar/2$ in the y direction with this since I can absolute value and square.*

Other responses to the survey identified the expression as an inner product, or one that represented a projection coefficient, but did not provide any interpretation. For example, one response simply read "${}_y\langle + | \psi \rangle$ is the inner product of $|+\rangle_y$ with $|\psi\rangle$." Students who identified the bra and ket individually labeled $\langle + |_y$ as related to a "spin up in the y direction" or as the "bra in the y basis" and labeled $|\psi\rangle$ as the "initial quantum state" or "general state ket."

Finally, returning to Q10, a small number of incorrect student responses included an application of the S_x operator. For example, writing an expectation value of S_x or writing the S_x operator acting on the state (Fig. 5). Course observations at our institutions and prior literature have shown that some students connect measurement to acting an operator on a state $S_x|\psi\rangle$ (Fig. 5) [1,8]. This response to Q10 could potentially be due to a conflation between measurement and changing basis representation, or simply a belief that the S_x operator should "do something" and a knowledge that it is connected to the basis states $|\pm\rangle_x$.

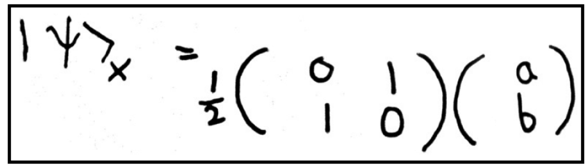

FIG. 5. A student's written response when asked to write the general form of a state in the basis of S_x. Here the student equates a state $|\psi\rangle_x$ to the S_x matrix acting on a generic column vector.

IV. DISCUSSION AND CONCLUSIONS

Basis and change of basis are key concepts in quantum mechanics with which students associate a variety of ideas and procedures. Our research has examined a broad range of data to extract a qualitative picture of student understanding of these ideas. Conceptual questions were asked on free response surveys while procedural questions were given on quizzes and exams. Interviews included both kinds of questions to provide additional insight into student thinking. By categorizing the different methods students use to change basis and the different ideas they associate with basis representation, we establish a research base for the creation of instructional materials.

Investigating students' ideas about the state following a change of basis revealed several categories of reasoning. First is the recognition that changing basis does not affect the physical system or state vector, possibly because the student recognizes that basis is merely a choice of representation, or change of basis is merely a reversible calculation. Second is the notion that changing basis alters the physical system (e.g., changing measurement probabilities), potentially due to confusion between change of basis and measurement. Third is the idea that changing basis can involve changing the label of a ket to reflect the basis it is represented in. These lines of reasoning are not mutually exclusive: we found that students may draw on two or even all three depending on context. That is, students' conceptualizations of the meaning of change of basis are not always robust.

Although when using bra-ket notation we do not add subscripts to a ket's label after changing basis, we can imagine why students might do so. There are instances in QM where subscripts *are* used to indicate choice of basis, namely, when using matrix notation. Moreover, subscripts on kets are typically only used to distinguish related basis states (e.g., $|+\rangle_x$ from $|+\rangle_y$); they are related to basis. Students also learn that, by convention, the z subscript may be omitted from the z basis states $|\pm\rangle$, so it is not unreasonable that they might conclude that any ket labeled without a subscript is implicitly in the z basis. Subscripts are also used for a similar purpose in classical physics contexts such as electricity and magnetism, where subscripts label the components of a vector.

Probing students' thinking about basis expansion coefficients revealed that many students connected the coefficients to measurement probabilities, but that the connection from coefficients to inner products can be more subtle. When asked about the expression, ${}_y\langle +|\psi\rangle$, most students described it as an inner product, but only some connected it to the expansion coefficient for the $|+\rangle_y$ basis state. Meanwhile, when asked for a generic representation of a state in a new basis, few students expressed the components as inner products, even if they did generate the generic representation [e.g., Eq. (3)]. It is possible that a more targeted survey question would have been more likely to prompt students to include inner product expressions in their answer, but previous research does support a disconnect between inner products and coefficients [1], including in a wave function context [9].

Moreover, we found that the labeling of coefficients as inner products is insufficient to determine whether or not a student would use projection to carry out a change of basis. A student might connect ${}_x\langle +|\psi\rangle$ to the basis expansion coefficient a in Eq. (3), which presents $|\psi\rangle$ in the x basis. However, the same student might *not* recognize that the same relationship holds if the state $|\psi\rangle$ was initially expressed in the z basis, possibly because of a belief that a change of basis changes the state physically.

