Explosive Instability due to Four-Wave Mixing

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It is known that an explosive instability can occur when nonlinear waves propagate in certain media that admit 3-wave mixing. The purpose of this Letter is to show that explosive instabilities can occur even in media that admit no 3-wave mixing. Instead, the instability is caused by 4-wave mixing: four resonantly interacting wave trains gain energy from a background, and all blowup in a finite time. Unlike singularities associated with self-focussing, these singularities can occur with no spatial structure—the waves blowup everywhere in space simultaneously. We have not yet investigated the effect of spatial structure on a 4-wave explosive instability.

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Among mathematical models that describe nonlinear wave propagation without dissipation, certain "universal" models stand out—each of these models appears when one takes a specific limit, and each arises in many physical situations. In all cases, one first linearizes the governing equations about some trivial state and obtains a (linearized) dispersion relation, $\omega(\boldsymbol{k})$, which relates the frequency (ω) of a signal to its wave number ($\boldsymbol{k}$). A 3-wave resonance is possible if the dispersion relation admits pairs $\{\omega_m, \boldsymbol{k}_m\}$ such that

$$\boldsymbol{k}_1 \pm \boldsymbol{k}_2 \pm \boldsymbol{k}_3 = \boldsymbol{0}, \qquad \omega_1 \pm \omega_2 \pm \omega_3 = 0. \tag{1}$$

For a given problem, (1) may or may not be possible. For example, in nonlinear optics, (1) occurs only in so-called χ_2 materials [1]; for surface water waves, (1) is impossible for pure gravity waves, but it occurs if both gravity and surface tension are included in the model [2]. When (1) occurs, then $\{A_1(\boldsymbol{x}, t), A_2(\boldsymbol{x}, t), A_3(\boldsymbol{x}, t)\}$, the slowly-varying complex amplitudes of three-wave modes, evolve according to the "three-wave equations": three coupled equations of the form (with $l, m, n = 1, 2, 3$ cyclically)

$$\partial_t A_m + \boldsymbol{c}_m \cdot \boldsymbol{\nabla} A_m = i\delta_m A_n^* A_l^*. \tag{2}$$

Here, $\boldsymbol{c}_m$ is the group velocity and δ_m is a real-valued interaction coefficient, each corresponding to $\{\omega_m, \boldsymbol{k}_m\}$ [3].

In the simplest model of 3-wave mixing, one ignores the spatial dependence of the interacting modes so that (2) reduces to three coupled, complex, ordinary differential equations (ODEs),

$$\frac{dA_1}{dt} = i\delta_1 A_2^* A_3^*, \qquad \frac{dA_2}{dt} = i\delta_2 A_3^* A_1^*, \qquad \frac{dA_3}{dt} = i\delta_3 A_1^* A_2^*. \tag{3}$$

If any two δ_m in (3) differ in sign, then one can show that all solutions of (3) are bounded for all time. But this is not the only possibility: situations in which $\{\delta_1, \delta_2, \delta_3\}$ all have the same sign occur in plasmas [4,5], in density-stratified shear flows [6,7], and for vorticity waves [8]. If all δ_m have the same sign, then nonzero solutions of (3) blowup in finite time [like $(t - t_0)^{-1}$], including solutions that start with arbitrarily small initial data. This is called the *explosive instability* [3,4]. All three waves grow together, so all three waves draw energy from a background source and blowup in unison. Thus, the relative signs of the δ_m in (2) and (3) signal whether such an energy source is available in the physical problem that (2) and (3) approximate.

The main point of this Letter is to show that explosive instabilities can occur even in situations where a 3-wave resonance is impossible. In that case, the simplest nonlinear interaction among wave modes is a 4-wave resonance, in which four pairs $\{\omega_m, \boldsymbol{k}_m\}$ satisfy

$$\boldsymbol{k}_1 \pm \boldsymbol{k}_2 \pm \boldsymbol{k}_3 \pm \boldsymbol{k}_4 = \boldsymbol{0}, \qquad \omega_1 \pm \omega_2 \pm \omega_3 \pm \omega_4 = 0. \tag{4}$$

A common special case, in which one wave mode interacts with two other modes at nearly the same frequency and wave number,

$$(\boldsymbol{k} + \delta\boldsymbol{k}) + (\boldsymbol{k} - \delta\boldsymbol{k}) - \boldsymbol{k} = \boldsymbol{k},$$
$$(\omega + \delta\omega) + (\omega - \delta\omega) - \omega = \omega,$$

leads to the nonlinear Schrödinger (NLS) equation for the slowly varying, complex amplitude of one wave mode [3]

$$i\partial_t A + \{\alpha_1\partial_x^2 + \alpha_2\partial_y^2 + \alpha_3\partial_z^2\}A + \gamma|A|^2 A = 0. \tag{5}$$

