New Post-Newtonian Parameter to Test Chern-Simons Gravity

Stephon Alexander and Nicolas Yunes

Institute for Gravitational Physics and Geometry, Center for Gravitational Wave Physics and Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802, USA

(Received 9 April 2007; published 10 December 2007)

We study Chern-Simons (CS) gravity in the parametrized post-Newtonian (PPN) framework through a weak-field solution of the modified field equations. We find that CS gravity possesses the same PPN parameters as general relativity, except for the inclusion of a new term, proportional to the CS coupling and the curl of the PPN vector potential. This new term leads to a modification of frame dragging and gyroscopic precession and we provide an estimate of its size. This correction might be used in experiments, such as Gravity Probe B, to bound CS gravity and test string theory.

DOI: 10.1103/PhysRevLett.99.241101 PACS numbers: 04.25.Nx, 04.50.+h, 04.80.Cc

Introduction.—Current astronomical observations, such as the apparent acceleration of the Universe, suggest a possible infrared modification to general relativity (GR). In the same spirit, another unresolved problem of cosmology, the cosmic baryon asymmetry, suggests a modification of general relativity via the inclusion of a Chern-Simons (CS) correction during the inflationary period [1]. This Chern-Simons correction is not an *ad hoc* extension, but it is actually motivated by both string theory, as a necessary anomaly-canceling term to conserve unitarity [2], and loop quantum gravity [3]. Recently, imprints of CS gravity have been investigated in the gravitational wave spectrum of the cosmic microwave background (CMB), where it was found to produce a circular, V-mode, polarization, albeit marginally detectable [4]. Motivated by observational signatures of string theory and loop-quantum gravity, we will explore and develop a new observational window to distinguish CS gravity from classical GR, which is of direct interest to gravitational experiments currently underway, such as Gravity Probe B (GPB) [5] and lunar ranging [6].

A proven avenue for testing alternative theories of gravity with current solar-system experiments is the parametrized post-Newtonian (PPN) framework [7]. This framework considers weak-field solutions of the field equations of the alternative theory and expresses them in terms of PPN potentials and parameters. The PPN potentials depend on the details of the system under consideration, while the PPN parameters can be mapped to intrinsic parameters of the theory. Predictions of the alternative theory can then be computed in terms of PPN parameters and compared to solar-system experiments, leading to stringent tests. One of the strengths of this framework is its generality: a single supermetric with certain PPN parameters can be constructed to reproduce and test several different alternative theories [7] (e.g., scalar-tensor, vector-tensor, bimetri, c and stratified theories). Other tests of alternative theories of gravity have also been proposed, some of which require a gravitational wave detection and shall not be discussed here [8–10].

In this Letter, we present a parametrized PPN expansion of CS gravity to allow for tests with current solar-system experiments. We discover that CS gravity *demands* the introduction of only one new term to the PPN supermetric and, thus, one new PPN parameter. This new term depends both on an intrinsic parameter of CS gravity, as well as on the curl of the PPN vector potential. Such a coupling of CS gravity to gravitational vector currents had so far been neglected. Furthermore, curl terms in the supermetric had also been neglected by the PPN community because other alternative theories had not required them. We find that this new term captures the key physical effect of CS gravity in the weak-field limit, leading to a modification of frame dragging that could be used to test this GR extension with GPB [5].

CS gravity in a nutshell.—CS gravity modifies GR via the addition of a new term to the action, namely [11,12],

$$S_{\rm CS} = \frac{1}{16\pi G}\int d^4x \frac{1}{4} f R^{\star}R, \tag{1}$$

where G is Newton's gravitational constant, f is a prescribed external quantity [13] (with units of squared length in geometrized units) that acts as a coupling constant, R is the Ricci scalar, and the star stands for the dual operation. The modified field equations can be obtained by varying the action with respect to the metric. These equations, in trace-reversed form, are

$$R_{\mu\nu} + C_{\mu\nu} = 8\pi(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T), \tag{2}$$

