Brownian Gyrator: A Minimal Heat Engine on the Nanoscale

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(Received 23 August 2007; published 5 December 2007)

A Brownian particle moving in the vicinity of a generic potential minimum under the influence of dissipation and thermal noise from two different heat baths is shown to act as a minimal heat engine, generating a systematic torque onto the physical object at the origin of the potential and an opposite torque onto the medium generating the dissipation.

DOI: 10.1103/PhysRevLett.99.230602 PACS numbers: 05.40.−a, 05.60.−k, 05.70.Ln

Introduction and summary.—The theory of heat engines is one of the main roots of modern thermodynamics and statistical physics. Recently, there has been a considerable renewed interest in the long-standing problem of determining the fundamental efficiency limit of a heat engine at maximum power [1]. Another exciting new perspective is the conceptual design of a cooling device on the nano- or even single-molecule scale by inverting a heat engine which is powered by Brownian motion [2]. The original and still paradigmatic setup of a heat engine consists of two heat baths in contact with a cyclically working "engine," generating work in the form of a torque. Here, we put forward the smallest and most primitive such engine one may think of: a single particle, gyrating around a generic potential energy minimum under the influence of friction and thermal noise forces from two simultaneously acting heat baths. Three typical examples are indicated in Fig. 1. While the particle itself is way too small to store or disburse any notable amount of angular momentum, it acts as a kind of catalyst. The particle in fact generates, with the help of the disequilibrium between the two baths, a systematic average torque onto the physical object at the origin of the potential and an opposite torque of the same magnitude by way of the dissipation mechanism onto one or both heat baths.

Such Brownian gyrators are minimal heat engines in so far as they are acting essentially like their macroscopic counterparts, but the engine itself does not need to be anything more than a structureless particle performing Brownian motion. Because of this simplicity they are, in principle, readily realizable by nanotechnological or even single-molecule techniques. Potential applications are manifold and obvious: wrapping and unwrapping of DNA and other polymers [3], driving wheels or screws of nanodevices [2] and synthetic molecular motors [4], stirring and mixing in micro- and nanofluidic devices [5], to name but a few. Though each single engine is "weak," when acting in parallel, a case naturally arising in the context of colloidal particles [6,7] or magnetic fluxons [8], the result may even be a macroscopic torque [9].

Model.—Focusing on the simplest case, we consider the motion of a point particle with two spatial degrees of freedom $\mathbf{x} := (x_1, x_2)$ in a static potential $U(\mathbf{x})$. We assume that the potential has a minimum at the origin $\mathbf{x} = \mathbf{0}$ and that large excursions are sufficiently rare to admit a parabolic approximation of the form

$$U(\mathbf{x}) = \sum_{i=1}^{2} \frac{u_i}{2} y_i^2, \qquad y_i := \sum_{k=1}^{2} O_{ik}(\alpha) x_k, \tag{1}$$

where $O(\alpha)$ is a 2×2 orthogonal matrix with elements $O_{11} = O_{22} = \cos\alpha$, $O_{12} = -O_{21} = \sin\alpha$, describing a rotation in the plane by an angle α. The transformed coordinates y_i thus correspond to the "principal axes" of the parabolic potential in (1) and u_1, $u_2 > 0$ are the corresponding "principal curvatures." Clearly, Eq. (1) represents the generic form of a potential minimum in 2 dimensions. Furthermore, one has $u_1 \neq u_2$ in the generic case, i.e., unless the system is rotationally symmetric about the origin.

To complete our heat engine, we need two heat baths which act onto the particle without resulting in a total equilibrium system. In the most common case, the heat engine is alternatingly brought into contact with two baths at two different temperatures. Since this requires a quite complicated machinery in practice, here we rather focus on the case that the particle is permanently in contact with both baths. In the following, we first discuss in detail the conceptually simplest theoretical model and only afterwards turn to the experimental realizations and more general system classes.

