

Framing difficulties in quantum mechanics

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Students' difficulties in quantum mechanics may be the result of unproductive framing rather than a fundamental inability to solve the problems or misconceptions about physics content. Using the theoretical lens of epistemological framing, we applied previously developed frames to seek an underlying structure to the long lists of published difficulties that span many topics in quantum mechanics. Mapping descriptions of published difficulties into errors in epistemological framing and resource use, we analyzed descriptions of students' problem solving to find their frames, and compared students' framing to the framing (and frame shifting) required by problem statements. We found three categories of error: mismatches between students' framing and problem statement framing inappropriate or absent shifting between frames and insufficient resource activation within an appropriate frame.

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I. INTRODUCTION

Researchers in student understanding of quantum mechanics have used a “difficulties” framework to understand student reasoning, identifying long lists of difficulties which span many topics in quantum mechanics. The goal of research in quantum difficulties is to determine common, repeatable incorrect patterns of students' reasoning [1–4]. Researchers refer to identified difficulties as universal patterns, since they occur across a wide range of student populations despite varying academic backgrounds [5].

Although the realms of quantum and classical mechanics are different—the classical world is simpler and more intuitive than the quantum world—researchers have long considered the possibility of difficulties in quantum mechanics being analogous to misconceptions in classical mechanics [6]. This similarity is due to both persistent misconceptions or difficulties in students' reasoning [7], and students not having enough preparation with the formalism of quantum mechanics [8].

Research has detailed lists of student difficulties in determining the time dependency of stationary, superposed, and degenerate eigenfunctions [3]; the effect of time dependency of different physical systems on the probability densities [3]; energy measurements of a quantum mechanical system [4]; concepts of the time-dependent Schrödinger equation [4] (TDSE); and the role of Hamiltonian physics

in determining energy [4]. As additional research in student difficulties investigates other topics in QM, we expect that many additional difficulties will be found.

However, we posit that these disparate difficulties can be unified through the lens of epistemological framing [9], errors in frame transitions [10], and errors in the content of a frame (e.g., with the resources framework [11]). This paper presents a secondary analysis of published difficulties in quantum mechanics through the lens of epistemological framing.

Our goal in this paper is to reanalyze students' difficulties in quantum mechanics. We apply a set of frames previously developed by our research team [12,13] to a long list of published difficulties in quantum mechanics in order to find an underlying structure to them. After developing our theoretical lens on our own video-based data, we turned to the published literature on student difficulties in quantum mechanics to seek an underlying structure to students' difficulties.

The choice of different theoretical frameworks is consequential for the kinds of data we collect, the way we analyze them, and the implications of our research for other researchers and for instructors. Difficulties focuses our attention on the ways in which students' wrong answers prevent them from succeeding in problem solving; framing focuses our attention on the pathways that students take as they navigate a problem. This difference of attention means that research using the two frameworks values two different kinds of evidence. Difficulties research values survey responses—possibly multiple choice—to carefully crafted questions that elicit known patterns of wrong answers. Framing research values records of students working on longer problems—possibly in groups—to see how their frames change in the course of the problem. As more topics are researched, the list of difficulties grows enormous;

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however, the list of frames can remain small. This feature suggests that framing, as a more parsimonious theory, can be more helpful to researchers seeking mechanisms for student understanding and instructors seeking to facilitate student reasoning during problem solving.

II. THEORETICAL FRAMEWORK

A. Difficulties

In a misconceptions or difficulties view, students apply an incorrect model of a concept across a wide range of situations independent of the context [6,14]. The core of conceptual understanding occurs by confronting the incorrect conception, and replacing it with a new concept. This unitary view of students' reasoning guides our attention as researchers toward the identification of topics with which students have difficulties at the cost of missing students' epistemological changes [15] because a difficulties view predicts a stable model of thinking that is repeatable, and does not account for sudden or contextual changes in the nature of student reasoning.

A large number of students showing the same wrong answer to the same question implies a widespread difficulty in a certain topic; if the same difficulty presents across multiple questions or over time, it is robust. There have been many difficulties identified in quantum mechanics over the last 20 years across many different topics. For example, there are several subtopic difficulties reported related to the topic of time dependence of the wave function: *incorrect belief that the time evolution of a wave function is always via an overall phase factor of the type $e^{-iEt/\hbar}$; inability to differentiate between $e^{-iHt/\hbar}$ and $e^{-iEt/\hbar}$; and belief that for a time-independent Hamiltonian, the wave function does not depend on time* [2].

Research into student difficulties is often focused on eliciting those difficulties in regular ways (possibly also involving the development of research-based conceptual assessments [16]), developing curricula to ameliorate those difficulties, and iteratively improving the curricula. Common methods for the identification and documentation of difficulties are outlined in Sec. III A; this paper is not concerned with the curriculum development or evaluation aspects of difficulties research.

Because difficulties research seeks to elicit regular, repeatable, wrong answers, much work on identifying difficulties relies on developing carefully phrased questions to elicit difficulties. This development effort, while deeply important to work using this framework, is often omitted or given short shrift in difficulties research publications. That omission is a consequence of the epistemic commitments of the framework: the framework assumes that student responses betray underlying and robust difficulties, and therefore presentations of those difficulties focus on students' ideas as more-or-less independent of the questions posed. However, any student response is an interaction

effect between the students' ideas and the question posed (as well as other contextual features such as question format and student identity). To properly consider which student ideas are elicited in a response, it is necessary to examine the interaction with the question posed.

B. Manifold views

An alternate view to a unitary difficulties view is a manifold "knowledge in pieces" view. In this view of student reasoning, we conceptualize student thinking as being highly context dependent and composed of small, reusable elements of knowledge and reasoning called "pieces." These pieces are not themselves correct or incorrect, but the ways in which students put them together to solve problems may be. By focusing on the pieces of student reasoning and how they fit together, this view of student reasoning foregrounds the seeds of productive student reasoning and not just incorrect answers. Theories in this family include phenomenological primitives [17] (p prims), resources [11], and symbolic forms [18].

A strong thread of research using knowledge in pieces is to investigate students' epistemologies. Epistemological resources [19] connect to conceptual [11] and procedural [20] resources in networks [21] to help students solve problems.

The mechanism that allows control of which subset of resources are activated locally in a given context is epistemological framing [22]. Framing shows the nature of students' knowledge that emerges from a coherent set of fine-grained resources which coherently and locally work together in a situation [23].

Epistemological frames reveal students' [9,22] ways of thinking and expectations. They govern which ideas students link together and utilize to solve problems. Students' epistemological framing is highly context sensitive. Being in the appropriate frame and shifting between frames are determining factors in students' success at problem solving [10,24,25]. Productive problem solving requires both an appropriate frame [24] and appropriate shifting between frames [13]. Careful observation of student behaviors, gaze, and discourse can provide clues for determining students' epistemological frames.

In contrast to difficulties research, which seeks to find the wrong answers that prevent students from being successful at problem solving, framing research tends to focus on the pathways that students take as they shift frames to solve problems. This emphasis on pathways rather than stopping points nudges framing research to more overtly consider the effects of problem statements, instructor interventions, and group mates discourse on students' problem solving.

In our prior work, we developed a set of four interrelated frames around the idea of math in physics (Fig. 1) [12,13]. We applied it to observational data to model students' framing in math and physics during in-class problem

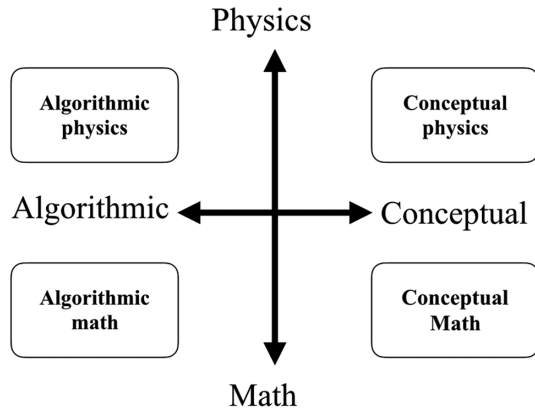


FIG. 1. Math-physics-algorithmic-conceptual theoretical framework. The horizontal axis indicates the algorithmic and conceptual directions. The vertical axis represents the math vs physics directions. Each quadrant is labeled. Figure originally from Ref. [12].

solving in two upper-division courses: quantum mechanics [12] and electromagnetic fields [13,26]. Briefly, our math-in-physics frames capture students' framing in math and physics, expanded through the algorithmic and conceptual space of students' problem solving. The four frames are algorithmic math, conceptual math, algorithmic physics, and conceptual physics. We briefly characterize each frame as follows:

Algorithmic mathematics frame: Students are in an algorithmic mathematics frame if they think about mathematics algorithmically, e.g., when students do pure mathematical manipulations, such as taking a derivative, or checking for sign errors in their procedural problem solving. One of the hallmarks of algorithmic problem solving is that it is fast. Students in this frame take several fast and trivial steps over a long period of time.