where $C_{\mu\nu}$ is a Cotton-like tensor, $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is a stress-energy tensor, with T its 4-dimensional trace, and Greek letters range over spacetime indices. The Cotton tensor encodes the CS modification to GR:

$$C_{\mu\nu} = -\frac{1}{\sqrt{-g}}[f_{,\sigma}\epsilon^{\sigma\alpha\beta}{}_{(\mu}D_{\alpha}R_{\nu)\beta} + (D_{\sigma}f_{,\tau})^{\star}R^{\tau}{}_{(\mu}{}^{\sigma}{}_{\nu)}], \tag{3}$$

where parentheses stand for symmetrization, g is the determinant of the metric, $\epsilon^{\sigma\alpha\beta\mu}$ is the Levi-Civita symbol

[14], D_α and colon subscripts stand for covariant and partial differentiation, respectively.

The CS correction to the action has been shown to lead to birefringence in the polarization of gravitational waves [16]. In this context, birefringence is a change in the amplitude of different polarization modes as the wave propagates. Recently, there have been proposals [17] of astrophysical tests of theories where gravitational waves with different polarization propagate at different speeds, but this is not the case in CS gravity. Nonetheless, such amplitude birefringence in gravitational waves could have a signature in the anisotropies of the CMB [1] and could explain baryogenesis during the inflationary epoch [4]. Given that the CS extension has been key in proposing a plausible explanation to some important cosmological problems, it seems natural to study CS gravity in the light of solar-system experiments.

Can we understand the CS correction in more physical terms? For this purpose, let us consider the CS coupling parameter f as a consequence of some external field that permeates all of spacetime, such as a model-independent gravitational axion. This field could depend on some intrinsic properties of spacetime, such as the fundamental string scale [18] or the existence of warped compactifications [19]. Furthermore, this field could also be coupled to regions of high curvature, such as binary neutron star systems, through standard model-like currents. These couplings have been proposed as enhancements to the CS modification, which would otherwise be suppressed by the Planck scale. For simplicity, in this Letter we shall concentrate on a CS coupling parameter that is spatially isotropic and whose only nonvanishing derivative is $\dot{f}$. These assumptions are made such that time-translation symmetry and reparametrization invariance are preserved in the modified theory [11].

Weak-field expansion of CS gravity.—Let us consider a system that is weakly gravitating, such that we can expand the metric about a fixed Minkowski background $\eta_{\mu\nu}$. In other words, let us write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ a small perturbation, and expand the Cotton tensor to second order in $h_{\mu\nu}$. We then obtain a complicated expression that can be schematically given by [20]

$$C_{\mu\nu} \sim N^{(1)}_{\mu\nu}[\epsilon \cdot h'''] + N^{(2)}_{\mu\nu}[\epsilon \cdot hh'''] + N^{(3)}_{\mu\nu}[\epsilon \cdot h'h''], \quad (4)$$

where primes stand for spatial or temporal derivatives and $\epsilon \cdot A$ is the full contraction of the Levi-Civita symbol with the tensor $A_{\mu_1,\ldots,\mu_n}$. Note that here we have not assumed any gauge conditions and, thus, Eq. (4) could be used in future work to calculate gravitational wave solutions to $\mathcal{O}(h)^2$. Equation (4) to linear order and in the Lorenz gauge $[h^{\alpha}_{\mu\alpha,} = h_{,\mu}/2,\ h \equiv \eta^{\mu\nu} h_{\mu\nu}]$ reduces to the previously known expression [11]

$$C_{\mu\nu} = -\frac{\dot{f}}{2} \epsilon^{0\alpha\beta}{}_{(\mu} \Box_\eta h_{\nu)\beta,\alpha} + \mathcal{O}(h)^2, \quad (5)$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $\Box_\eta$ is the D'Alambertian associated with it.

