Mobility of Discrete Solitons in Quadratically Nonlinear Media

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We study the mobility of solitons in lattices with quadratic ($\chi^{(2)}$, alias second-harmonic-generating) nonlinearity. Using the notion of the Peierls-Nabarro potential and systematic numerical simulations, we demonstrate that, in contrast with their cubic ($\chi^{(3)}$) counterparts, the discrete quadratic solitons are mobile not only in the one-dimensional (1D) setting, but also in two dimensions (2D), in *any* direction. We identify parametric regions where an initial kick applied to a soliton leads to three possible outcomes: staying put, persistent motion, or destruction. On the 2D lattice, the solitons survive the largest kick and attain the largest speed along the diagonal direction.

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Introduction.—In the past several years, tremendous progress has been made in studies of nonlinear dynamics in lattices [1]. To a large extent, this development was driven by direct physical applications, such as optical beams in waveguide arrays [2], Bose-Einstein condensates in deep optical lattices [3], transformations of the DNA double strand [4], and others.

A ubiquitous dynamical-lattice model is represented by the discrete nonlinear Schrödinger equation [1,2,5] with the cubic ($\chi^{(3)}$) nonlinearity. It has been used to model a variety of experimental settings, where it demonstrates the formation and interactions of discrete solitons and localized vortices [6], lattice modulational instability, buildup of the Peierls-Nabarro (PN) barrier impeding the motion of solitons, diffraction management, etc. [7].

Substantial activity has also been aimed at lattices with the quadratic ($\chi^{(2)}$) nonlinearity, which was originally introduced to describe the Fermi-resonance interface modes in multilayered systems based on organic crystals [8]. The interest in $\chi^{(2)}$ lattices was recently boosted by the experimental realization of discrete $\chi^{(2)}$ solitons in nonlinear optics [9]. A variety of topics have been studied in this context both theoretically and experimentally, including the formation of 1D and 2D solitons [10,11] (see also reviews [12]), modulational instability in waveguide arrays [13], few-site lattices [14], $\chi^{(2)}$ photonic crystals [15], cavity solitons [16], and multicolor localized modes [17].

A fundamental difference of $\chi^{(2)}$ continua from their $\chi^{(3)}$ counterparts is that they feature no collapse in 2D and 3D cases [18], which paves the way to create stable 2D [19] and 3D [20] quadratic solitons. On the other hand, due to the presence of collapse in 2D and 3D $\chi^{(3)}$ continua, lattice solitons may only exist with a norm exceeding a certain threshold [21], and are stable only if strongly localized (on a few lattice sites). Therefore, stable discrete 2D and 3D $\chi^{(3)}$ solitons are strongly pinned to the lattice and cannot move across it [22].

The absence of the trend to the catastrophic self-compression in the 2D $\chi^{(2)}$ medium suggests that the corresponding lattice solitons may be broad and therefore *mobile*, being loosely bound to the lattice. The aim of this work is to investigate the mobility of 1D and, especially, 2D solitons in $\chi^{(2)}$ lattices. Besides potential applications to photonics, the topic presents a fundamental interest, revealing a family of *mobile solitons* in 2D lattices. Thus far, the only example of mobility was provided by solitons in a 2D lattice with saturable nonlinearity [22] (the *Vinetskii-Kukhtarev* model [23], in which the mobility of 1D solitons was examined in Ref. [24]). Experimentally, the 1D mobility of strongly anisotropic 2D gap solitons was observed in a continuous photorefractive medium with a square photonic lattice [25].

In this work, we identify parametric regions admitting persistent motion of $\chi^{(2)}$ solitons on 1D and 2D lattices, and we investigate the *anisotropy* of the 2D soliton mobility. First, we introduce the model and discuss its PN barrier. Then, systematic numerical results for the soliton mobility in 1D and 2D lattices are reported.

