Cylindrical Invisibility Cloak with Simplified Material Parameters is Inherently Visible

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It was proposed that perfect invisibility cloaks can be constructed for hiding objects from electromagnetic illumination [J. B. Pendry, D. Schurig, and D. R. Smith, Science **312,** 1780 (2006)]. The cylindrical cloaks experimentally demonstrated [D. Schurig *et al.*, Science **314,** 977 (2006)] and theoretically proposed [W. Cai *et al.*, Nat. Photon. **1,** 224 (2007)] have however simplified material parameters in order to facilitate easier realization as well as to avoid infinities in optical constants. Here we show that the cylindrical cloaks with simplified material parameters inherently allow the zeroth-order cylindrical wave to pass through the cloak as if the cloak is made of a homogeneous isotropic medium, and thus visible. To all high-order cylindrical waves, our numerical simulation suggests that the simplified cloak inherits some properties of the ideal cloak, but finite scatterings exist.

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Recently there has been an increase of interest in realizing invisibility cloaks [1–8]. In particular, the coordinate transformation method proposed in [4] is noticed to be especially powerful for designing invisibility cloaks that can in principle completely shield enclosed objects from electromagnetic (EM) illumination and at the same time cause zero disturbance to the foreign EM field. In 2D case, the cylindrical cloak obtained through such technique has anisotropic, spatially varying optical constants. In addition, some of the material parameters have infinite values at the interior surface of the cloak. To facilitate experimental realization at microwave frequencies, Schurig *et al.* have simplified the material properties such that only one material parameter is gradient and the requirement on infinite material constant is lifted [9]. The authors claimed that, in comparison to the ideal cloak, the simplified cloak maintains the power-flow bending with the penalty of nonzero reflectance at the outer interface. Similar simplification on a cylindrical cloak at optical frequencies is also employed in Ref. [10]. In this Letter we provide a systematic theoretical study on simplified cylindrical cloaks. It is found that the bare device constructed using the simplified medium fails to be invisible. In addition, the device does not possess a spatial region that is completely in isolation from the outside world electromagnetically. Hence, perfect hiding with such a simplified cloak is not possible.

Consider a cylindrical cloak with its inner and outer boundaries positioned at $r = a$ and $r = b$, respectively. Both domains inside and outside the cloak are air. The structure is in general a three-layered cylindrical scatterer. We refer to the layers from inside to outside as layer 1, 2, and 3, respectively. Our analysis, similar to all previous works, is on normal incidence. That is, the **k** vector is perpendicular to the cloak cylinder axis. The EM wave is assumed to have TE polarization (i.e., the electric field only exists in the *z* direction). The TM polarization case can be derived by making $E \rightarrow -H$, $\varepsilon \rightarrow \mu$, and $\mu \rightarrow \varepsilon$ substitutions. By default, we choose the cloak's cylindrical coordinate as the global coordinate. In a homogeneous

material region (i.e., cloak interior and exterior) the general solution is expressible in Bessel functions. Within the cloak medium, the general wave equation that governs the E_z field can be written as

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{r}{\mu_\theta}\frac{\partial E_z}{\partial r}\right)\right] + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\mu_r}\frac{\partial E_z}{\partial \theta}\right) + k_0^2 \varepsilon_z E_z = 0, \quad (1)$$

where k_0 is the free-space wave number, μ_θ, μ_r and ε_r are polarization-dependent permeability or permittivity profiles of the cloak. The time dependence $\exp(i\omega t)$ has been used for deriving Eq. (1).

Ideally, the cloak is designed to compress all fields within a cylindrical air region $r < b$ into the cylindrical annular region $a < r < b$. A corresponding coordinate transformation leads to a set of anisotropic and *spatially variant* material constants in the cloak shell, as described in [9]. Such an annular cylinder can indeed provide perfect invisibility cloaking [11], but it requires infinite values of optical constants at the cloak's inner boundary. To circumvent the fabrication difficulty, the simplified parameters were used [9]. They are in the form of

$$\mu_r = \left(\frac{r-a}{r}\right)^2, \qquad \mu_\theta = 1, \qquad \varepsilon_z = \left(\frac{b}{b-a}\right)^2. \quad (2)$$

To achieve cloaking for TM waves, the same set of material parameters, but for ε_r, ε_θ and μ_z, is used [10]. However, we notice that the procedure for simplification of the material parameters adopted in [9] is questionable, as the derivation has assumed beforehand that μ_θ is a constant. It is obvious that the invariant μ_θ can be taken out of the differential operator in Eq. (1). Therefore, wave behavior within the cloak shell is altered as compared to that in an ideal cloak.