Here, $\{\alpha_m\}$ are real-valued constants obtained from $\omega(\boldsymbol{k})$, and γ is a real-valued interaction coefficient (provided the original problem has no dissipation). In optics, (4) occurs in χ_3 materials [1]. With one additional term, (5) becomes the Gross-Pitaevski equation, a commonly used model for Bose-Einstein condensates [9,10].

More complicated interactions, in which wave modes interact nonlinearly with themselves and also with other wave modes, lead to coupled NLS equations [11]. More complicated still are systems with self-interactions, cross-interactions, and 4-wave-mixing (with $m, p, q, r = 1, 2, 3, 4$ cyclically):

$$i(\partial_t A_m + \boldsymbol{c}_m \cdot \nabla A_m) + \sum_{l,n} \alpha_{m,l,n} \partial_{x_l} \partial_{x_n} A_m + A_m \sum_{n=1}^{4} \gamma_{m,n} |A_n|^2 + \delta_m A_p^* A_q^* A_r^* = 0. \quad (6)$$

The system in (6) has four such equations. In each equation, $\boldsymbol{c}_m$ is the group velocity and $\{\alpha_{m,l,n}\}$ are real-valued constants, all obtained from $\omega(\boldsymbol{k})$; $\{\gamma_{mn}\}$ are coefficients of NLS-type interaction terms; and $\{\delta_m\}$ are real-valued coefficients of the 4-wave mixing terms. The general form of this system of equations was first recognized in [12].

[The 4-wave mixing term in (6), $\delta_m A_p^* A_q^* A_r^*$, can be written in more than one way. By interchanging the roles of $(A_m \leftrightarrow A_m^*)$, one can change the sign of δ_m relative to that of $(i\partial_t A_m)$. We have chosen to write the four-wave mixing terms as in (6) and to allow the physical problem to dictate the signs of $\{\delta_j\}_{j=1}^4$. Alternatively, one could arrange for all $\{\delta_j\}$ to have the same sign, but then one or more of the factors in $A_p^* A_q^* A_r^*$ might be replaced by its complex conjugate. Our result below about the importance of the relative signs of $\{\delta_j\}$ should be interpreted in terms of the convention used in (6).]

In this Letter, we show that an explosive instability of the kind usually associated with 3-wave interactions can also occur because of 4-wave mixing. As with (3), a simpler model with 4-wave mixing than (6) is obtained by ignoring any spatial dependence of the interacting modes, so that (6) reduces to four coupled ODEs (with $m, p, q, r = 1, 2, 3, 4$ cyclically):

$$i\frac{dA_m}{dt} + A_m \sum_{n=1}^{4} \gamma_{m,n} |A_n|^2 + \delta_m A_p^* A_q^* A_r^* = 0. \quad (7)$$

Note that with no spatial dependence, the self-focussing kind of singularity usually associated with NLS-type systems [13,14] cannot occur.

Without the 4-wave mixing terms in (7), there is no blowup: one shows directly from (7) that for each m, if $\delta_m = 0$, then $|A_m|^2$ is constant. Hence, we now assume that all $\delta_m \neq 0$. Then, (7) admits three independent constants of the motion in the form of Manley-Rowe [15] relations:

$$J_m = \frac{|A_m|^2}{\delta_m} - \frac{|A_4|^2}{\delta_4}, \qquad m = 1, 2, 3. \quad (8)$$

It follows from (8) that if any two δ_m differ in sign, then every $|A_m|^2$ is bounded for all time. This result parallels the corresponding result for (3): all solutions of (3) are bounded if any two δ_m in (3) differ in sign. However, requiring that all the δ_m have the same sign in (7) is necessary but not sufficient for an explosive instability: it is also necessary that the δ_m be large enough relative to $\sum \gamma_{m,n}$, as we show next.