Algorithmic physics frame: Students are in an algorithmic physics frame if they think about physics algorithmically, e.g., when laying out physics definitions by using mathematical formalisms. Additionally, students might only use an algorithmic heuristic to find a physical relation without writing down mathematics and only stating their reasoning verbally. This further clarifies the difference between algorithmic mathematics and algorithmic physics frames.

Conceptual mathematics frame: Students are in a conceptual mathematics frame when they provide reasoning, based on the properties of mathematical functions. Instead of running through the math algorithmically, students reason based on the general class of information in mathematics, e.g., when students notice an integral is equal to zero, without explicit calculations, and only due to identification of the mathematical properties of the integrand.

Conceptual physics frame: Students are in a conceptual physics frame when they think about the features of a

physical system, or think conceptually about physical laws, or explain a concept. Students may use graphical representations to better visualize the physical system. By taking a conceptual approach, students create more sense-making opportunities with less need for writing several algebra-based steps.

Using this set of frames, we looked for moments where students' problem solving is impeded because they are in an unproductive frame or when a problem statement requires shifting between frames and students are unable to make that transition. We present a mapping of over thirty quantum mechanics difficulties from the literature [2–4,27] to our math-in-physics frames. This secondary analysis provides a deeper underlying structure to the reported difficulties and demonstrates the broad applicability of these frames to many kinds of quantum mechanics student data.

These categories represent a rotation of the basis set for student difficulties. We have remapped the space of difficulties, which is loosely grouped by physics topics, into the space of frames, seeking an underlying cognitive structure.

C. Interactions between problem statements and responses

Students' reasoning comes as responses to specific questions, and those questions strongly influence their framing. We examined problem statements for what frame(s) they initially promote. For example, consider these two problems:

- Using the time-independent Schrödinger equation (TISE), calculate the changes to E_0 , the ground state, as a given well shrinks from L wide to $L/2$ wide.
- What happens to the energy of the ground state when a finite square well gets narrower?

The first problem encourages students to think mathematically ("calculate") and algorithmically (by hinting at a procedure). The second is more conceptual, specifying neither numbers nor procedures. These two problems, because of their different phrasing, may elicit different student difficulties, or the difficulties they elicit may appear in different proportions because of their phrasing. Depending on which difficulties the researcher focuses on, the researcher might prefer one problem statement over another.

A major element of difficulties research is to carefully craft problem statements so as to best elicit student difficulties. To honor this careful work, our secondary analysis of difficulties considers difficulties as they are paired to problem statements. It is unusual for reports of difficulties research to closely examine how the interaction between question posed and students' ideas creates the student response (even as actual research work to elicit difficulties requires careful attention to question craft, the reports do not focus on it), so our insistence on examining questions paired with responses may seem strange to some difficulties researchers. However, it is required by the

framing framework, which sees students' responses as fundamentally context dependent.

III. METHODOLOGY

A. Difficulty identification

Because this paper reinterprets existing data sets using new theory, we first review where the data come from and how they were originally analyzed using a difficulties framework.

Researchers in difficulty studies have multiple methods for data collection, both quantitative and qualitative. The populations of students in these studies are drawn from advanced undergraduate courses and first-year graduate courses at several different U.S. universities. Students are administered a written test (as part of their course work or for research purposes), usually at the beginning of the semester [2] or after relevant instruction [3,4]. Some students also participate in think-aloud interviews intended to both develop the test and discover common responses to it. Data analysis on the interviews and written responses extracts common difficulties despite the differences in the students' backgrounds. The results of the analysis from interviews and tests are consistent. Several cycles of test development and administration adjust the questions to best elicit student difficulties and ensure validity and reliability.

The first paper covered a broad range of data: administered surveys to upper-level undergraduate and graduate students simultaneously at several university across the U.S., including 100–200 students; administered surveys to students in a typical upper-level class size at state universities; and conducted think-aloud interviews with students for in depth analysis [2].

The original data in two papers [3,4] were collected at the University of Washington (UW), where undergraduate physics students are required to take between one and three quantum mechanics courses. The first course (sophomore level) covers the first five chapters of McIntyre's textbook [28]; the second and third courses (junior level) cover all of Griffiths' textbook [29]. Students were given a written pretest before relevant tutorial instruction, but after lecture instruction. In some of the tasks a variation of the questions was given to the students, but those question variations were not published.

In the fourth paper [27], survey data of first-year graduate students were collected from seven different universities. Researchers also conducted interviews with fifteen students at the University of Pittsburgh.

The research groups at both the University of Pittsburgh and the University of Washington have long histories of difficulties research in quantum mechanics and other physics subjects, and their expertise in developing questions, developing tests and curricula, and identifying difficulties is second to none. We chose their papers for secondary analysis because they represent the best that difficulties research in quantum mechanics has to offer.

B. Mapping students' difficulties to framing

We posit that many student difficulties in quantum mechanics may be due to unproductive framing in problem solving, because students' current frame may not help them with actual problem solving, because students find themselves temporarily unable to shift to a more productive frame, or because they cannot activate productive resources within their current frame.

We mapped descriptions of published difficulties into errors in epistemological framing and resource use.

Framing is context dependent, and a problem statement is one of the very first contexts students interact with which can influence students thought processes and even their future decision making during a problem-solving scenario. A student's frame can be affected by different external frames. For example, in an individual problem-solving setting, the problem statement is one of the influential contexts. By contrast, if we consider a group problem-solving setting, the context can possibly expand from the problem statement to other students' frames in the group. Furthermore, if the problem solving occurs in an interactive classroom setting, where the instructor occasionally intervenes to give a hint or further explain concepts to the groups, then the instructor frame can also affect and interact with students' frames. In this study, the external factor that we are focused on is the problem statement, as the difficulties are elicited mostly in (i) individual problem-solving settings via think-aloud interviews with little or no intervention on the part of the interviewer, or in (ii) written-mode settings such as surveys.

We considered the problem statement as the "jumping off" point for student framing, reasoning that students initial problem framing is probably strongly influenced by framing in the problem statement. From published descriptions of student responses—including their written responses, where available—we identified students response frames and compared them to the frame of the problems to categorize errors.

Because this is a secondary analysis, we take the difficulty as the unit of analysis, not an individual student's response. This is a practical choice on our part, as some authors do not identify the frequency of each difficulty, and we did not have access to all of the descriptions of students' problem solving. The numbers reported for the error rates indicate how many difficulties fall in each category and do not indicate how many students have difficulties in each category. This kind of analysis is strange in the knowledge-in-pieces research tradition, as it severely hampers us from looking at what students do that is correct or productive; difficulties-focused research does not report productive ideas, only incorrect ones.

C. Methodology for secondary analysis

1. Selection criteria

We gathered published works that describe student difficulties in quantum mechanics from Physical Review

Special Topics Physics Education Research, Physical Review Physics Education Research, and the American Journal of Physics. We identified four papers and thirty-six student difficulties in quantum mechanics.

From these papers, we sought difficulties in which the authors had sufficiently described their problem statements (or instructor interactions) for us to determine initial problem framing, excluding those difficulties whose problem statements were omitted, or where variations on a problem statement were alluded to but not presented.

There were times when our research team came to a consensus that there was not enough information to determine difficulties' probable framings. (Difficulties on problems for which the problem statement was not reported in enough detail are excluded from our analysis altogether). Out of thirty-six difficulties, twenty-seven difficulties remained. Nine difficulties did not have enough information for us to figure out what the framing could have been. We excluded these difficulties from further analysis, simply because there was not enough context to determine students' reasoning frames.

2. Coding

We examined students' responses with respect to the features of four frames in our math-in-physics set and coded the student framing present. We started with student responses to problems that matched our qualitative data [12]. Descriptions of student responses—and the resulting difficulties identified by researchers—matched our observational qualitative data well. The detailed analysis of examples from our own observational data is not within the scope of this study, but can be found in Sec. VI A of Ref. [12]. In that study, using the lens of our developed theoretical framework, we identified students' frames and transitions in frames from analyzing in-class group problem-solving activities, as video recorded in a senior-level quantum mechanics class [12].

Emboldened, we extended our coding of student responses to difficulties not present in our qualitative data. As much as possible, we investigated students' statements (or equations, on occasion) to identify the nature of their reasoning and frames used in students responses. For example, a response that is just a piece of an equation, or an equation that is used as a plug-n-chug tool, suggests that the student used an algorithmic frame to generate their response, whereas a response that coordinates energy and probability descriptions suggests that the student argued from a physical principle and is in a conceptual physics frame.