Before proceeding with the PPN solution of the modified field equations, we must discuss the stress-energy source that we shall employ. Here we model this tensor as that of a perfect fluid (cf., e.g., [7]). Such a stress-energy tensor is sufficient to obtain the PPN solution of the modified field equations for solar-system experiments, where the internal structure of the fluid bodies shall be neglected to lowest order by the effacement principle [21].

The stress energy considered here requires the strong equivalence principle (SEP) to hold in CS gravity [1]. This principle states that the outcome of all local gravitational experiments is independent of the experimenter's reference frame. In other words, the motion of test particles is exclusively governed by the spacetime metric, through the divergence of the stress-energy tensor. In CS gravity, the divergence of Eq. (2) leads to

$$D^\alpha C_{\alpha\beta} = \frac{\dot{f}\delta^{\beta 0}}{8\sqrt{-g}} R^\star R = 8\pi D^\alpha T_{\alpha\beta}, \quad (6)$$

where $D^\alpha G_{\alpha\beta} = 0$ by the Bianchi identities. The right-hand side of Eq. (6) vanishes in vacuum and, thus, the SEP holds provided $R^\star R = 0$, which is known as the Pontryagin constraint. In fact, in the original formulation of CS gravity [11], this constraint was independently required to preserve time-translation symmetry and spatial reparametrization invariance. We shall later see that the solution to Eq. (2) found here automatically satisfies this constraint to $\mathcal{O}(h)^2$ and, thus, the SEP holds.

Weak-field solution.—Let us first study the weak-field solution to the modified field equations in Lorenz gauge. The formal first-order solution of Eq. (2), with Eq. (5) used for the Cotton tensor, is simply [15]

$$h_{\mu\nu} = -16\pi \Box_\eta^{-1}[\bar{T}_{\mu\nu} - \dot{f}\epsilon^{k\ell i}(\delta_{i(\mu} T_{\nu)\ell,k} - \tfrac{1}{2}\delta_{i(\mu}\eta_{\nu)\ell} T_{,k})], \quad (7)$$

where $\bar{T}_{\mu\nu}$ is the trace reversed $T_{\mu\nu}$. Note that this formal solution has the property that as $\dot{f} \to 0$ it reduces to that predicted by the post-Newtonian (PN) expansion of GR [21]. In fact, such a solution is the cornerstone of the PN formalism and would be essential if one were to pursue such an expansion of CS gravity.

Let us now proceed with the PPN solution of the modified field equations. The PPN formalism differs from the PN Lagrangian formulation for inspiraling compact binaries [21] by the use of a different gauge, the harmonic one [22]. In the PPN formalism, one usually employs a PPN gauge, designed such that the spatial part of the metric is diagonal and isotropic. These conditions can be enforced perturbatively via [7]

$$h_{jk,}{}^{k} - \tfrac{1}{2}h_{,j} = \mathcal{O}(4), \qquad h_{0k,}{}^{k} - \tfrac{1}{2}h^k{}_{k,0} = \mathcal{O}(5), \quad (8)$$

where $h^k{}_k$ is the spatial trace of the metric perturbation and the symbol $\mathcal{O}(A)$ stands for PN remainders of order

$\mathcal{O}(1/c)^A$, with c the speed of light [15]. One can show that Eq. (8) is related to the Lorenz gauge via an infinitesimal gauge transformation. The solution to the CS modified field equations in PPN gauge is given by

$$\begin{aligned} g_{00} &= -1 + 2U - 2U^2 + 4\Phi_1 + 4\Phi_2 \\ &\quad + 2\Phi_3 + 6\Phi_4 + \mathcal{O}(6), \\ g_{0i} &= -\tfrac{7}{2}V_i - \tfrac{1}{2}W_i + 2\dot{f}(\nabla \times V)_i + \mathcal{O}(5), \\ g_{ij} &= (1 + 2U)\delta_{ij} + \mathcal{O}(4), \end{aligned} \tag{9}$$

where $\{U, \Phi_1, \Phi_2, \Phi_3, \Phi_4, V_i, W_i\}$ are PPN potentials (see, e.g., [7] for definitions and discussion of these potentials). Both the PPN potentials and parameters take the same values in CS gravity as in GR. Equation (9) is a solution to 1 PN order, since from it one could calculate the point-particle Lagrangian to $\mathcal{O}(4)$. As one can check, this solution satisfies the Pontryagin constraint [15].