Theoretical setup.—Following Ref. [11], we introduce a system of equations for the fundamental-frequency (FF) and second-harmonic (SH) waves, $\psi_{m,n}(t)$ and $\phi_{m,n}(t)$, on the 2D lattice:

$$i\frac{d}{dt}\psi_{m,n} = -(C_1\Delta_2\psi_{m,n} + \psi^{\star}_{m,n}\phi_{m,n}), \qquad (1)$$

$$i\frac{d}{dt}\phi_{m,n} = -\frac{1}{2}(C_2\Delta_2\phi_{m,n} + \psi^2_{m,n} + k\phi_{m,n}), \qquad (2)$$

where $\Delta_2\psi_{m,n} \equiv \psi_{m+1,n} + \psi_{m-1,n} + \psi_{m,n+1} + \psi_{m,n-1} - 4\psi_{mn}$, C_1 and C_2 are the (real) FF and SH lattice-coupling constants, and k is the mismatch parameter. Equations (1) and (2) conserve the Hamiltonian,

$$H = \sum_{m,n} C_1(|\psi_{m+1,n} - \psi_{m,n}|^2 + |\psi_{m,n+1} - \psi_{m,n}|^2) + (C_2/2)(|\phi_{m+1,n} - \phi_{m,n}|^2 + |\phi_{m,n+1} - \phi_{m,n}|^2) - (1/2)[\psi_{m,n}^2 \phi_{m,n}^\star + (\psi_{m,n}^\star)^2 \phi_{m,n} + k\phi_{m,n}^2], \quad (3)$$

and the norm (Manley-Rowe invariant, i.e., total power), $I = \sum_{m,n}(|\psi_{m,n}|^2 + 2|\phi_{m,n}|^2)$.

Stationary solutions of Eqs. (1) and (2) are of the form $\{\psi_{m,n}(t), \phi_{m,n}(t)\} = \{e^{-i\omega t}u_{m,n}, e^{-2i\omega t}v_{m,n}\}$, with $u_{m,n}$, $v_{m,n}$ real for fundamental solitons, and complex for other patterns, such as vortices [11]. To set the discrete solitons in motion, one must overcome the *PN barrier*, i.e., the energy difference between static solitons which are centered on a lattice site and between sites. This can be defined in two ways: either as the difference in the Hamiltonian (3) for *fixed norm* (see, e.g., [24] and references therein), or, according to Ref. [26], as the difference in the *free energy*, $F = H - \omega I$, between intersite and on-site solutions, for a *fixed* ω.

A lengthy asymptotic analysis demonstrates that for large values of the coupling constants (quasicontinuum regime) the dominant dependence of the above-defined free-energy difference W_0 on $C \equiv C_1 = C_2$ (in the most relevant case of equal FF and SH coupling constants) is of the form $W_0 = aC\exp(-bC)$, where a and b are constants for given ω and k (we do not present a formal derivation of this rather general formula, which is, instead, supported below by comparison with numerical results). Then, in the 1D case, the effective PN potential for the coordinate ξ of the soliton's center of mass is $F = (W_0/2)\cos(2\pi\xi)$. In the 2D case, the PN potential may be approximated fairly accurately by the combination of the two 1D terms, $F = (W_0/2)[\cos(2\pi\xi) + \cos(2\pi\eta)]$. A sufficiently strong kick applied to the discrete soliton should allow it to overcome the pinning PN barrier and set in motion through the lattice. Accordingly, we begin the presentation of our simulations by identifying the dependence of the effective barrier on the lattice discreteness, and then examine the strength of the kick necessary to overcome it and initiate persistent motion of the soliton.

Numerical Results.—We examine the dependence of the free-energy difference on the lattice discreteness, for a typical set of parameter values, e.g., $\omega = -0.25$, $k = 0.25$. The free-energy barrier, ΔF, between intersite and on-site positions of the 2D discrete soliton (shifted in one lattice direction) is shown as a function of the coupling strength C in the left panel of Fig. 1. The dashed line shows the best fit of this dependence to the above asymptotic formula, showing that the latter remains *quantitatively correct* even for very discrete lattices, up to $C \simeq 0.3$. The right panel additionally verifies the accuracy of the formula for the 2D PN potential, by displaying the free-energy difference ΔF for the soliton placed intersite in both directions (i.e., along the diagonal) with twice the value of ΔF for the respective shift in one direction only.