If all δ_m in (7) have the same sign, then change variables $\{A_m(t) = \sqrt{|\delta_m|} B_m(t), \Gamma_{m,n} = \gamma_{m,n} |\delta_n|, \delta = \mathrm{sgn}(\delta_1)\sqrt{\delta_1 \delta_2 \delta_3 \delta_4}\}$ to obtain an equivalent system of ODEs ($m, p, q, r = 1, 2, 3, 4$ cyclically):

$$i\frac{dB_m}{dt} + B_m \sum_{n=1}^{4} \Gamma_{m,n} |B_n|^2 + \delta B_p^* B_q^* B_r^* = 0. \quad (9)$$

This system of ODEs is Hamiltonian, with conjugate variables $\{B_m, B_m^*, m = 1, 2, 3, 4\}$ and Hamiltonian $H = H_1 + H_2$, where

$$H_1 = -\frac{i}{2} \sum_{m,n=1}^{4} \Gamma_{m,n} B_m B_m^* B_n B_n^*,$$
$$H_2 = -i\delta(B_1 B_2 B_3 B_4 + B_1^* B_2^* B_3^* B_4^*) \quad (10)$$

Direct computation shows that H is a constant of the motion. In addition, in these variables, (8) becomes

$$J_m = |B_m|^2 - |B_4|^2, \qquad m = 1, 2, 3. \quad (11)$$

And one can verify directly that the usual Poisson bracket of any two of (H, J_1, J_2, J_3) vanishes, so these constants are said to be in involution. Then, it follows that the system of four complex ODEs in (9) is completely integrable in the sense of Liouville [16].

Next, we show that solutions of (9) blowup in finite time if

$$\left| \sum_{m,n=1}^{4} \Gamma_{m,n} \right| \leq 4|\delta|. \quad (12a)$$

In terms of the variables in (7), its solutions blowup in finite time if all four δ_m have the same sign and

$$\left| \sum_{m,n=1}^{4} \gamma_{m,n} |\delta_n| \right| \leq 4\sqrt{\delta_1 \delta_2 \delta_3 \delta_4}. \quad (12b)$$

In either notation, these are the criteria for an explosive instability due to 4-wave mixing. They comprise the main result in this Letter. Assuming (12) holds, a four-parameter family of exact, singular solutions of (9) is

$$B_m(t) = \frac{c e^{i\theta_m}}{(t_0 - t)^{(1/2) + i\phi_m}}, \qquad m = 1, 2, 3, 4 \quad (13)$$

where $\{c, t_0, \theta_m, \varphi_m\}$ are real-valued constants, and

$$\Theta = \sum_{m=1}^{4} \theta_m = \arccos\left\{-\frac{1}{4\delta} \sum_{m,n=1}^{4} \Gamma_{m,n}\right\}, \quad (14a)$$

$$c^2 = \frac{1}{2\delta \sin\Theta}, \quad (14b)$$

$$\phi_m = c^2 \left\{ \sum_{n=1}^{4} \Gamma_{m,n} + \delta \cos\Theta \right\}. \quad (14c)$$

[These results hold for $t_0 > t$; for $t > t_0$, one changes the sign of $(t_0 - t)$ in (13), and the sign of c^2 in (14b) and (14c).] The four free constants in (13) are t_0, and any three of the four θ_m. Then the last θ_m must be chosen to satisfy

(14a). Substitution of (13) into (9) shows that this form of solution is possible if and only if (12) holds. One can also verify by substituting (13) into (10) and (11) that $\{H, J_1, J_2, J_3\}$ all vanish for any solution in this family.

Next, we show that when (12) holds, *all* solutions of (9) blowup in finite time. To do so, we may consider the solution in (9) to be the first term in a Laurent series, near $(t = t_0)$, and seek solutions of (9) in the form (for $m = 1, 2, 3, 4$)

$$B_m(t) = \frac{ce^{i\theta_m}}{(t_0 - t)^{(1/2)+i\phi_m}} \times [1 + \alpha_m(t - t_0) + \beta_m(t - t_0)^2 + \ldots]. \tag{15}$$

In (15), $\{c, t_0, \theta_m, \varphi_m\}$ are defined as above, and $\{\alpha_m, \beta_m, \ldots\}$ are complex numbers. Substituting (15) into (9) and requiring that the complex coefficient of each power of $(t - t_0)$ vanish shows that most of the coefficients in this expansion are fixed, with four exceptions: the real parts of three α_m can be chosen arbitrarily, as can the imaginary part of one β_m. Thus, the family of solutions in (15) contains eight free, real constants. [For example, one can choose the 8 free constants to be $\{t_0, \theta_1, \theta_2, \theta_3, \mathrm{Re}(\alpha_1), \mathrm{Re}(\alpha_2), \mathrm{Re}(\alpha_3), \mathrm{Im}(\beta_4)\}$.] Therefore, the family of solutions in (15) is the *general* solution of (9), so all solutions of (9) with nonzero initial data blowup in finite time, provided only that (12) holds.