In the simplest case, at least one of the two heat baths interacts with the particle along a preferential direction, which can be identified with the x_1 axis without loss of generality. Generically, this direction is not related in any particular way to the principal axes of the potential in (1), and hence their relative angle α is not bound to take any particular value *a priori*.

Whereas one heat bath thus solely influences the particle motion along the x_1 axis, the second one may either be of isotropic character or acting only on x_2. For the sake of

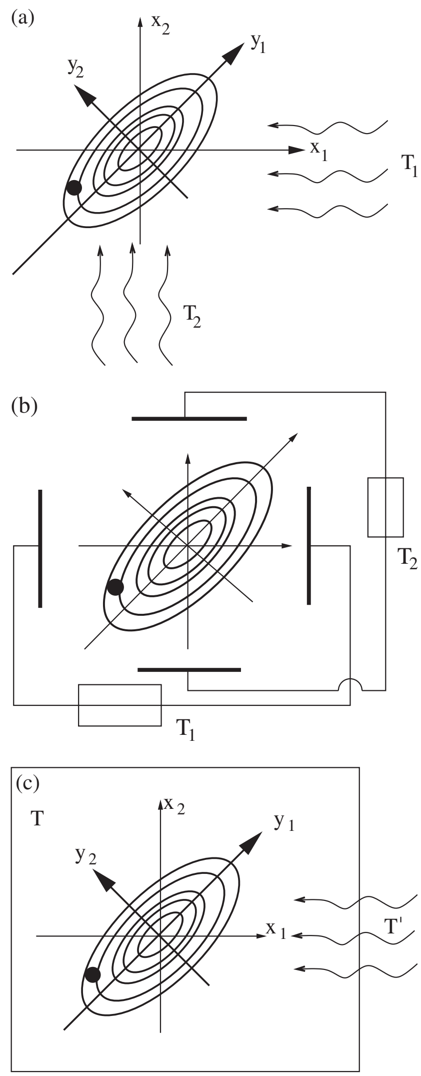

FIG. 1. Schematic sketch of various types of Brownian gyrators acting as heat engines. The black dot represents the Brownian particle, moving in the x_1-x_2 plane. The contour lines indicate a typical parabolic potential (1) with principal axes y_1 and y_2. (a)–(c) illustrate different realizations of the two heat baths. (a) Both heat baths act on the charged particle by way of blackbody radiation at different temperatures T_1 and T_2, irradiating along the x_1 and x_2 axes, respectively. The dissipation mechanism is provided by radiation damping into the vacuum. (b) "Electrical heat baths," realized by two resistors at different temperatures T_1 and T_2, coupled to the charged particle by means of two plate condensers. Each of them transfers the random voltage fluctuations of one resistor to the particle along a preferential direction and gives rise to dissipation via the resistor when the particle moves and hence induces a current in the electrical circuit. Replacing the condenser plates by Helmholtz coils gives rise to "magnetic baths" interacting with, e.g., a paramagnetic particle [9]. Replacing the condenser plates by piezo elements gives rise to "acoustomechanical" baths [14]. (c) Only one heat bath (with temperature T') is of the anisotropic type as in (a) and (b). The second heat bath (with temperature T) consists of the usual fluid environment of the Brownian particle.

simplicity, we first focus on the latter case; see Figs. 1(a) and 1(b). Modeling the thermal bath effects as usual [10,11] by Gaussian white noise and a concomitant dissipation proportional to the velocity, and neglecting inertia effects, we arrive at the following overdamped Langevin equations for the particle dynamics in the plane:

$$\eta_i \dot{x}_i(t) = -\frac{\partial U(\mathbf{x}(t))}{\partial x_i} + \sqrt{2\eta_i k_B T_i}\,\xi_i(t), \qquad i = 1, 2. \tag{2}$$

Here, k_B is Boltzmann's constant, T_i is the temperature of the ith bath, $\xi_i(t)$ are independent, δ-correlated Gaussian noises, and the coupling strength between particle and bath i is quantified by the friction coefficient η_i [10,11].