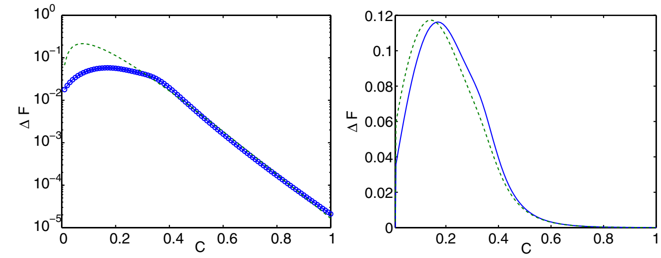

FIG. 1 (color online). The left panel shows the dependence on the coupling strength of the free-energy difference ΔF between intersite and on-site positions of the 2D discrete soliton (shifted in one lattice direction), for $\omega = -0.25$ and $k = 0.25$. The dashed line is the best fit, for large C, to the analytical prediction, $aC\exp(-bC)$, see text. The right panel shows the comparison of ΔF and $2\Delta F$ for the soliton set intersite, respectively, in both directions (solid line), and in one direction only (dashed line).

We now turn to the motion of the soliton. In both 1D and 2D cases, we used lattices with periodic boundary conditions to allow indefinitely long progressive motion. Dynamical simulations were initialized by application of a kick to the numerically exact stationary solitons, $\{u_{m,n}^{(0)}, v_{m,n}^{(0)}\}$, i.e., considering $\{\psi_{m,n}, \phi_{m,n}\} = \exp[i(S/C_{1,2})(m\cos\theta + n\sin\theta)]\{u_{m,n}^{(0)}, v_{m,n}^{(0)}\}$, where S and θ determine the size and (in the 2D case) orientation of the kick.

Examples of persistent motion and destruction of the 1D lattice soliton, induced by a moderate and strong kick, respectively, are displayed in Fig. 2. Systematic results, produced by variation of S and $C = C_1 = C_2$, are summarized in Fig. 3. Destruction of the soliton was registered if it eventually lost more than 30% of the initial norm. For the coupling strengths corresponding to Fig. 2, the initial kicks of different sizes S give rise to two outcomes only: stable motion or destruction. However, for weaker couplings (stronger discreteness), "localization" is also possible:

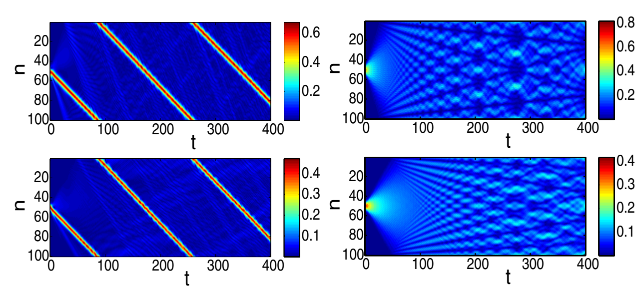

FIG. 2 (color online). Space-time contour plots of $|\psi_{m,n}|^2$ and $|\phi_{m,n}|^2$ for the FF and SH fields (top and bottom panels) in the 1D lattice with periodic boundary conditions, for $C_1 = C_2 = 1$, $\omega = -0.25$, $k = 0.25$. The kick, $S = 0.4$ or 3.0 (left and right parts), sets the soliton in stable motion, or destroys it, respectively.