Analysis of quiz and exam data given as part of regular coursework revealed the ways in which students found the representation of a given state in a different basis. An efficient method for finding the new basis expansion coefficients involved calculating inner products between the state and the new basis states. We refer to this method as projection (or "orthogonal projection"). Some students also treated the new eigenstates as a system of equations and attempted to rewrite them, or substituted the new eigenstates into the generic equation, $|\psi\rangle = a|+\rangle_n + b|-\rangle_n$, and compared to the given state. Both of these latter solutions involved extended algebra, which led to various calculation errors or was sometimes abandoned by students. Nonetheless, these algebraic approaches may be appealing for students who are more comfortable working with systems of equations than with projection.

The projection, substitution, and system of equations methods all lead to the correct answer when executed properly. One incorrect method that students used was to calculate the probabilities associated with measurements in the new basis (i.e., of the S_n observable) and use the square roots of these probabilities as the new coefficients. We also found that students articulated the connection between coefficients and measurement probabilities more often than describing coefficients in terms of basis expansion or connecting them with inner products, which may explain why some students employed this incorrect method for changing basis.

Inconsistency in student responses over several questions exemplifies a larger pattern: students may demonstrate the distinct procedural and conceptual elements of basis and change of basis in independent contexts without connecting them together to form a comprehensive understanding. For example, a student might successfully implement a basis change calculation and still argue that the state has changed physically. Another student might articulate the connection between coefficients in a basis expansion and projection in one context, but not recognize projection as a method for changing basis in another. Recognizing projection as a method is supported by both a connection between coefficients and inner product, and an identification of the state being the same regardless of which basis it is represented in. The latter is what allows the inner product to be calculated regardless of the basis used for the calculation. As discussed in the next section, instruction could explicitly encourage students to make these connections towards the goal of forming a more robust understanding.

V. INSTRUCTIONAL IMPLICATIONS AND FUTURE WORK

Our investigation uncovered a range of student thinking on the meaning of basis and change of basis, and identified the variety of methods—varying in efficiency, correctness, and generality—that students employ when changing basis. There are a number of ideas employed by students that proved productive. On the other hand, our investigation also revealed some confusion about notation or physical interpretation.

Our knowledge of students' ideas informs a set of learning goals for instruction targeted at basis and change of basis, especially in the context of spin-½ systems. These learning goals are not a set of observed misconceptions that we believe instruction should correct; in fact, these goals encourage productive lines of reasoning observed in our research. By studying the span of student thinking on these topics, we have identified areas on which instruction should focus. After instruction, students should be able to

Recognize that changing basis does not change the state or any associated measurement probabilities. Some students described basis as a means of representation akin to unit or coordinate systems when explaining their correct answers to questions about change of basis; instruction can promote these productive analogies. Instruction can also clarify the distinction between changing basis and making a measurement, as a conflation between these two concepts arose in student responses. Viewing change of basis as a reversible calculation is another useful idea that arose in student responses, which instruction can emphasize.

Use projection as a method for changing basis. In order to implement projection for this purpose, and in order to fully use basis representation as a tool for tackling quantum mechanical problems, students must connect the inner products they take for computing measurement probabilities with the coefficients in a basis expansion. Moreover, they must recognize that the same state may be represented in any basis. Our results show a variety of challenges related to these aspects of change of basis, indicating that instruction should attend directly to these ideas. Projection is the most general and usually most efficient method of changing basis, but students employed a range of procedural approaches to changing basis. After instruction, students should be able to identify projection as a preferred method for changing basis.

Identify the coefficients in a basis expansion (a) physically as probability amplitudes and (b) mathematically as inner products. Our results suggest that the connections between these ideas are important: some students produced a generic expression for a basis expansion but did not produce the projection inner products for computing the expansion coefficients. Instruction should emphasize these connections, which can also help students recognize that a state encodes all probabilities at once, contrary to the idea that a state is associated with just one pair of spin-up or spin-down probabilities, as arose in some student responses. Instruction should also clarify the meaning of subscripts in Dirac notation—students should recognize that $|+\rangle$ and $|+\rangle_y$ are physically different states.

As part of this project, our research team has developed an instructional tutorial for quantum mechanics targeted at change of basis in the context of spin-½ systems, as mentioned in Sec. II. This tutorial draws on our findings to improve student understanding of change of basis via analogy to two-dimensional Cartesian space [35,36]. In a future project we plan to expand our investigation to student understanding of basis and change of basis in the context of continuous vector spaces (such as the position basis for a quantum mechanical particle). Additional research is also needed to examine how students connect the notion of a basis with its corresponding measurement operator (i.e., observables) as well as student use of Hilbert space representations other than basis, such as the Bloch sphere.

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$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for the spin-up state along the z direction, meaning that matrices expressed in other bases would include a subscript.