Solution.—We first discuss in some more detail the forces and torques connected with the dynamics (2). Denoting by $\mathbf{e}_i$ the unit vector along the ith coordinate axis, the three relevant forces are the dissipation $\mathbf{f}_\eta(t) := -\sum_{i=1}^2 \mathbf{e}_i \eta_i \dot{x}_i(t)$, the potential force $\mathbf{f}_U(t) := -\nabla U(\mathbf{x}(t))$, and the fluctuation force $\mathbf{f}_\xi(t) := \sum_{i=1}^2 \mathbf{e}_i \sqrt{2\eta_i k_B T_i}\,\xi_i(t)$. Hence, (2) is tantamount to the force balance $\mathbf{f}_\eta(t) + \mathbf{f}_U(t) + \mathbf{f}_\xi(t) = \mathbf{0}$. On the average over many realizations of the noise we thus obtain $\langle \mathbf{f}_\xi(t)\rangle = \mathbf{0}$ and hence $\langle \mathbf{f}_U(t)\rangle = -\langle \mathbf{f}_\eta(t)\rangle$, corresponding to the following elementary physics: Since the particle momentum is by definition considered as negligible in the overdamped limit, the force exerted by the potential U is compensated on the average by the friction forces. A similar consideration applies to the three torques of the form $\mathbf{f} \times \mathbf{x}$: The thermal fluctuations do not give rise to any systematic torque, $\langle \mathbf{f}_\xi(t) \times \mathbf{x}(t)\rangle = \mathbf{0}$, and hence the torque of modulus $M(t)$ and direction $\mathbf{e}_3 := \mathbf{e}_1 \times \mathbf{e}_2$ which the particle exerts on the potential U (or the physical object at the origin of that potential) is on the average exactly equal to the opposite torque $-M(t)\mathbf{e}_3$ which the particle exerts via the friction forces on the thermal environment (or the physical objects containing the baths):

$$\langle \mathbf{f}_U(t) \times \mathbf{x}(t)\rangle = -\langle \mathbf{f}_\eta(t) \times \mathbf{x}(t)\rangle = M(t)\mathbf{e}_3. \tag{3}$$

Next we turn to the Fokker-Planck equation [10,11] equivalent to (2),

$$\frac{\partial P(\mathbf{x}, t)}{\partial t} = -\sum_{i=1}^{2} \frac{\partial J_i(\mathbf{x}(t), t)}{\partial x_i}, \tag{4}$$

where $P(\mathbf{x}, t)$ is the probability density to find the particle at position $\mathbf{x}$ at time t and $\mathbf{J} = (J_1, J_2)$ the corresponding probability current density with components

$$J_i(\mathbf{x}, t) = -\left[\frac{1}{\eta_i}\frac{\partial U(\mathbf{x})}{\partial x_i} + \frac{k_B T_i}{\eta_i}\frac{\partial}{\partial x_i}\right] P(\mathbf{x}, t). \tag{5}$$

The Fokker-Planck equation (4) and (5) is complemented by natural boundary conditions $P(\mathbf{x}, t) \to 0$ and $J_i(\mathbf{x}, t) \to 0$ for $x_i \to \pm\infty$. Given $P(\mathbf{x}, t)$, the torque modulus $M(t)$ from (3) readily follows according to

$$M(t) = \int P(\mathbf{x}, t)\left(x_1 \frac{\partial U(\mathbf{x})}{\partial x_2} - x_2 \frac{\partial U(\mathbf{x})}{\partial x_1}\right) dx_1 dx_2. \quad (6)$$

After initial transients have died out, the system approaches a unique, steady probability density as $t \to \infty$ [10–12]. For the parabolic potential (1), this unique steady state solution of the Fokker-Planck equation (4) and (5) can be obtained in closed analytical form. Since the calculations are straightforward but rather tedious and the expressions quite bulky and not very illuminating, they are not explicitly given here. Rather we immediately present the resulting torque (6) in the steady state, reading

$$M = \frac{k_B(T_1 - T_2)(u_1 - u_2)\sin 2\alpha}{u_1 + u_2 - \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}(u_1 - u_2)\cos 2\alpha}. \quad (7)$$

Note that for symmetry reasons, any value of the potential force $\mathbf{f}_U(t)$ occurs with the same probability as its inverse in the steady state. On the average, we thus have $\langle \mathbf{f}_U(t) \rangle = \mathbf{0}$, implying as usual that the resulting torque (7) remains unchanged for any other choice of the reference rotation axis in (3).