To investigate how problem frame affects student frame, we coded for frames promoted by problem statements. We looked at the keywords and the givens in the problem statement to identify the starting frame(s) suggested by the problem statement. Some problem statements, particularly multipart problems, require students to start in one frame

and shift to another one (for example, see Sec. IV A). In those cases, we coded for which sequence(s) of frames would yield correct answers.

Through intensive discussion among multiple researchers, we coded for which frame(s) a problem statement promoted, and which frame(s) were evident in students' reasoning. For some difficulties, responses or the descriptions of students' reasoning did not contain enough detail to figure out students' framing. Our goals in these discussions were to come to agreement about our inferences of student reasoning. As our discussion reached consensus and our codebook stabilized, two independent raters coded both the rating of the problem statements and the ratings of the students' responses (or the descriptions of students' reasoning), with an agreement rate of > 90% for both kinds of coding.

3. Error type determination

Once problem statements were coded for frames promoted and student responses were coded for frames used, we classified students' difficulties into three categories:

Transition error: When a problem statement requires shifting from one frame to another, and students are unable to make a productive transition. (Fig. 2)

Displacement error: When a problem statement promotes one frame, but students' reasoning places them in another frame. (Fig. 3)

Content error: When students appear to be framing the problem correctly, but are not activating appropriate resources to solve it. (Fig. 4)

This naming scheme relies heavily on the metaphor of framing as a location in a plane. In other words, transition error is *going to* the wrong place, displacement error is *being in* the wrong place, and content error is being in the right place but using the wrong ideas. We have illustrated

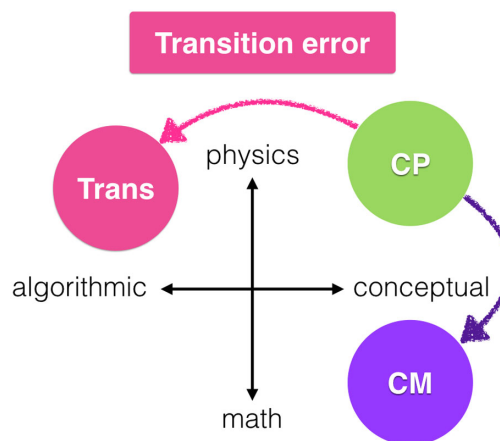


FIG. 2. Transition error: Conceptual physics frame (CP), Conceptual mathematics frame (CM), Transition error (Trans). This student starts in CP and should transition to CM, but has instead transitioned to algorithmic physics.

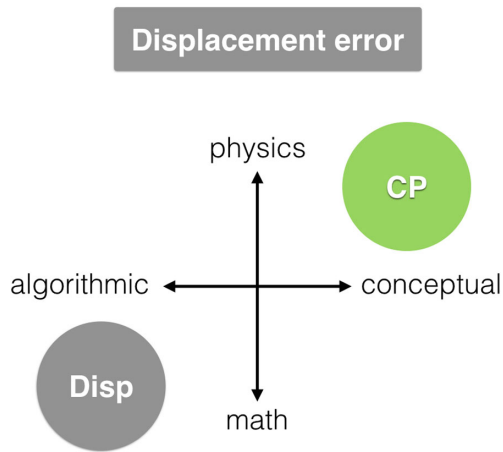


FIG. 3. Displacement error: Displacement error (Disp). This student should be in CP, but is in algorithmic math.

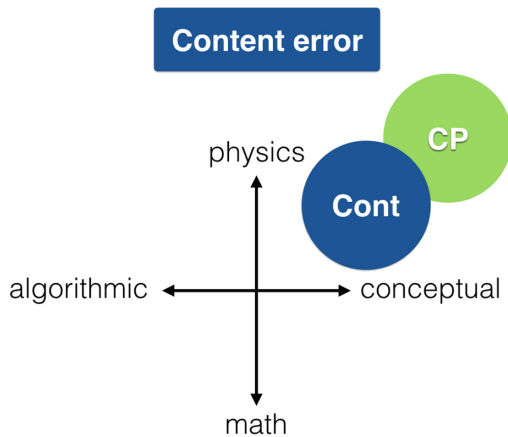


FIG. 4. Content error: Content error (Cont). The student is in CP appropriately, but is using inappropriate or insufficient resources.

each error visually in Figs. 2–4, by considering a hypothetical problem statement that promotes the conceptual physics frame as the starting frame, and requires transition to the conceptual mathematics frame.

Figure 2 demonstrates a transition error for a hypothetical student that does not go to the right place (conceptual mathematics frame). Figure 3 depicts a displacement error when the student is in the wrong place (algorithmic mathematics frame). Figure 4 shows a content error when a student is in the right place (conceptual physics frame), but using wrong ideas.

4. Limitations

Many student responses to these questions are correct, and our secondary analysis of student difficulties cannot capture those responses. This is a fundamental limitation of difficulties-based research: it seeks to describe the ways

students are wrong, not the ways that their responses are reasonable.

Some difficulties may have arisen due to multiple types of error. This is a limitation of secondary analysis—we do not have full reports of student reasoning—and of the survey-style free-response data on which many of the original difficulties are based. For this reason, we classified some difficulties as arising from multiple error types. With sufficiently detailed data, we believe that each difficulty-displaying student response can be classified into a single error type.

Additionally, some surveys were multiple choice. While the original researchers based the choices on common student reasoning, and that reasoning could have showed evidence of student framing, the multiple-choice answers themselves are often insufficiently detailed to determine students' framing. As much as possible, we coded researchers' descriptions of student difficulties, but sometimes we simply did not have enough information.

In the following sections, we show examples of each kind of error, arguing from published literature that difficulties can be categorized by framing error type. Within each type, we tabulate published difficulties. Because some problems require transitions between frames and some do not, we classify difficulties first by the kind of problem they come from and second by the kinds of errors they produce. More specifically, for each framing error type, the error categorization is first provided for problems that require transitions, followed by simpler problems that do not require transitions. The table arrangements also appear in the same order. Error types are labeled by capital letters; the small letters stand for the kind of problem.

IV. TRANSITION ERROR

Transition errors occur when a problem statement expects students to shift between two frames, and the students either do not shift, or shift into an unproductive frame. In this section, we first motivate the idea of transition errors through extended analysis of one example, then tabulate all difficulties that exhibit transition errors.

A. Transition error example

The first example illustrates a transition error that arises from interpreting a graph of wave function vs position [3] (Fig. 5). The problem asked students to explain if the probability of finding the particles within a marked region depends on time or not.

The probability density depends on time if the modulus square of the wave function depends on time. The wave functions are given at time $t = 0$. The authors mention that the problem requires students to think about the time-dependent phase of each term in the superposition wave function. This encourages students to frame the question as thinking about what it means to be in a superposition of

A system consists of a particle in an infinite square well of width a . Two possible wave functions for the system at time $t=0$ s are shown at below. Both wave functions are entirely real at the instant shown.

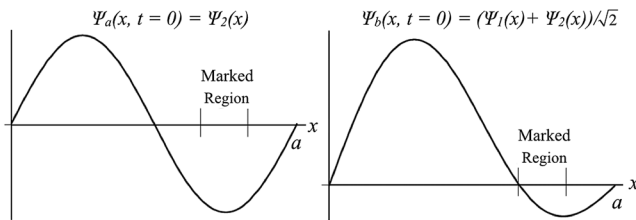


FIG. 5. Does the probability of finding a particle in the marked region depend on time? In this problem, student difficulties indicate transition error and displacement error. Figure originally from Ref. [3].

states, what are the energies of each term in the superposition, and how does the system evolve over time. Framing the problem in this manner suggests conceptual physics as the initial frame.

Students may start this problem by thinking of the time evolution operator, which is determined by the Hamiltonian of the system. After students recognize the correct time-phase factors, they need to coordinate mathematical representations to show how the phase factors determine the time dependence of the probability density. In the context of quantum mechanics with more novel mathematical formalism, students can use mathematics in epistemologically different ways to map their physical understanding to a mathematical representation.

In algorithmic problem solving, the mathematical process is broken into many smaller algebraic steps and taken over a longer period of time. Whereas, in conceptual problem solving, a mathematical justification can account for all, or part of the algorithmic steps. Sometimes, in a problem, students need to make a transition between algorithmic and conceptual mathematics to fully coordinate all the features of a physical system into a mathematical representation. Our set of math-in-physics frames has two possible transitions from conceptual physics into mathematics frames:

Algorithmic math: In algorithmic math, a student would manipulate the modulus square of the superposed wave function explicitly and algorithmically, finding that the time dependence of the pure terms falls out, and the time dependence of the cross terms persists.