Since the material parameters in Eq. (2) are azimuthally invariant (which is also true for the ideal parameter set), we can use the variable separation $E_z = \Psi(r)\Theta(\theta)$. Equation (1) can then be decomposed into

$$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0, \tag{3}$$

$$\frac{d}{dr}\left(\frac{r}{\mu_\theta}\frac{d\Psi}{dr}\right) + k_0^2 r\varepsilon_z\Psi - m^2\frac{1}{r\mu_r}\Psi = 0, \tag{4}$$

where m is an integer. The solution to Eq. (3) is $\exp(im\theta)$.
Equation (4) is a second-order homogeneous differential
equation. Two independent solutions are expected. At this
moment, we assume the solution to Eq. (4), for a fixed m,
can be written in general as $\mathcal{A}_m Q_m + \mathcal{B}_m R_m$, where $\mathcal{A}_m$
and $\mathcal{B}_m$ are constants. Q_m and R_m are functions of r. Now
valid field solutions in three layers (denoted by super-
scripts) can be described as

$$E_z^1 = \sum_m \mathcal{A}_m^1 J_m(k_0 r)\exp(im\theta), \tag{5}$$

$$E_z^2 = \sum_m \{\mathcal{A}_m^2 Q_m + \mathcal{B}_m^2 R_m\}\exp(im\theta), \tag{6}$$

$$E_z^3 = \sum_m \{\mathcal{A}_m^3 J_m(k_0 r) + \mathcal{B}_m^3 H_m^{(2)}(k_0 r)\}\exp(im\theta), \tag{7}$$

$H_m^{(2)}$ is the Hankel function of the second kind, which
represents the outward-traveling cylindrical wave. The
J_m and $H_m^{(2)}$ terms in the 3rd layer are physically in corre-
spondence to the incident and scattered waves, respec-
tively. Hence, the scattering problem becomes to solve
for, most importantly, $\mathcal{A}_m^1$ (transmitted field) and $\mathcal{B}_m^3$
(scattered field) subject to a given incidence $\mathcal{A}_m^3$ [12].
Comparatively, $\mathcal{A}_m^2$ and $\mathcal{B}_m^2$ are physically less interesting.
The coefficients are solved by matching the tangential
fields (E_z and H_θ) at the layer interfaces. Because of the
orthogonality of the function $\exp(im\theta)$, the cylindrical
waves in different orders decouple. We hence can examine
the transmission and scattering of the cloak for each indi-
vidual order number m.

By substituting the simplified material parameters into
Eq. (4), we obtain

$$(r-a)^2\frac{d^2\Psi}{dr^2} + \frac{(r-a)^2}{r}\frac{d\Psi}{dr}$$
$$+ \left[(r-a)^2\left(\frac{b}{b-a}\right)^2 k_0^2 - m^2\right]\Psi = 0. \tag{8}$$

Equation (8) has two nonessential singularities at $r = 0$
and $r = a$ for $m \neq 0$. It is worthwhile to mention that, with
the ideal parameters, Eq. (4) can be written instead as

$$(r-a)^2\frac{d^2\Psi}{dr^2} + (r-a)\frac{d\Psi}{dr}$$
$$+ \left[(r-a)^2\left(\frac{b}{b-a}\right)^2 k_0^2 - m^2\right]\Psi = 0. \tag{9}$$

When $m = 0$, Eq. (8) can be further simplified to

$$r^2\frac{d^2\Psi}{dr^2} + r\frac{d\Psi}{dr} + r^2 k_0^2\left(\frac{b}{b-a}\right)^2\Psi = 0. \tag{10}$$

This is the zeroth-order Bessel differential equation. Its
nonessential singularity remains at $r = 0$. Equation (10)
suggests that an incoming zeroth-order cylindrical wave
would effectively see the simplified cloak as a homoge-
neous isotropic medium whose effective refractive index is
$n_{\text{eff}} = \frac{b}{b-a}$. Its transmission through the cloak shell is
therefore determined by the etalon effect of the finite
medium.