Discussion.—The closed, general expression (7) for the average torque of a Brownian gyrator in the steady state is the main result of our Letter. For $T_1 = T_2$ we are dealing with an equilibrium system in (2) and hence the average torque must vanish due to the second law of thermodynamics. If $u_1 = u_2$ the potential (1) is rotationally symmetric and hence there cannot be any preferential direction of rotation and the torque must vanish. If $\sin 2\alpha = 0$ then the principal axes of the parabolic potential agree with the directions along which the two heat baths in (2) are acting; hence a net torque is again ruled out by symmetry. In any other case a finite torque (7) results (the denominator is always positive since u_i, $\eta_i > 0$). A particularly simple behavior arises for equal coupling strengths $\eta_1 = \eta_2$. In this case, the maximal torque is reached at $\alpha = \pi/2 + n\pi$. In general, the optimal angles will be slightly different. Typically, the maximal torque is roughly given by the difference between the thermal energies associated with the two baths, $k_B T_1 - k_B T_2$, indicating that the Brownian gyrator transforms its random motion into a systematic torque quite effectively. We, however, remark that speaking about efficiencies in the usual sense is not possible as long as one does not know the resulting rotation speed of the thermal baths relative to the ''carrier'' of the potential U, which is beyond the scope of the general model (2).

The basic physical origin of the preferential rotation of the Brownian particle in one direction can be most easily understood in the limit that one temperature vanishes, say $T_2 = 0$ in (2). Now, let us assume the particle has reached (in whatever way) a position on the x_1 axis; i.e., $x_2(t) = 0$. In the generic case that the x_1 axis does not coincide with a principal axis of the parabolic potential in (1), there will be a nonvanishing deterministic force $-\partial U(x_1(t), 0)/\partial x_2$ proportional to $x_1(t)$ acting on the particle along the x_2 direction. Since the noise along this direction vanishes in (2), we can conclude that the particle is able to cross the x_1 axis only in one direction for all positive $x_1(t)$ values and in the opposite direction for all negative $x_1(t)$ values. In other words, the particle rotates around the origin in a preferential direction. It is plausible that qualitatively an analogous behavior is expected also for finite T_2 (different form T_1), though the details will be more complicated.

Some of the above basic physical principles governing the behavior of a Brownian gyrator are similar to those governing so-called thermal ratchets and Brownian motors [11]. Yet, to the best of our knowledge, neither of those ratchet systems is immediately comparable to the setup treated here, nor are the general concepts in the context of ratchet effects of any help to gain easier or deeper insight in the present case.

Experimental realizations.—The setup from Fig. 1(a) involving anisotropic blackbody radiation is of considerable conceptual interest. The main practical problems are the weak coupling of a charged particle to the electromagnetic irradiation and the vacuum, and that the model (2) itself is a very crude description of the real system. Yet, the basic concept may well be of relevance for various dynamical processes in astrophysics, space physics, and laboratory plasmas; see [13] and references therein. More easy to realize in the lab is the setup from Fig. 1(b), whose heat baths consist of simple resistors at different temperatures. If the so generated thermal fluctuations are still too weak, an electronic amplification is straightforward [14]. The shortcoming of this setup is that a conversion of the torque into a relative rotation between potential and baths is not desirable, since that would change the angle α in (1).