Chern-Simons gravity introduces a correction to the metric in the vectorial sector of the metric perturbation. This correction is proportional to the first time derivative of the CS coupling parameter $\dot{f}$ and to the curl of the PPN vector potential V_i. In principle, there is also a CS coupling to the other PPN vector potential W_i, but this contribution is already accounted for because $\nabla \times W_i = \nabla \times V_i$. Since this is the only modification to the metric, the PPN parameters of CS gravity are identical to those of classical GR, with the exception of the inclusion of a new term in g_{0i}. In fact, defining the CS correction as $\delta g_{0i} = g_{0i} - g_{0i}^{\rm GR}$, with $g_{0i}^{\rm GR}$ the GR prediction, we get

$$\delta g_{0i} = \chi M(\nabla \times V)_i, \tag{10}$$

where we have defined a new PPN parameter, $\chi \equiv 2\dot{f}/M$, with M the characteristic mass scale of the source inducing the vector potential. This new PPN parameter is rescaled by M to make it dimensionless and coordinate independent. The rescaling choice might seem arbitrary, but since $\dot{f}$ has units of mass it can be interpreted as some CS mass scale, yielding $\chi \propto m_{\rm CS}/M$ as a ratio of masses with a clear physical meaning.

Until now, a PPN potential of the type of Eq. (10) had not been considered, nor had any experimental constraints been placed on χ. Clearly, any experiment that samples the vectorial sector of the metric perturbation, and thus, the frame-dragging effect, could achieve such a constraint.

Astrophysical tests.—Consider a system of A nearly spherical bodies in the standard PPN point-particle approximation, where the PPN vector potential is [7]

$$V^i = \sum_A \frac{m_A}{r_A} v_A^i + \frac{1}{2}\sum_A \left(\frac{J_A}{r_A^2} \times n_A\right)^i, \tag{11}$$

with m_A the mass of the Ath body, r_A the field point distance to the Ath body, $n_A^i = x_A^i/r_A$ a unit vector pointing to the Ath body, v_A the velocity of the Ath body, and J_A^i the spin-angular momentum of the Ath body. When the number of bodies $A = 2$, Eq. (11) is the vector potential for a binary of spinning compact objects, while when there is only one body present, $A = 1$, it represents the potential outside a moving spinning body. For such a vector potential, the CS correction to the metric becomes

$$\delta g_{0i} = 2\sum_A \frac{\dot{f}}{r_A}\left[\frac{m_A}{r_A}(v_A \times n_A)^i - \frac{J_A^i}{2r_A^2} + \frac{3}{2}\frac{(J_A \cdot n_A)}{r_A^2} n_A^i\right], \tag{12}$$

where the $\cdot$ and $\times$ operators are the flat-space inner and cross products. Note that the CS correction couples both to the spin and orbital angular momentum of the system.

The full gravitomagnetic sector of the metric becomes

$$\begin{aligned} g_{0i} = \sum_A \Big[&-\frac{7}{2}\frac{m_A}{r_A} v_A^i - \frac{m_A}{6r_A}(v_A - v_A^{(\rm eff)})^i \\ &- \frac{1}{2} n_A^i \frac{m_A}{r_A}(v_A^{(\rm eff)} \cdot n_A) - 2\left(\frac{J_A^{(\rm eff)}}{r_A^2} \times n_A\right)^i \Big], \end{aligned} \tag{13}$$

where we have introduced an effective velocity and angular momentum through

$$v^i_{A(\rm eff)} = v_A^i - 6\dot{f}\frac{J_A^i}{m_A r_A^2}, \quad J^i_{A(\rm eff)} = J_A^i - \dot{f} m_A v_A^i. \tag{14}$$

When the spin-angular momentum J_A vanishes, g_{0i} is identical to that of a spinning moving object, with the spin induced by the CS coupling to the orbital angular momentum. Such a coupling leads to an interesting physical interpretation: if we model the field that sources the CS coupling as a fluid that permeates all spacetime, the CS modification to the metric is nothing but the "dragging" of such a fluid [15], whose strength is proportional to the first derivative of the CS coupling parameter.