When $m \neq 0$, the wave solution is governed by Eq. (8).
By comparing Eqs. (8) and (9), we see that at radial
positions $r \gg a$, Eq. (8) asympototically resembles
Eq. (9) due to $r - a \approx r$. This indicates that all high-order
cylindrical waves tend to behave similarly in both media at
$r \gg a$ positions, and hence the importance of parameter b,
at which the cloak medium is truncated. In the following,
we will derive the scattering coefficient s_m subject to
individual cylindrical wave incidences. The scattering co-
efficient in each cylindrical order is defined as $s_m = |\mathcal{B}_m^3/\mathcal{A}_m^3|$. Analytic derivation of s_m can be done if two
solutions to Eq. (8) (i.e., Q_m and R_m) are known in closed
form. However, despite the analogueness to Eq. (9), Eq. (8)
fails the analytic Frobenius method [13]. Here we tackle
the problem through the finite-element method. The field
outside the cloak is computed numerically, and then is used
for deriving coefficients $\mathcal{A}_m^3$ and $\mathcal{B}_m^3$ through a fitting
procedure. s_m is known in turn. Solutions with different
azimuthal orders are obtained by varying the azimuthal
dependence of a circular current source outside the cloak.
The scattering problem is numerically manageable since
functionals $\varepsilon_z\mu_\theta$ and $\frac{\mu_\theta}{\mu_r}$ in Eq. (4), unlike in the ideal cloak
case, are both finite and do not possess any removable
singularity for the simplified medium. The commercial
software COMSOL is deployed to carry out calculations.
For our case study, we fix $a = 0.1$ m and operating fre-
quency $f = 2$ GHz. The performance of the cloak is ex-
amined as b is increased from 0.2 m. Similar parameters
are also found in [14].

The scattering coefficients in different azimuthal orders
as a function of b are shown in Fig. 1. As expected, the
zeroth-order scattering coefficient is quite distinct from
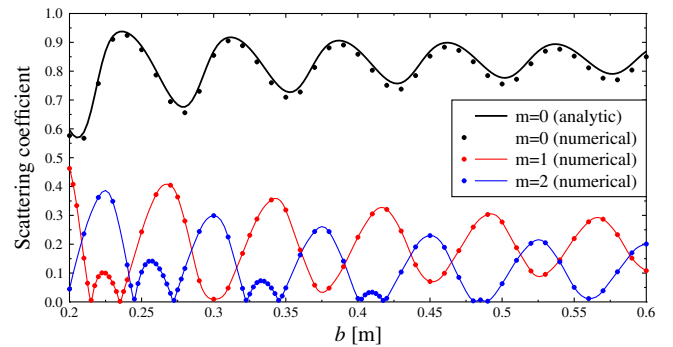

FIG. 1 (color online).   Variation of the scattering coefficients,
examined in each cylindrical wave order, as a function of b. For
$m = 1, 2$, the curves are fitted using Savitzky-Golay smoothing
filter according to numerically derived data points (dots).

others due to the different governing wave equation. In fact, analytic solution exists when $m = 0$, as field in the cloak medium are Bessel functions [11]. The excellent agreement between the numerical and analytic results for zeroth-order scattering coefficient confirms the validity and accuracy of our approach. Using the analytic technique, the zeroth-order scattering coefficient is found to converge to 0.867 with respect to b. Despite that the effective index of the cloak approaches to 1 as b increases, the phase variation of the zeroth-order wave within the cloak medium is increasing, as $k_0 \frac{b}{b-a}(b-a) = k_0 b$. This explains why the scattering coefficient converges to a value other than 0. The existence of zeroth-order scattering coefficient effectively disqualifies the cloak to be completely invisible.

Compared to the zeroth-order scattering coefficient, the high-order scattering coefficients (only those for $m = 1, 2$ are shown in Fig. 1) are noticed to be in a similar oscillatory fashion, and in general much smaller. The scattering coefficient tends to converge to a value closer to zero when the order number m increases. Over certain ranges of b value (e.g., around $b = 0.225$ m for $m = 1$), the computed $\mathcal{B}_m^3$ changes sign and hence the resulted scattering coefficient seems to be flipped from a negative value. We should attribute the relatively small high-order scattering coefficients to the cloak's partial inherence of the ideal cloak based on coordinate transformation [15]. However, our numerical result shows that the high-order scattering coefficients do not converge to zero even when the cloak wall is very thick.