Conceptual math: In conceptual math, a student initially could use a conceptual mathematical shortcut: the exponential term multiplied by its complex conjugate sets the product equal to one. However, this solution leads to neglecting the role of the cross terms.

Because the problem starts in a conceptual physics frame, it may be easier or more appealing to transition first into conceptual math rather than algorithmic math; in our observational data, this is the transition we observed [12] from the student (Eric). Conceptual mathematics was Eric's initial framing on a comparable problem. Using conceptual mathematics reasoning just as Eric did can only explain that the pure terms lose their time dependence. Eric then made a transition to algorithmic math, allowing him to mathematically read out that in computing the modulus square of the probability density there are cross terms and for those terms the time dependence persists.

Emigh *et al.* [3] describe student reasoning in response to the same task:

Student While it is true that the general wave function is of the form

$$\sqrt{\frac{1}{2}}\phi_1 e^{-iE_1 t/\hbar} + \sqrt{\frac{1}{2}}\phi_2 e^{-iE_2 t/\hbar},$$

again the function we are interested in is $P(x) = |\phi|^2$, which loses its time dependence.

The first part of this statement shows that the student has correctly used the ideas of the problem statement frame to note the different energies of each energy eigenstate in the wave function. The second part of the statement suggests that the student coordinates the physics and conceptual math to recall that the probability density is the modulus square of the wave function. However, the student does not do any further algorithmic calculations, instead arguing that the probability “loses the time dependence.” This is congruent with the conceptual math reasoning above. Eric's initial frame is similar to the student in the Emigh *et al.* study in that he only accounts for part of the solution. This leads to an error without Eric being aware of it. According to both the difficulties framework and knowledge in pieces, at this moment an error has occurred in the problem-solving procedure. However, by using a framing lens we can extend our analysis further to a level that incorporates the role of other existing external factors in a real problem-solving situation such as a group problem-solving classroom setting. It is only after the instructor gives the correct final answer to the class that Eric becomes aware of his error, which had momentarily made him “get stuck.” Eric is able to “get unstuck” by making a transition to the algorithmic frame and paying attention to the features of the problem that are easy to see in the new frame.

About 10%–20% of the students ($N = 416$) from Emigh *et al.*, applied the same kind of reasoning to argue that “time drops out” or “the probability is squared and the time won't matter” [3]. These arguments indicate that students do not feel a need to actually do math, because their conceptual math frame has solved and justified their time-dependent answer. While Emigh *et al.* interpreted these

responses as a difficulty—students’ “tendency to treat all wave functions as having a single phase”—we interpret it as an example of error in frame transition (Table VII).

B. Difficulties that exhibit transition errors

We found two published difficulties for which students exhibit only transition errors (Table I).

In the first difficulty *mathematical representations of nonstationary state wave functions* (T1) in Table I, the students were asked if different wave functions $A \sin^3(\pi x/a)$, $A[\sqrt{\frac{2}{5}}\sin(\pi x/a) + \sqrt{\frac{3}{5}}\sin(2\pi x/a)]$, and $Ae^{-[(x-a/2)/a]^2}$ can be proper candidates for an infinite square well of width a with boundaries at $x = 0$ and $x = a$. This problem requires students to start from a conceptual physics frame to extract the boundary condition information and read out that the potential is infinite at the boundary conditions and the wave function has to go to zero to satisfy the continuity of the wave function at the boundaries. This problem requires transition as students may need to plug in the values for the boundary conditions, conduct some algorithmic steps, and figure out if the solution satisfies the boundaries of the problem. One type of incorrect response suggests that many of the students reasoned that two conditions must be satisfied. First, wave functions should be smooth, single valued, and satisfy the boundary conditions of the physical system. Second, it should be possible to write the wave function as a superposition of stationary states, or the wave function should satisfy the time-independent Schrödinger equation (TISE) [27]. A typical response of the students looks like

Student $A \sin^3(\pi x/a)$ satisfies b.c. but does not satisfy Schrödinger equation that is, it cannot represent a particle wave. The second one is a solution to S.E. it is a particle wave. The third does not satisfy b.c. The author mentions that students do not note that even the superposition wave function $\{A[\sqrt{\frac{2}{5}}\sin(\pi x/a) + \sqrt{\frac{3}{5}}\sin(2\pi x/a)]\}$ does not satisfy the TISE. We think that

TABLE I. Difficulties that exhibit transition error only. These difficulties are labeled “T” for transition errors. The ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.

No.	Name	Ref.
T1	Mathematical representations of nonstationary state wave function; this difficulty emerges when determining possible wave functions for a system.	[2]
T2	Belief that the wave function is time independent; this difficulty emerges when interpreting the time phases that arise from the time-dependent Schrödinger equation.	[3]

this student is in the frame of the problem since they match the boundary conditions with each wave function to see if they satisfy the boundaries of the physical system. However, we do not have sufficient information to conclude how this student is working on this problem toward a solution. We do not know if the student is taking some algebraic steps to match the boundary condition, or if they are only reasoning verbally. We think that, by making a transition to the conceptual mathematics frame, this student can activate ideas regarding expansion of the function $\sin^3(\pi x/a)$ in terms of the energy eigenstates.

A second difficulty is that students believe the wave function is time independent because it satisfies the TISE. Students who generate these responses provide a mathematical basis for their answer. The author mentions that students think that the superposition wave function satisfies the TISE.

Student [both wave functions] satisfy the time-independent Schrödinger equation so Ψ_1 and Ψ_2 do not have time dependence.

Although the solution to the TISE does not depend on time, the TISE solution is incomplete because this problem is time dependent. This difficulty is categorized as a transition error, as students need to shift to a conceptual frame (either conceptual physics or conceptual mathematics) to complete the problem. Shifting to conceptual physics may lead them to think in terms of the independent eigenfunctions of space and time; shifting to conceptual math may lead them to think about missing orthogonal functions.

This problem may also exhibit displacement errors. To better allow the reader to compare displacement error with the transition errors described above, we present our analysis of two possible displacement errors emerging from this problem statement in the Sec. VA.

V. DISPLACEMENT ERROR

Displacement errors arise when students are meant to be in one frame, but instead operate in another. In this section, we first describe a displacement error that a student may exhibit when attempting the extended example problem in the prior section. Then we tabulate difficulties which exhibit displacement errors for problems with and without transitions.

A. Displacement error example

For the same task as shown in Fig. 5 (the probability of finding a particle in the marked region), we present two possible displacement errors.

Some students considered that just the linear combination $A[\sqrt{\frac{2}{5}}\sin(\pi x/a) + \sqrt{\frac{3}{5}}\sin(2\pi x/a)]$ or a pure sinusoidal wave function are allowed; but the $A \sin^3(\pi x/a)$ is not allowed and “only simple sines or cosines are allowed” as proper wave functions. Some other students mentioned that for a particle in a box, only the wave functions in the form

of $A \sin(n\pi x/a)$ are allowed and the $Ae^{-[(x-a/2)/a]^2}$ wave function is only allowed for a simple harmonic oscillator.

We consider that students with this type of response are not in the frame of the problem as they are not thinking about the characteristics of the boundary conditions. Instead, they just recall what the solutions for physical systems of a particle in a box or a harmonic oscillator look like. They assert that they know how the answer should look, having worked out the problem before. While the students might not necessarily attempt the algorithmic processes to arrive at this conclusion during the interview, they are relying on the fact that they have done these calculations before and can recall the conclusion. This tendency stems from having previously worked through the problem of a particle in a box, which fits into the algorithmic physics frame. We categorized students with this type of response as having made a displacement error.

Other difficulties displacement errors are possible. For example, this student writes the time dependence of the wave function instead of finding the probability, which is an incomplete answer:

$$\sqrt{\frac{1}{2}}e^{-iEt/\hbar}(\psi_1 + \psi_2). \quad (1)$$

This short answer segment suggests that the student is in an algorithmic frame; there is no other information about student reasoning (such as narration or a graph provided by the student). This student has not picked up the conceptual framing intended by the problem statement. Starting from the conceptual physics frame could help the student to conceptually think about superposition of wave functions and the different energy terms instead of a single time-dependent phase. The authors of the original paper [3] do not provide the percentage of students that answered in this way. However, they mention that the *tendency to consider just a single phase wave function* for a superposition state is very common. About 25% of their students ($N = 223$) on a final exam showed the same difficulty on a version of the same task. Students were given the time-dependent wave function, and were asked about the “time dependence

of the probability of a particular outcome of a position measurement.”

This difficulty is classified as a displacement error: the student is in the wrong frame initially, and does not transition to a more productive frame.