The CS correction to the metric computed here couples to the three-velocity of the sources, which could suggest the possibility that this effect is not coordinate invariant. However, this velocity-dependence comes directly from the PPN vector potential V^i [Eq. (11)]. Therefore, the CS correction to the metric is as coordinate invariant as the GR PPN metric itself, since this also depends on V^i [Eq. (9)] [7]. Observables, on the other hand, must be constructed in a coordinate-invariant way, which is sensitive to the choice of basis vectors. This choice in general depends on the experiments that look for such observables [23] and a more formal analysis of such coordinate issues should be carried out elsewhere.

We can now compute the correction to the frame-dragging effect in CS gravity and compare it to the Lense-Thirring effect. Consider then a free gyroscope in the presence of the gravitational field of Eq. (13). The gyroscope will acquire the precessional $\Omega^i = (\nabla \times g)^i$, where $g^i = g_{0i}$. Therefore, the CS modification to the precession angular velocity, defined via $\delta\Omega^i = \Omega^i - \Omega^i_{\rm GR}$, where $\Omega^i_{\rm GR}$ is the GR prediction, is given by

$$\delta\Omega^i = -\sum_A \dot{f}\frac{m_A}{r_A^3}[3(v_A \cdot n_A)n_A^i - v_A^i], \tag{15}$$

while the full Lense-Thirring term is

$$\Omega^i_{\rm LT} = -\frac{1}{r_A^3}\sum_A J^i_{A({\rm eff})} - 3n^i_A(J_{A({\rm eff})}\cdot n_A)^i, \qquad (16)$$

which vanishes for static sources. As before, the CS correction has the effect of modifying the classical GR prediction via the replacement $J^i_A \to J^i_{A({\rm eff})}$.

If experiments [5] detect frame dragging and find it in agreement with the GR prediction, we can immediately test CS gravity. In order to place a bound, however, a careful analysis must be performed, using the tools developed in [23] and properly accounting for the experiment's frame. Nonetheless, we can construct a crude, order-of-magnitude estimate of the size of such a bound. In order to do so, we assume the Newtonian limit $\mathcal{O}(J) \sim \mathcal{O}(mRv)$, with m the total mass of the system, R the distance from the gyroscope to the gravitational source (for GPB, $R \sim$ 7000 km [23]), and v the orbital velocity, such that we can model the CS correction as $|\Omega| \sim \Omega^{\rm GR}(1 + \dot{f}/R)$ [24]. Then, a 1% accuracy in the frame-dragging measurement relative to the GR prediction (nominal for GPB [23]) translates, roughly, into the bound $\dot{f} \lesssim 10^{-3}$ s.

Let us conclude with a discussion of the scaling of the order of magnitude of the CS correction. From Eqs. (13) and (15), we can see that the CS correction is of $\mathcal{O}(3)$ if $\dot{f}/r_A$ is of order unity, which implies that it is enhanced in regions of high curvature, precisely where the PPN and post-Newtonian analysis does not hold. Such scaling also suggests that the CS effect might be larger in highly dynamical systems that are not weakly gravitating, such as compact object binaries. If frame dragging were measured in such systems to sufficient accuracy, then possibly a much better bound could be placed on CS gravity.

Conclusions.—We have calculated the weak-field expansion of CS gravity and solved the field equations in the PPN formalism. We have found that CS gravity has the same PPN parameters as GR, except for the inclusion of a new term in g_{0i}, which can be parametrized in terms of a new PPN quantity. We have seen that this new term leads to a correction to the frame-dragging effect, thus allowing for the first solar-system test of CS gravity.