Besides the requirement of zero scattering (invisibility), a device also need to possess a spatial region which is in complete EM isolation from the exterior world in order to be an invisibility cloak. Therefore it is meaningful to know how much field penetrates into the simplified cloak subject to a foreign EM illumination. Again, the problem is studied by examining the individual cylindrical wave components separately. The transmission coefficient, defined as $t_m = |\mathcal{A}_m^1/\mathcal{A}_m^3|$, is used to characterize the field transmission. When $m = 0$, the amount of field transmitted into the cloak interior can be analytically derived, which is shown in Fig. 2 as a function of b. The transmission is noticed to be oscillatory, and converging to 1 as b increases. Numerical calculation is also superimposed for validation. When $m \neq 0$, the FEM calculations show that the field inside the cloak is almost zero. The corresponding transmission coefficients are exclusively smaller than 0.005 and, hence, are not plotted in Fig. 2. This indicates that the contour $r = a$ provides an insulation between its enclosed domain and the exterior domain, but only for all high-order cylindrical waves. Therefore, any objects placed inside the cloak are exposed to the zeroth-order cylindrical wave component. Reversely, the zeroth-order wave component of an EM source placed within the cloak (or scattered wave by objects inside the cloak) will transmit out. As a result, objects enclosed by a simplified cloak is sensible by a foreign detection unit.
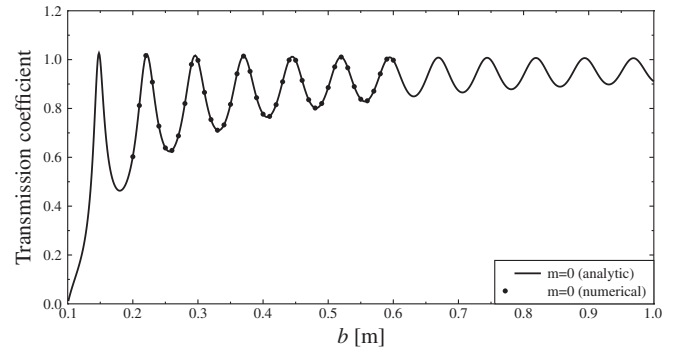

FIG. 2. The zeroth-order transmission coefficient.

Next, we numerically demonstrate (in COMSOL) scattering by the simplified cloaks with a plane-wave incidence. The incident plane wave travels from left to right and has the amplitude of 1. When $b = 0.2$ m, the E_z snapshot, E_z norm, and the scattered E_z snapshot are plotted in Fig. 3(a1)–(a3), respectively. It is noticed that the amplitude of the scattered field is about one half of the incident field. From the scattered field distribution, high-order Bessel terms constitute a significant portion. Scattering aside, the field inside the cloak shell is seen to have only zeroth-order Bessel term. We then increase b to 0.5 m, and the corresponding numerical results are plotted in Fig. 3(b1)–(b3). Compared to the previous case, scattered field is reduced roughly by half in amplitude, and is now dominated by the zeroth-order Bessel term. The E_z norm is closer to be uniform outside the cloak, indicating better invisibility. The overall smaller scattering as well as the dominance of the zeroth-order scattering for the second cloak agree well with our derivation of the scattering coefficients in Fig. 1. When b is changed from 0.2 to 0.5 m, the transmitted E_z field at the center of the cloak increases from 0.6032 to 0.8855 in norm, also in agreement with Fig. 2.