B. Difficulties which exhibit displacement errors

In problems that require frame shifting (like the problem in Fig. 5), we found five difficulties which exhibit only displacement error (Table II).

One difficulty from Emigh’s study [3] (Problem in Fig. 5), *confusion between the time dependence of wave functions and probability density* (Ds2) in Table II, shows that students correspond the time dependence of one quantity to another such that both physics quantities obtain the same time evolution. Between 5% and 20% of the students in their data ($N = 416$) have provided this type of reasoning.

Student The wave function is time independent. Thus, its probability density does not change. If the wave function is time dependent, then [its] probability density would change in time too.

This student does not calculate the modulus square either via an algorithmic mathematical frame, a conceptual mathematical frame, or both. Additionally, the student does not think conceptually about the different energy eigenvalues of each term in the superposition. This student is not in the frame of the problem, which is a conceptual physics frame. Instead, the student is in the algorithmic physics frame. This student is using a simple algorithmic heuristic: If this thing (wave function) is not changing, the other thing (probability density) is also not changing; if this (wave function) were changing then its (probability density) would also be changing. The student is applying this algorithmic piece of reasoning to the physical quantities of (wave function) and (probability density), without considering the physics of those quantities deeply and conceptually. The student is applying algorithmic reasoning to make a quick conclusion about the relation between two physical quantities. One of the hallmarks of algorithmic thinking is that it is fast and nonreflective. It is possible to

TABLE II. Difficulties that exhibit displacement error in problems that require frame shifting. These difficulties are labeled “D” for displacement errors and “s” because their problems require shifting. The ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.

No.	Name	Ref.
Ds1	Incorrect belief that $H\psi = E\psi$ holds for any possible wave function ψ	[2]
Ds2	Confusion between the time dependence of wave functions and probability density	[3]
Ds3	Belief that for a time-independent Hamiltonian, the wave function does not depend on time	[2]
Ds4	Tendency to associate the time dependence of energy measurements with properties of stationary states	[4]
Ds5	Tendency to treat every superposition as having multiple distinct phases; this difficulty emerges when interpreting the time phases that arise from the time-dependent Schrödinger equation.	[3]

TABLE III. Difficulties that exhibit only displacement error in simpler problems which do not require frame shifting. These difficulties are labeled D for displacement errors and “n” because their problems require no shifting. The ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.

No.	Name	Ref.
Dn1	Difficulties related to outside knowledge in student understanding of energy measurements	[4]

totally answer this problem conceptually and be wrong. But, we do not have further information such as the student’s tone or before and after this argument to investigate this possibility.

The piece of algorithmic physics reasoning given appears to indicate that the student is recalling, but it may not be true that all errors of this type arise from recollection.

For the stationary state wave function on the same problem (Fig. 5), about 5% of the students think that the stationary state wave function is time independent.

Student This is a stationary state so the wave function will not evolve with time.

This piece of data suggests that this student is not initially in the conceptual frame of the problem. The student might have previously derived that some properties, such as probability density, are time independent for a stationary state. However, they do not accurately remember the conclusion. The incorrect notion that the wave function, rather than the probability density, is time independent further implies that the student is in an algorithmic physics frame and is trying to recall a fact about stationary states.

In simpler problems that do not require frame shifting, we find one difficulty which exhibits displacement error (Table III).

VI. CONTENT ERROR

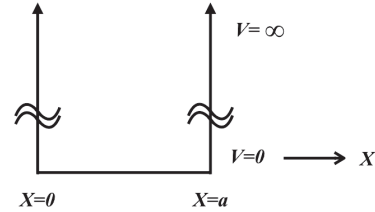
A third kind of error occurs when students are in the appropriate frame intended by the problem statement, but have not activated enough of (or the correct) resources to complete the problem. We term this kind of error “content error.” In this section, we illustrate content error with one example difficulty and then tabulate content errors.

A. Content error example

To illustrate content error, we draw an example from Singh’s study (Fig. 6) [27]. This example comes from the interview data of first-year graduate students. For this problem, students are given the problem in Fig. 6, which asks them to calculate the expectation value of the superposition of the ground state and the first excited stationary state of the system.

Although 67% of the students were able to answer part 2 correctly, only 39% were able to answer part 3 correctly,

The wave function of an electron in a one dimensional infinite square well of width a at time $t=0$ is given by $\Psi(x,0)=\sqrt{2/7} \phi_1(x) + \sqrt{5/7} \phi_2(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are the ground state and first excited stationary state of the system. ($\phi_n(x) = \sqrt{2/a} \sin(n\pi x/a)$, $E_n = n^2\pi^2\hbar^2/(2ma^2)$ where $n=1,2,3,\dots$)



1. Write down the time dependent wave function.
2. You measure the energy of an electron at time $t = 0$. Write down the possible values of the energy and the probability of measuring each.
3. Calculate the expectation value of the energy in the state (x, t) above.

FIG. 6. Calculate the expectation value of the energy in the state $\psi(x, t)$. On this problem, student difficulties display content error. Figure originally from [27].

and many were not able to use the information to apply in part 3. Instead, students explicitly calculated the integrals of the expectation value. We analyze their description of a student’s response to part 3. The problem statement starts students in an algorithmic frame (directing them to “calculate”). The frame is algorithmic physics, rather than algorithmic math, because the students must first start by recalling some facts and equations about expectation values and wave functions.

The student writes down the TISE as $\hat{H}\phi_n = E_n$ without ϕ_n on the right-hand side of the equation, but correctly writes ϕ_n as the sum of ϕ_1 and ϕ_2 on the left-hand side. This is an appropriate initial framing to this problem, but it is missing a key piece of content. This mistake results in an incorrect answer in terms of ϕ_1 and ϕ_2 . At this point, the student is not confused that their answer does not make sense because they are unaware of their error. The interviewer points to the part of the solution with the missing element, but the student is still unable to find their mistake. Finally, the interviewer explicitly gives the right TISE, $\hat{H}\phi_n = E_n\phi_n$ to the student. The student can then review the math conceptually in their solution by applying the orthonormality properties of the eigenstates, simplifying the integration, and getting a correct answer. It seems that all they need is a correct TISE, and they are able to frame the problem appropriately and continue to a successful solution. We do not consider this example as a case of a simple typographic error on the student’s part, because the instructor notifies the student about their error several

TABLE IV. Difficulties that exhibit only content error in problems which require shifting frames. These difficulties are labeled “C” for content errors; and s because their problems require shifting. The ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.

No.	Name	Ref.
Cs1	Tendency to treat wave functions for bound systems as traveling waves	[3]
Cs2	Tendency to treat time-dependent phase factors as decaying exponentials	[3]
Cs3	Difficulties related to outside knowledge in student understanding of energy measurements	[4]

times, but the student believes that their written TISE is fine. After finding the correct answer, the student is able to reflect on their answer and even conceptually reason about the expectation value.

The interviewer continues by asking the student if they can think of the response to part 3 in terms of the response of part 2. The student responds “Oh yes ... I never thought of it this way... I can just multiply the probability of measuring a particular energy with that energy and add them up to get the expectation value because expectation value is the average value.” The interviewer’s intervention to explicitly connect parts 2 and 3 prompts the student to think more physically in terms of the underlying concept of expectation value. They can relate the concept of expectation value to the parameter of the physical system such as energy eigenvalues, and probability of measuring each. This difficulty is categorized as a content error because the student is in the frame intended by the problem but is not able to find the correct answer until the interviewer provides more content. In this example, the student in a brief but important interaction with the interviewer receives feedback and is able to continue from there to incorporate that given piece of information to revise the solution. Some students may also need help with the incorporation of the

given information into their existing knowledge network; to add, remove, alter, or refine an idea(s) in their mind [19,21]. This view can help instructors to better identify the moments during a problem-solving situation to provide a piece of content to students or to help students to activate a piece of content and account for variation in students’ reasoning.

B. Difficulties which exhibit content error

Among problems that require transitions, three difficulties are classified as content error only (Table IV).

For Cs2, *tendency to treat time-dependent phase factors as decaying exponentials*, the authors [3] provided a student’s reasoning:

Student Since the wave equation will gain a $e^{-E_2t/\hbar}$ term to represent its evolution as time goes on, the probability of finding the particle in the marked area will decrease [...] since the square of its wave equation will decrease as well.

This student is in the same frame as that promoted by the problem statement (Fig. 5), which is conceptual physics, as discussed in Sec. IV A. The only difference is that the student’s response is with regard to the stationary wave function. The student has determined the energy of the stationary state E_2 and knows how to perform the appropriate calculations to find the probability density. However, their exponential term has a (negative) real power instead of an imaginary one. We interpret this as a content error: the student has activated incorrect resources and reasoned from them.