Because the solutions in (13) all occur with $\{0 = H = J_1 = J_2 = J_3\}$, it follows that the four new constants in (15) must determine $\{H, J_1, J_2, J_3\}$. One can show that for $m = 1, 2, 3$,

$$J_m = 2c^2[\mathrm{Re}(\alpha_4) - \mathrm{Re}(\alpha_m)]. \tag{16}$$

Then $\mathrm{Im}(\beta_4)$ determines the value of H, but we have found no simple way to write this relation.

It is known that the self-focussing (or "wave collapse") singularity of an NLS-type equation occurs only for a range of H [13]. The singularity in (15) occurs for any (real) values of $\{H, J_1, J_2, J_3\}$, provided only that (12) holds, so the two kinds of singularities differ in this respect. They also differ because spatial structure plays an essential role in a self-focussing singularity, but it plays no role whatsoever here.

It remains to show that the solutions of (9) must be nonsingular if (12) is not satisfied, so that (12) is both necessary and sufficient for an explosive instability. Suppose that $|B_4(t)| \to \infty$ as $t \to t_0$. Then, it follows from (11) that all four $|B_m(t)|$ must grow at the same rate. Hence, as $t \to t_0$, the dominant terms in (10) are

$$H_1 = -\frac{i}{2}\sum_{m,n=1}^{4}\Gamma_{m,n}|B_4|^4 + O(|B_4|^2),$$

$$H_2 = -2i\delta|B_4|^4\cos(\varphi) + O(|B_4|^3),$$

where $\varphi(t)$ is some (unknown) phase. These dominant terms must balance as $t \to t_0$, so necessarily

$$\left|\frac{1}{2}\sum_{m,n=1}^{4}\Gamma_{m,n}\right||B_4|^4 = |2\delta|B_4|^4\cos(\varphi)| \le 2|\delta||B_4|^4 \tag{17}$$

in this limit. Dividing by $|B_4|^4$ shows that no explosive singularity can occur for $|\sum_{m,n=1}^{4}\Gamma_{m,n}| > 4|\delta|$. This completes the proof.

Explosive instabilities due to 3-wave mixing have been known for 30 years [4–8,17,18]. To our knowledge, no explosive instability caused by 4-wave mixing has ever been observed in a physical system. The analysis above indicates that it should be possible. As with 3-wave mixing, an explosive instability in a 4-wave system requires a background source of energy so that all four-wave modes can grow in intensity together. And as with 3-wave mixing, the indication that such a background source is available is that all four δ_m in (6) or (7) have the same sign. One difference between the two processes is that for 4-wave mixing, this agreement in signs of the δ_m in (7) by itself does not guarantee an explosive instability—the interaction coefficients must also satisfy (12).

This analysis establishes the existence of a new family of singular solutions of (6), which approximates many physical systems. However, it leaves unanswered other questions: (i) The singular solutions in (13) and (15) have no spatial structure. How does spatial structure in the initial data for (6) affect these singularities? For an explosive instability due to 3-wave mixing, the corresponding question was answered in part by Kaup [18]. (ii) The energy source that drives these singular solutions is implicit in (6), through the coefficients in the equation. What is the nature of this energy source in a concrete physical example? (iii) Equations (6) approximate many physical systems under a set of assumptions, including one that wave amplitudes are not too large. When a solutions of (6) becomes singular, then the mathematical model is no longer valid. When a solution of (6) becomes singular, what happens in any of the actual physical problems that (6) approximates? (iv) The explosive instability, discussed here, should not be confused with a wave-collapse singularity, even though both are singularities that can arise from smooth initial data in finite time. In wave collapse, a finite amount of energy becomes concentrated in a smaller and smaller spatial region, and the singularity that occurs is extremely localized in space. This process is quite different from an explosive instability, in which all of the interacting waves acquire more and more energy from an unlimited (external) supply, and the blowup occurs everywhere in space simultaneously. To our knowledge, the system of equations in (6) is the first known example of a family of physically relevant models whose solutions can blowup in a finite time in two completely distinct ways, depending on signs of coefficients. We have shown that the explosive instability requires that all four δ_m have the same sign; we conjecture that a wave-collapse kind of singularity *cannot* occur if all four δ_m have the same sign.

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