For completeness, we also show the scattering by a near-to-ideal cloak in Fig. 3(c1)–(c3). The material parameters of the cloak are denned by $\mu_r = \frac{r-a}{r}$, $\mu_\theta = \frac{r}{r-a}$, and $\varepsilon_z = (\frac{b}{b-a})^2 \frac{r-a}{r}$, with $a = 0.1$ m and $b = 0.5$ m. To avoid the $r = 0.1$ m critical contour, we let the inner boundary of the cloak be positioned at $r = 0.101$ m. In [11] we have confirmed analytically that the performance of an ideal cloak is extremely sensitive to the position of the cloak's inner surface. In particular, the zeroth-order cylindrical wave will experience considerable reflection and transmission as it meets the cloak shell. This is confirmed in Fig. 3(c1)–(c3). The scattered field is almost purely the zeroth-order cylindrical wave. Even at such a small mislocation of the inner boundary, the amount of field leaking into the cloak is handsome, valued at 0.5466 in norm. By comparing panels (c1)–(c3) and (b1)–(b3) in Fig. 3, we see that the improved cloak made from simplified medium inherits some merits of the near-to-ideal cloak, but with overall higher scattering, especially in the monopole wave component.
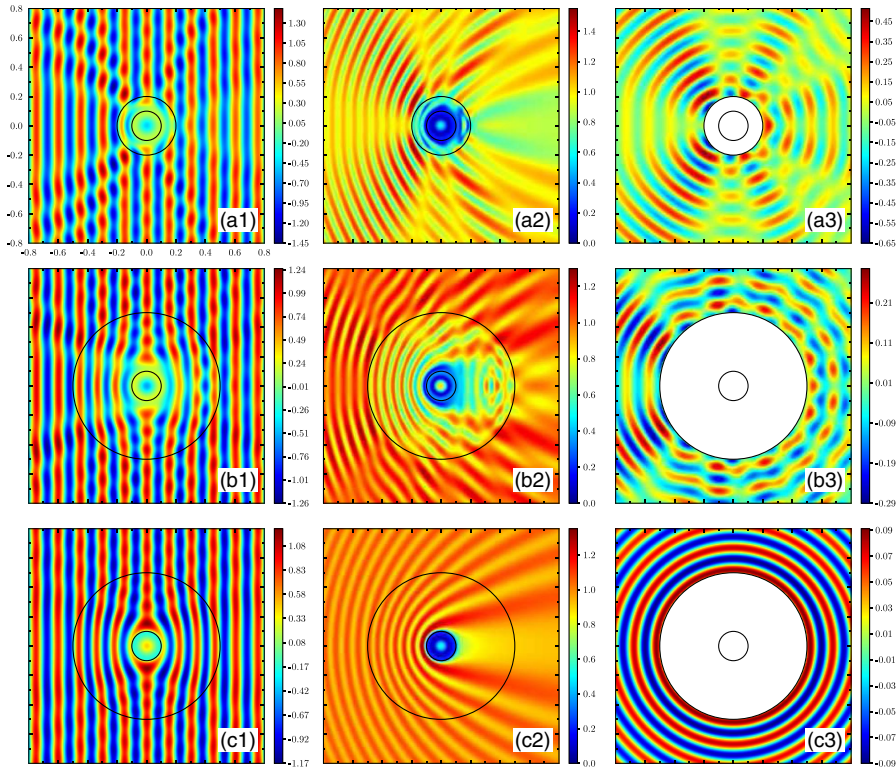
FIG. 3 (color online). (a1)–(a3) E_z snapshot, E_z norm, and scattered E_z field, respectively, for a simplified cloak with $a = 0.1$ m and $b = 0.2$ m. (b1)–(b3) Same fields but for a simplified cloak with $a = 0.1$ m and $b = 0.5$ m. (c1)–(c3) Same fields but for a near-to-ideal cloak with $a = 0.101$ m and $b = 0.5$ m.

In conclusion, we have theoretically studied the cylindrical cloaks with simplified material parameters. Such a simplified cloak is shown to inherit some properties of an ideal cloak based on coordinate transformation of Maxwell equations. However the penalty of using the simplified cloak is more than just nonzero reflectance at the cloak boundary. The monopole component of the incoming wave will always experience relatively high scattering. High-order cylindrical waves also experience finite (although smaller) scattering even when the cloak's wall is kept very thick, as suggested by our numerical simulation. Besides that the device itself is visible, it cannot completely shield EM field, due to penetration of the monopole field component. Hence cloaking of objects will not be perfect. Lastly, considering that the monopole field treats a simplified cloak as a conventional glass tube, detection of any object placed within the cloak can be greatly enhanced by using a EM source with the maximum energy on its monopole component.

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