In simpler problems that do not require transition, we found an additional nine difficulties (Table V), that exhibit content error. The difficulty, *incorrect belief that the time evolution of a wave function is always via an overall phase factor of the type $e^{-iEt/\hbar}$* (Cn1) in Table V shows that students are performing a content error. The problem (Fig. 6) asks students to find the time-dependent wave function $\Psi(x, t)$ for a system in an initial state of

TABLE V. Difficulties that exhibit content error in simpler problems. These difficulties are labeled C for content errors; and n because their problems require no shifting. The ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.

No.	Name	Ref.
Cn1	Incorrect belief that the time evolution of a wave function is always via an overall phase factor of the type $e^{-iEt/\hbar}$	[2]
Cn2	Determining the outcomes of a subsequent energy measurement	[4]
Cn3	Difficulties with the possible outcomes of a measurement	[2]
Cn4	Failure to recognize that the time evolution of an isolated system is determined by the Schrödinger equation: “Decay reasoning”	[4]
Cn5	Belief that the wave function will return to its initial state	[3]
Cn6	Failure to recognize that the time evolution of an isolated system is determined by the Schrödinger equation: “Diffusion reasoning”	[4]
Cn7	Difficulties with time development of the wave function after measurement of an observable	[2]
Cn8	Difficulties in distinguishing between three-dimensional space and Hilbert space	[27]
Cn9	Inability to differentiate between $e^{-iHt/\hbar}$ and $e^{-iEt/\hbar}$	[2]

superpositions of the ground state and first excited states, $\Psi(x, t = 0) = \sqrt{\frac{2}{7}}\phi_1(x) + \sqrt{\frac{5}{7}}\phi_2(x)$. The equations of the eigenfunctions and the eigenvalues are given in the problem statement. The frame of this problem is algorithmic physics. The problem statement asks the student to write down the wave function as opposed to figure out the wave function. The frame of the problem requires the student to recall the time phase factor and follow simple algorithmic steps to assign the readout energy eigenvalues from the problem statement into the time phase factor for each term and write the time-dependent wave function in terms of ϕ_1 and ϕ_2 . About one-third of the students ($N = 202$) in this study wrote

$$\text{Student } \Psi(x, t) = \psi(x, 0)e^{-iEt/\hbar}.$$

The frame of the student is algorithmic physics, which is the frame of the problem. The student has written down a time phase factor, but still needs to include more content and take more steps. This student does not read the information regarding the energy eigenvalues from the given equations in the problem statement, and does not attempt to write the answer in terms of ϕ_1 and ϕ_2 . Therefore, this difficulty is categorized as a content error as the student is not reading enough content from the problem statement.

The difficulty, *inability to differentiate between $e^{-iHt/\hbar}$ and $e^{-iEt/\hbar}$* (Cn9, Table V), occurs when students misapply the energy eigenstate instead of the Hamiltonian operator in the time evolution operator [2]. The problem asks students to find the time-dependent wave function $\Psi(x, t)$ for a system in an initial state of superpositions of the ground state and first excited states, $\Psi(x, t = 0) = \sqrt{\frac{2}{7}}\phi_1(x) + \sqrt{\frac{5}{7}}\phi_2(x)$. The equations of the eigenfunctions and the eigenvalues are given in the problem statement. We frame this problem as algorithmic physics as it requires the student to recall the time phase factor and follow simple algorithmic steps: assign the readout energy eigenvalues from the problem statement into the time phase factor for each term and write the time-dependent wave function in terms of ϕ_1 and ϕ_2 . The authors mention that students write an intermediate state for $\Psi(x, t)$:

$$\begin{aligned} \Psi(x, t) = \Psi(x, t = 0)e^{-iEt/\hbar} &= \sqrt{\frac{2}{7}}\phi_1(x)e^{-iE_1t/\hbar} \\ &+ \sqrt{\frac{5}{7}}\phi_2(x)e^{-iE_2t/\hbar}. \end{aligned}$$

Since the student proceeded from an intermediate state, we presume that the student does not attempt to rederive the relationship between a space portion and a time portion of a wave function. This student is in the frame of the problem by reading out the energy eigenvalues E_1 and E_2 and assigning each energy into the time phase factors. However, the

intermediate step does not convey any algorithmic process or physical meaning and cannot lead to the final step.

This problem is classified as a content error since the student uses the wrong idea that the symbol H as the Hamiltonian operator and the symbol E as the energy eigenvalue of the system are the same. This is evidenced by further probing by the interviewer revealed the difficulty differentiating between the Hamiltonian operator and its eigenvalue.

The difficulty in *distinguishing between three-dimensional space and Hilbert space* (Cn8) in Table V indicates that students have difficulty differentiating vectors in real 3D space from vectors in Hilbert space, such that students may not be able to distinguish between the 3D space describing the gradient of the magnetic field in the z direction, and the 2D Hilbert space for describing a spin- $\frac{1}{2}$ particle. The question is about the Stern-Gerlach experiment: “a beam of electrons propagates along the y direction into the page, and are in the spin state of $(|\uparrow_z\rangle + |\downarrow_z\rangle)/\sqrt{2}$. The beam is sent through a Stern-Gerlach apparatus (SGA) with a vertical magnetic field gradient. Sketch the electron cloud pattern that you expect to see on a distant phosphor screen in the x - z plane. Explain your reasoning.” Because of the magnetic field gradient in the z direction, the beam of electrons will experience a force and become deviated. However, electrons due to having an intrinsic angular momentum, which is their spins, split only into two directions along the z axis and form two spots on the screen.

The frame of this question is conceptual physics, because it encourages students to think about “what is going on” in the physical apparatus. The problem statement requires different readouts about the direction of the magnetic field gradient, or the direction of the electron beam. Students are asked to use graphical representation and justify their reasoning. Only 41% of the students ($N = 202$) answered correctly, and the rest of the students predicted that there will be only a single spot on the screen. A typical response of a student looks like

Student All of the electrons that come out of the SGA will be spin down with expectation value $-\hbar/2$ because the field gradient is in $-z$ direction.

This student is thinking conceptually by reading out information about the direction of the magnetic field from the problem statement (“ $-z$ direction”) and connecting that to the idea of spin- $\frac{1}{2}$ and thinking that this measurement has only one outcome and thus the expectation value is $-\hbar/2$. However, the student needs to more carefully read out from the problem statement that the state of the system before the measurement is in the combination of two states of spin-up $|\uparrow_z\rangle$ and spin-down $|\downarrow_z\rangle$, and the state of the system is not just prepared in one state of down $|\downarrow_z\rangle$ to stay unchanged after the measurement. This problem is categorized as a content error since the student’s reasoning is missing some content that blocks a correct answer. However, the description of the student’s reasoning is not enough to identify

which exact piece of content is missing. It could be helpful to the students to think about what it means for a beam of electrons to be in the combination of states spin-up and spin-down before passing through an SGA, or to think about what it means for an electron to have an intrinsic angular momentum.

The difficulty *determining the outcomes of a subsequent energy measurement* (Cn2) is mostly limited to a content error. Students are able to use the ideas of the problem statement and operate in the frame of the question, but are activating the wrong resources to productively and correctly solve the problem.

For example, a student is asked about the outcomes of an energy measurement after a previous measurement on the system of a particle in an infinite square well in an initial state $\Psi(x, 0) = 0.6\Psi_1(x, 0) + 0.8i\Psi_2(x, 0)$. Part A of the question asks “What value or values would a measurement of the energy yield?” Part B of the question asks what would be the result of a second energy measurement after time t_2 . In response to part B [4]

Student The particle is described by a wave function with elements in both eigenstates. Although a measurement of energy collapses it to one, the possibility of the other still exists, so a second measurement could get the other E .

The frame of this question is conceptual physics, which requires students to think about the idea that repeating an energy measurement does not change the state of the system. Repeating an energy measurement only yields the same result as the first measurement, since the system is already collapsed to one of the energy eigenstates and is isolated from its surroundings.

This student has activated several ideas about energy measurement on a physical system in a superposition wave function. In the first part of the response, the student acknowledges that a particle “is described by a wave function with elements in both eigenstates,” and also “a measurement of energy collapses it to one.” These ideas are both correct. With this being the case, it may be difficult to understand why the student arrives at the wrong answer despite seeming to have correct ideas about the system.

This student acknowledges the fact that when a system collapses it has only one energy, but they also activate the idea that the probability of other energy, “ E ,” “still exists” and associates this possibility with the second measurement on the system.