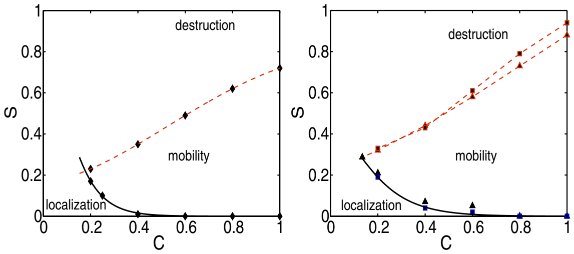

FIG. 3 (color online). A diagram in the plane of the coupling strength, $C = C_1 = C_2$, and kick size, S, showing outcomes of kicking the quiescent soliton in the 1D (left panel) and 2D (right panel) lattice, for $\omega = -0.25$ and $k = 0.25$. Solid lines show exponential fitting curves, $S = 6.02389\sqrt{C_2}\exp(-13.7684C_2)$ and $S = 2.56619\sqrt{C_2}\exp(-9.0043C_2)$, for the 1D and 2D cases, respectively. In the 2D case, triangles and rectangles denote the numerical results for $\theta = 20°$ and $\theta = 45°$, respectively. (localization means immobility).

for S taken below a (lower) critical value, $S_{\rm cr}^{(0)}$, the kicked soliton survives without acquiring any velocity. This occurs if the kinetic energy, $E_{\rm kin} \sim S^2$, initially imparted to the soliton, is insufficient to overcome the PN barrier. The dependence of W_0 on $C = C_1 = C_2$ and its comparison to S^2 explain the shape of the separatrix of the localization area in Fig. 3. Thus, general features of both the 1D and 2D situation are summarized as follows: (i) for $S < S_{\rm cr}^{(0)}$, the soliton remains quiescent; (ii) for $S_{\rm cr}^{(0)} < S < S_{\rm cr}$, it sets in persistent motion; (iii) for $S > S_{\rm cr}$, it is destroyed.

The 2D setting is especially interesting for two reasons. First, as noted above, in the 2D case the mobility is a nontrivial feature, which is impossible in the $\chi^{(3)}$ lattices; second, it is interesting to study the *anisotropy* of the mobility, i.e., its dependence on the orientation of the kick relative to the lattice. Figure 4 shows two examples of stable motion: one along the lattice diagonal, and, to the best of our knowledge, the first example of motion in an arbitrary direction on the lattice.

In Fig. 5 we summarize the dependence of the mobility on the strength, S, and direction, θ, of the initial kick. As said above, the localization of the 2D soliton (no motion) is observed at $S < S_{\rm cr}^{(0)}$, while for $S > S_{\rm cr}^{(0)}$ the outcome is either stable motion, or destruction, at $S > S_{\rm cr}$. Analysis of numerical data confirms that $S_{\rm cr}^{(0)}$ exponentially decays with the increase of $C_1 = C_2$ (cf. Fig. 3), in accordance with the analytical asymptotic dependence of W_0. On the other hand, the dependences of $S_{\rm cr}$ and velocity on θ shown in Fig. 5 demonstrate that the propagation direction *along the diagonal* is easiest to sustain the motion on the square lattice, as $S_{\rm cr}$ is largest in this direction, and the motion is fastest for given $S < S_{\rm cr}$. Of course, the lattice soliton cannot move straight along the diagonal (since lattice sites are not directly coupled along the diagonals); actually, it periodically splits along the two lattice directions and recombines at the site located diagonally across from the splitting point.

We have also examined the situation with $C_2 < C_1$ and obtained similar results, but with larger $S_{\rm cr}^{(0)}$, which may also be explained by the asymptotic expression for W_0, which depends exponentially on C_2 in this case. In the special case of $C_2 = 0$ (no lattice coupling in the SH field), moving solitons cannot be generated, which is easy to understand, too: with $C_2 = 0$, Eq. (2) yields $v_{m,n} = -u_{m,n}^2/(4\omega + k)$, and the substitution of this in Eq. (1) makes the model equivalent to one with the cubic nonlinearity, where traveling 2D discrete solitons do not exist.

