Friedmann Equations and Thermodynamics of Apparent Horizons

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With the help of a masslike function which has a dimension of energy and is equal to the Misner-Sharp mass at the apparent horizon, we show that the first law of thermodynamics of the apparent horizon $dE = T_A dS_A$ can be derived from the Friedmann equation in various theories of gravity, including the Einstein, Lovelock, nonlinear, and scalar-tensor theories. This result strongly suggests that the relationship between the first law of thermodynamics of the apparent horizon and the Friedmann equation is not just a simple coincidence, but rather a more profound physical connection.

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The derivation of the thermodynamic laws of black holes from the classical Einstein equation suggests a deep connection between gravitation and thermodynamics [1]. The discovery of the quantum Hawking radiation [2] and black hole entropy which is proportional to the area of the event horizon of the black hole [3] further supports this connection and the thermodynamic (physical) interpretation of geometric quantities. The interesting relation between thermodynamics and gravitation became manifest when Jacobson derived the Einstein equation from the first law of thermodynamics by assuming the proportionality of the entropy and the horizon area for all local acceleration horizons [4].

In cosmology, like in black holes, for the cosmological model with a cosmological constant (called de Sitter space), there also exist Hawking temperature and entropy associated with the cosmological event horizon, and thermodynamic laws of the cosmological event horizon [5]. In de Sitter space, the event horizon coincides with the apparent horizon (AH). For more general cosmological models, the event horizon may not exist, but the AH always exists, so it is possible to have Hawking temperature and entropy associated with the AH. The connection between the first law of thermodynamics of the AH and the Friedmann equation was shown in [6]. Now, we must ask if this interesting relation between gravitation and thermodynamics exists in more general theories of gravity, like Brans-Dicke (BD) theory and nonlinear gravitational theory. In [7], the gravitational field equations for the nonlinear theory of gravity were derived from the first law of thermodynamics by adding some nonequilibrium corrections. In this Letter, we show that equilibrium thermodynamics indeed exists for more general theories of gravity, provided that a new masslike function is introduced.

To show our claim, we begin by reviewing the thermodynamics of the AH with the use of the Misner-Sharp (MS) mass in Einstein and BD theories of gravity, whereby we find the equilibrium thermodynamics fails to hold for the BD theory. The Einstein equation can be rewritten as the mass formulas with the help of the MS mass $\mathcal{M}$. The energy flow through the AH dE is related with the MS mass. Since the MS mass $\mathcal{M}$, the Hawking temperature T_A, and the entropy S_A of the AH are geometric quantities, the first law of thermodynamics of the AH can be thought of as a geometric relation. Therefore, we expect the geometric relation to hold in other gravitational theory if it holds in Einstein theory. To achieve this, we replace the MS mass $\mathcal{M}$ by a masslike function M which is equal to the MS mass $\mathcal{M}$ at the AH; we then show that the connection between the first law of thermodynamics of the AH and the gravitational equations holds in scalar-tensor and nonlinear theories of gravity without adding nonequilibrium correction.

For a spherically symmetric space-time with the metric $ds^2 = g_{ab}dx^a dx^b + \tilde{r}^2 d\Omega^2$, using the MS mass $\mathcal{M} = \tilde{r}(1 - g^{ab}\tilde{r}_{,a}\tilde{r}_{,b})/2G$ [8], the $a - b$ components of the Einstein equation give the mass formulas [9,10]

$$\mathcal{M}_{,a} = 4\pi\tilde{r}^2(T_a^b - \delta_a^b T)\tilde{r}_{,b}, \tag{1}$$

where the unit spherical metric is given by $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and $T = T_a^a$. From now on, all the indices are raised and lowered by the metric g_{ab} and the covariant derivative is with respect to g_{ab}. The AH is