The second part of this statement can be considered as correct if no measurement has been actually performed on the system (similar to the context of the problem in part A). For a system in a superposition state, if the system is prepared n times in the exact same way and each time a measurement is made on the system, one can find the number of times that the energy measurement yields E_1 , and the number of times that the energy measurement yields E_2 . However, as soon as an energy measurement is

made, the system collapses into one of the energy eigenstates and repeating the energy measurement yields the same result as the first measurement.

This student somehow decides that their knowledge of state collapse is not applicable here and a measurement possibly yields multiple possible energy values. This student uses the word “although” to put these two ideas in opposition to each other. This student needs to activate more content from the information that the problem statement provides about the system before and after an energy measurement as well as repeated energy measurements to the system.

In the study by Singh *et al.* [2,27] they showed that students have difficulty with the *time development of the wave function after measurement of an observable* (Cn7). Students were asked about the wave function a long time after measurement of energy E_2 for an electron in an infinite square well. Some of the students stated similar responses that “If you are talking about what happens at the instant you measure the energy, the wave function will be ϕ_2 , but if you wait long enough it will go back to the state before the measurement.” The first part of the response suggests that the student is able to correctly relate the measured energy eigenvalue to the associated eigenstate of the system ϕ_2 by activating the resource of an instant measurement. However, the student does not further investigate the idea that long after the measurement only a phase will be added to the eigenstate, which does not change the state of the system to any other combination of eigenstates; the system will not “go back to the state before the measurement.”

VII. DIFFICULTIES WHERE MORE THAN ONE ERROR TYPE IS POSSIBLE

For some difficulties, multiple error types are possible. Additional details of student reasoning could resolve these ambiguities, but these details are either not gathered (survey data) or not available to us (interview data) for secondary analysis.

The first difficulty with the *confusion between the time dependence of probabilities of energy measurements and other quantities* (DCs1) in Table VI indicates a displacement or content error. The task asks about the time dependence aspect of energy probability measurements on a particle in a quantum mechanics harmonic oscillator system in the initial state, $\psi = (i/\sqrt{3})\psi_0 - (\sqrt{2}/\sqrt{3})\psi_1$. A displacement error occurs when the student associates the time dependence aspect of the probabilities of energy measurements to the time-independent properties of the “probability density” or the “wave function.” In this type of answer the student usually recalls some properties of the physical quantities without any justification, since they think their reasoning is correct the way it is first recalled.

Student It [the energy probability] depends on the probability density. If it’s time independent then no, if time dependent then yes.

TABLE VI. Difficulties that exhibit both displacement and content error in problems which require transitions. These difficulties are labeled “DC” for displacement and content errors and s because their problems require shifting. The ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.

No.	Name	Ref.
DCs1	Confusion between the time dependence of probabilities of energy measurements and other quantities	[3]
DCs2	Belief that the wave function will spread out over time	[3]

A content error occurs when the student is able to determine some of the features of the physical system by being in the frame intended by the problem; however, they are not considering all aspects of the problem context. As in the example mentioned in the study by Emigh *et al.* [3], the student begins by stating, that “A linear combo of stationary states is not stationary.” This student is mindful that a superposition of eigenstates is not a stationary state. The student is also able to differentiate between the energy levels of each eigenstate and give a description of the system by stating “The system will oscillate around E_0 and E_1 .” This student activates the idea that the state of the system is in the combination of two states. This piece of reasoning leads to the activation of another piece of the idea that both energies can be available; in the student’s words, the system can “oscillate around” two energies. This student should activate further resources in accordance to the problem statement that asks “Are there times when the probability of measuring E_1 is zero and the probability of measuring E_0 is one?”

Table VII indicates, that students’ difficulties in *tendency to treat all wave functions as having a single phase* (DTs1) can be mapped as a displacement error or a transition error. A displacement error indicates that the student has not attended to the frame of the question to blend the information effectively with the corresponding concepts in the task. This is similar to the described example in Sec. V from the Emigh *et al.* study [3]. The other possible

TABLE VII. Difficulties that exhibit both displacement and transition errors in problems which require frame shifting. These difficulties are labeled “DT” for displacement and transition errors and s because their problems require shifting. The ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.

No.	Name	Ref.
DTs1	Tendency to treat all wave functions as having a single phase	[3]
DTs2	Finding the probability of an energy measurement from the wave function	[3]

TABLE VIII. Difficulty that exhibits both content and transition errors in a problem that requires frame shifting. This difficulty is labeled “CT” for content and transition errors and s because its problem requires shifting.

No.	Name	Ref.
CTs1	Tendency to misinterpret the real and imaginary components of the wave function	[3]

error occurs when the student frames the task properly and is able to coordinate between frames; however, they fail to productively transition between frames to remove all the barriers (Sec. IV A).

The difficulty *tendency to misinterpret the real and imaginary components of the wave function* (CTs1 in Table VIII) shows that some students establish a conceptual discussion in a math frame to relate the real and imaginary parts of the wave function in the complex plane. However, viewing the problem as purely conceptual (e.g., in the conceptual math frame) prevents students from noting other related ideas in the problem statement. Shifting to the algorithmic math frame may help the student to recall other related facts to successfully solve the problem. Alternately, moving to the conceptual physics frame may help the student to better map the activated mathematical ideas to the problem situation.

Table IX shows that students can exhibit different types of errors in *interpreting the meaning of the expectation value* (DCTs1). As discussed in the Sec. VI, the task required the student to start by recalling physics relations to calculate the expectation value. The student makes a content error when they activate an unstructured piece of their knowledge related to the TISE. This error is corrected by the interviewer [27]. The displacement error occurs when the student is outside of the problem statement frame (algorithmic physics), and writes down just a mathematical expression $\langle \Psi | \hat{H} | \Psi \rangle$, which lacks blended information from the physical space.

An example of transition error is when the student is able to blend the physical meaning of the probability of the energy values with the related coefficients and then translates the problem into procedural steps. However, the student might leave an extra coefficient ($\frac{1}{2}$) in the final answer, $(\frac{2}{7}E_1 + \frac{5}{7}E_2)/2$. This error can be removed by reviewing the solution and thinking purely conceptually about the quantity of the expectation value [27]. For this

TABLE IX. Difficulties that exhibit all types of error in problems which require frame shifting. This difficulty is labeled “DCT” because it may involve all three types of error and s because the problem requires shifting.

No.	Name	Ref.
DCTs1	Interpreting the meaning of expectation value	[27]

TABLE X. Difficulties that exhibit both displacement and content errors in simpler problems. This difficulty is labeled DC for displacement and content errors and n because the problem does not require shifting.

No.	Name	Ref.
DCn1	Time dependence of expectation values: recognizing the special properties of stationary states—distinguishing between stationary states and eigenstates of operators corresponding to observables other than energy	[2]

example only the student’s final answer was provided in the paper; no further narration from the student was available, which leaves uncertainty in our analysis.

Table X shows students’ difficulties with the calculation of time-dependent expectation values in the context of Larmor precession for problems that do not require transition between frames. In this problem, the magnetic field is along the z axis, which gives the Hamiltonian as $\hat{H} = -\gamma B_0 \hat{S}_z$. Since the particle is initially in an eigenstate of the z component of spin angular momentum operator, the expectation value of any operator Q will be time independent.

Difficulties with *recognizing the special properties of stationary states* could result in a content error, as students similar to this case state, that for a stationary state the commutation of the Hamiltonian and the operator Q is nonzero, thus “its expectation value must depend on time” [2].

Student Since \hat{S}_x does not commute with \hat{H} , its expectation value must depend on time.

Although the student is able to apply Ehrenfest’s theorem correctly, the student does not note that being in a stationary state changes the Hamiltonian in the time-dependent phase factor from an operator ($e^{-iHt/\hbar}$) to a number ($e^{-iEt/\hbar}$), which commutes with the operator Q . In addition, *difficulties with distinguishing between stationary states and eigenstates of operators other than energy* could result in a displacement error as students think that “if a system is initially in an eigenstate of \hat{S}_x , then only the expectation value of S_x will not depend on time” [2].

VIII. ERROR RATES

Figure 7 shows all the possible ways that descriptions of published difficulties can be mapped into errors in framing and resource use. Each number refers to the number of difficulties in that error category, not the number of students in that category.

This figure shows that all the error categories and combinations of those categories are populated.