Most attractive from the experimental viewpoint seems to be the setup from Fig. 1(c): One heat bath is given—as usual in the context of Brownian motion—by a fluid environment of the particle without any kind of anisotropy. For later use, we denote its temperature by T and the Stokes friction coefficient by η. They both appear as usual [10,11] in both components of the 2-dimensional dynamics (2). Only the second bath continues to emit its thermal fluctuations along a preferred direction onto the particle, say along the x_1 dynamics. Paradigmatic examples are [15] the (almost) blackbody irradiation from the sun or some analogous artificial device in the lab or any kind of anisotropic (almost) white noise in the original mechanoacoustical sense, emitted, e.g., by a loudspeaker in the experimental work [14]. Denoting its temperature and dissipation coefficient by T' and η', respectively, the resulting 2-dimensional dynamics is again of the overdamped Langevin type and can be readily brought into the form (2) by means of the identifications $\eta_2 = \eta$, $T_2 = T$, $\eta_1 = \eta + \eta'$, and $T_2 = (\eta T + \eta' T')/(\eta + \eta')$. The resulting torque (7) thus takes the form

$$M = \frac{\frac{\eta'}{\eta+\eta'} k_B (T' - T)(u_1 - u_2)\sin 2\alpha}{u_1 + u_2 - \frac{\eta'}{2\eta+\eta'}(u_1 - u_2)\cos 2\alpha}. \tag{8}$$

Typically, the coupling strength η to the fluid will be very much larger than the dissipation coefficient η' due to the anisotropic second heat bath. In order that the relevant strength

$$g := k_B T' \eta' \tag{9}$$

of the concomitant anisotropic fluctuations is non-negligible in spite of the small η' value, the temperature T' must be very much larger than T. Then the resulting torque from (8) simplifies in very good approximation to

$$M = \frac{g}{\eta} \frac{u_1 - u_2}{u_1 + u_2} \sin 2\alpha. \tag{10}$$

Here, any nonzero value of M indicates that the particle is transferring torque from the potential U to the dissipative mechanism, while its own angular momentum always remains negligible. The effect of that pair of opposite torques will be to generate rotations of the physical object carrying the potential and the fluid around the particle into opposite directions, just in the way any engine is commonly supposed to operate.

The experimental realization of the potential U is possible in many straightforward ways, e.g., by means of electro- or magnetostatic forces [6], dielectrophoretic effects (including light forces as exploited in optical traps) [7], pinning centers of fluxons [8], etc.

Outlook.—We close with some generalizations and perspectives. First of all, basically the same behavior is expected when working in 3 rather than 2 dimensions.

It is more challenging to understand better the role of the anisotropy of at least one of the heat baths with the main goal of possibly abandoning this condition. The first purpose of this anisotropy is of a mainly technical character. Namely, within the usual modeling of thermal fluctuations as Gaussian white noise and the concomitant dissipation proportional to the instantaneous velocity, see (2), two isotropic baths are effectively equivalent to one single equilibrium environment with properly adapted effective friction and temperature. Hence the anisotropy is indispensable in order to obtain a nonequilibrium system within this standard modeling of the thermal baths. However, there are, in principle, many possibilities—some of mainly conceptual interest, others of practical relevance but mathematically more difficult to handle—to model a thermal bath in a different way, e.g., by means of correlated Gaussian noise and concomitant memory friction [11]. Two of these baths at different temperatures can no longer be mathematically transformed to one single equilibrium bath and hence in this regard the anisotropy is no longer needed. A second more fundamental role of the anisotropy is to break the symmetry between gyrating clock- and counterclockwise. Clearly, breaking this symmetry is an indispensable prerequisite of making the Brownian gyrator work. Hence, in the presence of two isotropic heat baths, this symmetry must be broken in some other way. The most straightforward possibility is via the potential U, e.g., by keeping terms up to cubic order in the expansion (1) of the potential about its minimum.

An interesting extension of the present work will be to explore the collective phenomena due to many interacting Brownian gyrators, in particular, the similarities and differences as compared to collective effects of rotating molecular motors in membranes [16] and rotating magnetic disks confined to a two-dimensional interface [17].

This work was supported by the Alexander von Humboldt Stiftung and the Deutsche Forschungsgemeinschaft under SFB Contract No. 613 and RE Contract No. 1344/3-1.

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