The CS correction is clearly enhanced in the nonlinear regime, where the stress-energy tensor diverges. This regime, however, is precisely where the PN approximation and PPN framework break down. Therefore, an accurate analysis of the size of the CS correction relative to the GR prediction in the nonlinear regime will have to await full numerical simulations of modified GR.

We wish to thank Clifford Will, Roman Jackiw, Pablo Laguna, and Ben Owen for insightful discussions. This work was supported by NSF Grant No. PHY-05-55-628 and the Center for Gravitational Wave Physics, funded by the NSF via Grant No. PHY-01-14375.

Note added.—After submission of this work, a paper was submitted that expands on the analysis presented here to account for extended sources [25]. After a detailed study, a bound is placed on the CS coupling parameter with data from LAGEOS and GPB.

[1] S. H. S. Alexander, M. E. Peskin, and M. M. Sheik-Jabbari, Phys. Rev. Lett. **96**, 081301 (2006).
[2] J. Polchinski, *Superstring Theory and Beyond* String Theory Vol. 2 (Cambridge University Press, Cambridge, England, 1998).
[3] A. Ashtekar, A. P. Balachandran, and S. Jo, Int. J. Mod. Phys. A **4**, 1493 (1989).
[4] A. Lue, L.-M. Wang, and M. Kamionkowski, Phys. Rev. Lett. **83**, 1506 (1999).
[5] A discussion of the history, technology and physics of Gravity Probe B can be found at http://einstein.standford.edu.
[6] T. W. Murphy, Jr., K. Nordtvedt, and S. G. Turyshev, Phys. Rev. Lett. **98**, 071102 (2007).
[7] C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, England, 1993).
[8] C. M. Will, Phys. Rev. D **57**, 2061 (1998).
[9] C. M. Will and N. Yunes, Classical Quantum Gravity **21**, 4367 (2004).
[10] E. Berti, A. Buonanno, and C. M. Will, Phys. Rev. D **71**, 084025 (2005).
[11] R. Jackiw and S. Y. Pi, Phys. Rev. D **68**, 104012 (2003).
[12] D. Guarrera and A. J. Hariton, Phys. Rev. D **76**, 044011 (2007).
[13] This field can also be interpreted as a dynamical one, in which case it possesses equations of motion. These equations dynamically enforce the Pontryagin constraint [12].
[14] We here correct a mistake in [15], where the Levi-Civita tensor was used instead of the symbol, which in the linearized theory are equivalent up to terms of $\mathcal{O}(h)^2$.
[15] S. Alexander and N. Yunes, Phys. Rev. D **75**, 124022 (2007).
[16] S. Alexander and J. Martin, Phys. Rev. D **71**, 063526 (2005).
[17] R. J. Gleiser and C. N. Kozameh, Phys. Rev. D **64**, 083007 (2001).
[18] R. H. Brandenberger and C. Vafa, Nucl. Phys. **B316**, 391 (1989).
[19] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
[20] The full expansion of the Cotton tensor to second order in the metric perturbation is given in [15].
[21] L. Blanchet, Living Rev. Relativity **9**, 4 (2006), http://www.livingreviews.org/lrr-2006-4, and references therein.
[22] This gauge is defined by $\partial_\mu \tilde{h}^{\alpha\mu} = 0$, with the metric perturbation defined via $\tilde{h}^{\alpha\beta} = \eta^{\alpha\beta} - (-g)^{1/2} g^{\alpha\beta}$.
[23] C. M. Will, Phys. Rev. D **67**, 062003 (2003).
[24] This can be seen by inserting $\mathcal{O}(J) \sim \mathcal{O}(mRv)$ into Eq. (16), with the definition of the effective angular velocity given in Eq. (14), and factoring out $1 + \dot{f}/R$.
[25] T. L. Smith, A. L. Erickcek, R. R. Caldwell, and M. Kamionkowski, arXiv:0708.0001 [Phys. Rev. D (to be published)].