By starting with our set of math-in-physics frames and focusing on the context-dependent artifacts such as the problem statement, we can reveal a more fine-grained

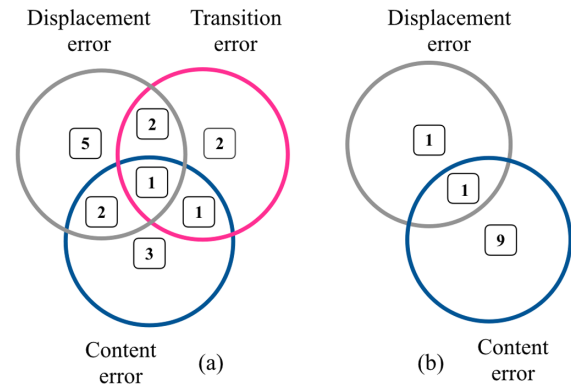


FIG. 7. The number of difficulties mapped to error categories, a) for questions that require shifting, b) for questions that do not require shifting.

structure to students’ cognitive processes, which are evoked in response to the keywords and cues in the problem statement.

For a problem that requires a transition, we expect to see a breakdown to displacement, content, and transition errors. If a problem does not require transition we expect to see a breakdown to displacement and content error. In addition, when a problem requires transition we expect to see more displacement errors because the transition inherent in the problem means students are more likely to be confused about which frames to begin with.

Figures 8 and 9 give an overview of the occurrence of the three error types—displacement, transition, and content—among all the topics. Figure 8 shows that displacement error is the most frequent among problem statements which require transition. By most frequent, we do not mean to imply that more students make that error; we mean that the displacement error category has more difficulties in it. This distinction is important because the underlying rates of each difficulty differ in the population of students. It could be that displacement difficulties are more common among

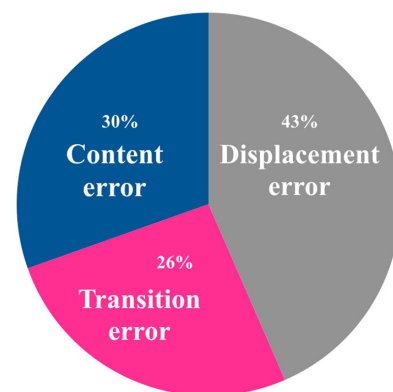


FIG. 8. Displacement, transition and content error categories of difficulty topics for questions that require transition. The total percentage is not 100 due to rounding error.

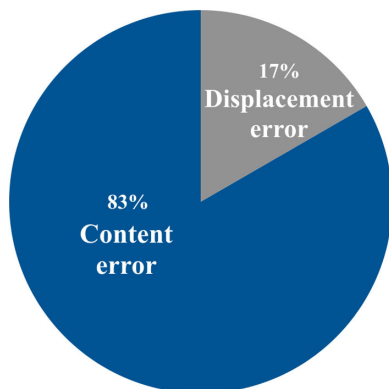


FIG. 9. Displacement and content error categories of difficulty topics for questions that do not require transition.

problems that require transition because those problems are harder (overall high rate of difficulties), or that there are simply more possible ways in which students displayed regular wrong responses to those problems. A great deal of effort goes into designing and testing questions that will reveal or cause student difficulties, so it is possible that these error rates are an artifact of the kinds of questions most likely to produce difficulty-like responses.

IX. CONCLUSION AND IMPLICATIONS

A difficulties view models students' conceptual understanding of fundamental ideas. Viewing knowledge as a large-grained construct makes it possible to find prevalence patterns of difficulties within a topic over a large number of students, which helps the research scale to quantitative studies. In this view, student success is determined by applying a concept to different physical situations correctly.

From a knowledge-in-pieces view, students' ideas are fine-grained, and success is determined by how they navigate in the problem-solving space; choosing the appropriate frame, activating the relevant ideas within a frame, and transitioning out of a frame into other productive frames to coordinate different kinds of knowledge. This emphasis on pathways in problem solving helps the research scale better to studies across different topics.

Students (similar to Eric's example and the example in the Emigh *et al.* study [3] in Sec. IV A) may rely on their initial frame which may cause an error. However, students' awareness of their error due to external factors can help them to reframe the problem. This awareness can be due to an interaction with an instructor or due to a disagreement among the members of a group that arises in the context of group problem solving. Because of the context-dependent nature of ideas, the knowledge in pieces view is better able to account for students' unseen conditions and unexpected moments of getting stuck, aha moments, and getting unstuck, and arriving at disagreements with an instructor or a group mate during a problem-solving procedure. The knowledge-in-pieces framework is also better able to

account for students' thought processes at a social (non-individual) level, such as students' in-class group problem solving, students working on a lab experiment, or in a tutorial session.

The goal of this paper is to reinterpret research on student difficulties in quantum mechanics through the lens of epistemological framing, as part of the family of knowledge-in-pieces theory, particularly using the set of four math-in-physics frames [12] previously applied to our observational classroom data of upper-division student problem solving. As our work in electromagnetic theory [13,26] and quantum mechanics [12] has shown, the framing space is also useful in describing student ideas as they solve problems. Additionally, these four frames can account for students' correct as well as incorrect reasoning.

We seek an underlying structure to the kinds of difficulties that other researchers have identified, and this mission necessarily moves us away from the particular details of the topics those difficulties are tied to. Our framework splits the underlying thought processes behind student errors into three different categories as displacement error, transition error, and content error. Displacement error reveals a student's unproductive frame of the situation. Content error shows what pieces of knowledge have yet to be activated to understand all the ideas incorporated in the problem frame. Transition error shows that students are able to activate resources in one frame, but then cannot make a productive transition in frame to continue with problem solving. We excluded nine difficulties from further analysis due to not having enough information to figure out what the framing could have been.

Our analysis of secondary data is hampered by the very nature of that data: we do not have access to full student reasoning because sometimes primary sources do not report it, and sometimes the nature of their survey data precludes them from collecting it. We expect that, were sufficiently detailed data available, all of the student reasoning attributed to difficulties in quantum mechanics could be analyzed using these frames.

It is possible that if we had access to the original student data under a certain difficulty, that difficulty could possibly map to all of the framing errors, as shown in Table IX for one of the difficulties. We suggest that this is unlikely both statistically and theoretically.

To better illustrate this point, we use a metaphor of multidimensional space. If we attribute all twenty-seven surviving difficulties to a 27-dimensional space and the framing errors to a three-dimensional space, we can then reproject difficulties into framing errors. Inasmuch as our video-based data overlap with the data presented in the difficulties papers, we notice that our data map one difficulty to one framing error. While it is technically true that each difficulty could map to all framing errors, that interpretation is inconsistent with our video-based data and statistically implausible for the remainder of the data.

Even though the difficulty space can be more easily scaled when adding new identified difficulty topics, the framing space is more parsimonious than the difficulties space in understanding student ideas within and across topics. This is due to the diverse nature of knowledge in pieces approach; which supports that different students have different ways of thinking and use of their knowledge [30]. We suggest—but cannot robustly support—that many difficulties will largely map to single framing errors. This is a potential avenue for future collaboration between difficulties-based teams and framing-based teams. At a community level, the difficulties framework provides insight into students' difficulties with specific topics and a pieces approach provides insight into dealing with students' emerging difficulties for different classroom settings in the moment, both at an individual and at a social level.

Choosing framing over difficulties has implications for both future research and for instruction. For research, it is an open question as to whether these four frames—conceptual physics, conceptual math, algorithmic physics, and algorithmic math—constitute the optimal basis set for epistemological frames in student understanding of quantum mechanics. However, they do form a more compact basis set than is possible (let alone extant) with difficulties, as they are applicable across topics in a way which difficulties cannot be. Because framing focuses on the pathways of problem solving, rather than students' answers and reasoning, researchers using framing generate different kinds of data and value different kinds of student responses than researchers using difficulties.

Instructors' awareness of student error categories may help them scaffold students' reasoning more effectively, as instructors can tip students into different frames [10,26] or gently nudge students to use additional resources [2,31] to resolve content errors. Epistemologically aware tutorials at the introductory level [32] have been shown to outperform difficulties-based tutorials [33] in student understanding of Newton's third law [34,35]. More broadly, supporting students' epistemologies in the classroom may have far-reaching implications for retention and persistence [36–38].

Curriculum development work in quantum mechanics at the upper division is exclusively in a difficulties-based mode, though some epistemologically aware work has occurred at the introductory level in quantum mechanics [39].

Curriculum developers could take up framing as a guiding theoretical framework for professional development at the upper division. Difficulties promotes a model of student reasoning as fractured among long lists of topically centered wrong ideas. A faculty member must memorize a different list of difficulties for each topic and each course, substantially increasing their overhead in teaching. In contrast, a faculty development program using framing might emphasize a smaller set of frames to guide problem solving across topics, which may be easier for faculty to learn and apply in their teaching in different courses than long lists of difficulties.

These differences in implications for instruction are particularly interesting for quantum mechanics because the conceptual content is epistemologically difficult [40]—yet conceptually fascinating [41]—for students and because faculty who teach quantum courses usually teach other courses as well.

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Correction: The department name in the first affiliation was incomplete and has been fixed.