Conclusions.—We have examined the mobility of solitary waves in 1D and 2D lattices with the quadratic nonlinearity. The solitons can easily be set in stable motion in 1D, and they remain mobile on the 2D lattice, contrary to what is the case for the cubic nonlinearity. In the 2D case,

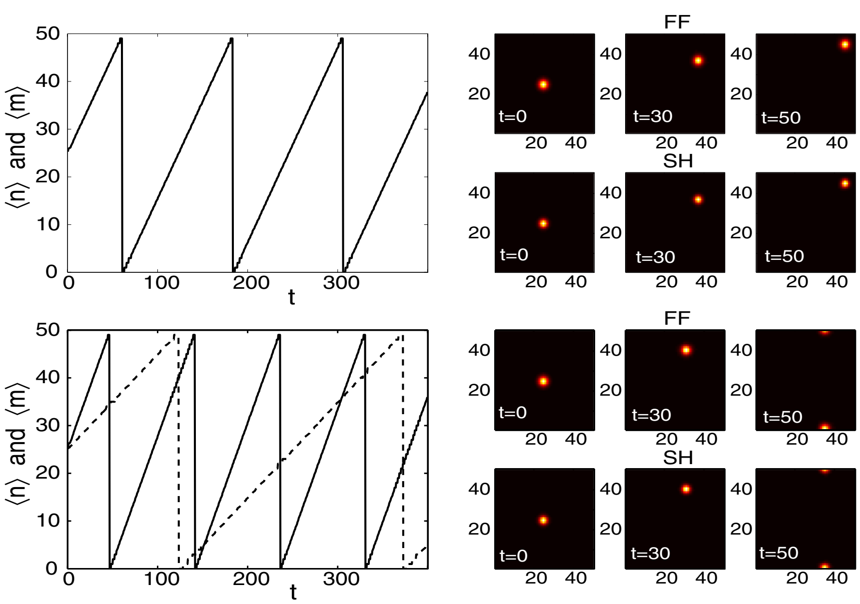

FIG. 4 (color online). Motion of the discrete soliton in a 2D periodic lattice (same parameters as in Fig. 2). Depicted are the cases of propagation along the diagonal (45°, top panels) and at an arbitrarily chosen direction (20°, bottom panels). In the left panels, dashed and solid lines show coordinates of the soliton's center, ξ and η, as functions of time (in the case of the diagonal motion, $\xi = \eta$). In the right panels, snapshots of the moving soliton in the FF (top) and SH (bottom) fields at $t = 0$, 30, 50 are shown.

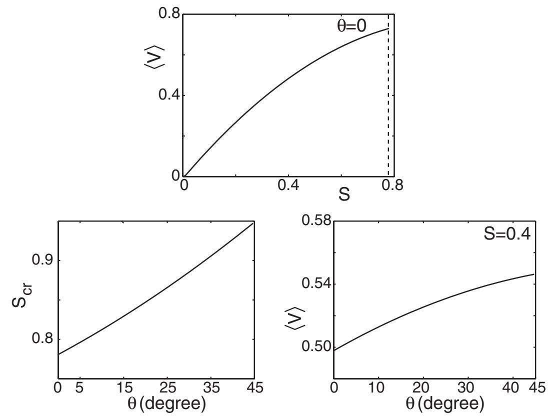

FIG. 5. Characteristics of the soliton motion in the 2D periodic lattice, for $C_1 = C_2 = 1$, $k = 0.25$, $\omega = -0.25$. The top panel shows the velocity versus strength S of the kick directed along the lattice bonds (at angle $\theta = 0$); the vertical dashed line indicates the value S_{cr}, beyond which the soliton is destroyed. The bottom left panel depicts S_{cr} as a function of orientation θ of the initial kick. For given $S = 0.4$, the velocity is shown vs θ in the bottom right corner.

we have demonstrated the possibility of motion of solitons in an arbitrary direction, the motion along the diagonal being easiest. An explanation of key features, such as the border between the pinned and mobile states, and the anisotropy of the mobility, can be provided by the consideration of the effective Peierls-Nabarro potential.

It may be interesting to extend the analysis to other 1D and, especially, 2D models, where mobile solitons may be expected, such as systems with competing nonlinearities (the cubic-quintic model [27] or the Salerno model with competing on-site and intersite cubic terms [28]). A full proof of the existence of traveling lattice solitons is a challenging computational [26] and mathematical [29] problem.

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