$$\tilde{r}_A = ar_A = (H^2 + k/a^2)^{-1/2}. \tag{2}$$

At the AH, the MS mass $\mathcal{M} = 4\pi\tilde{r}_A^3\rho/3$, which can be interpreted as the total energy inside the AH. Now we use the (approximate) generator $k^a = (1, -Hr)$ of the AH, which is null at the horizon, to project the mass formulas. Since $k^a\tilde{r}_{,a} = 0$, at the AH we find that

$$-dE = -k^a\nabla_a\mathcal{M}dt = d(\tilde{r}_A)/G = T_A dS_A, \tag{3}$$

where the horizon temperature is $T_A = 1/(2\pi\tilde{r}_A)$ and the horizon entropy is $S_A = \pi\tilde{r}_A^2/G$. On the other hand, using the mass formulas (1), we get the energy flow through the AH

$$\begin{aligned} -dE &= -k^a\nabla_a\mathcal{M}dt = -4\pi\tilde{r}^2 T_a^b\tilde{r}_{,b}k^a dt \\ &= 4\pi\tilde{r}_A^3 H(\rho + p)dt. \end{aligned} \tag{4}$$

Therefore, the Friedmann equation gives rise to the first law of thermodynamics $-dE = T_A dS_A$ of the AH. From the above definitions, we see that the relation $-dE = T_A dS_A$ is a geometrical relation which depends on the only assumption of the Robertson-Walker metric. To connect the geometrical quantity dE with the energy flow through the AH, we need to use the Friedmann equations. Therefore, for any gravitational theory, if we can write the gravitational field equation as $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ and regard the right-hand side as the effective energy-momentum tensor, then we find the energy flow through the AH, whereby we derive the first law of thermodynamics of the AH $-dE = T_A dS_A$. For example, in the Jordan frame of the scalar-tensor theory of gravity, if we take the right-hand side of gravitational field equation as the total effective energy-momentum tensor, then the Friedmann equation can be regarded as a thermodynamic identity at the AH [11].

The connection between the first law of thermodynamics and the Friedmann equation at the AH was also found for gravity with Gauss-Bonnet term, the Lovelock theory of gravity [6], and the braneworld cosmology [12]. For a general static spherically symmetric and stationary axisymmetric space-times, it was shown that the Einstein equation at the horizon gives rise to the first law of thermodynamics [13,14]. For the Lovelock gravity, the interpretation of gravitational field equation as a thermodynamic identity was proposed in [15].

Alternatively, the mass formulas (1) can be written as the so-called unified first law $\nabla_a \mathcal{M} = A\Psi_a + W\nabla_a V$ [16,17], where $W = (\rho - p)/2$ and $\Psi_a = T_a^b \tilde{r}_{,b} + W\tilde{r}_{,a}$. Projecting the unified first law along the direction tangent to the AH (or trapping horizon in Hayward's terminology), the first law of thermodynamics $d\mathcal{M} = TdS + WdV$ can be derived, where the horizon temperature and entropy are given, respectively, by $T = \Box\tilde{r}/(4\pi)$ and $S = A/(4G)$. Based on this result, the connection between the Friedmann equation and the first law of thermodynamics of the AH with the work term was widely discussed for Einstein gravity, Lovelock's gravity, the scalar-tensor theory of gravity, the nonlinear theory of gravity, and the braneworld scenario [18–23].

This connection between the Friedmann equation and the first law of thermodynamics of the AH suggests the unique role of the AH in thermodynamics of cosmology. This may be used to probe the property of dark energy [10,24]. For example, if we assume that the temperature of the dark components is $T = bT_A$, then use the relation $T = (\rho + p)/s = (\rho + p)a^3/\sigma$, we find that the total energy density of the dark components is given by

$$\rho = \rho_\Lambda + \rho_0\left(\frac{a_0}{a}\right)^6 + 2\sqrt{\rho_0\rho_\Lambda}\left(\frac{a_0}{a}\right)^3, \tag{5}$$

where $\rho_0 = \sigma^2 b^2 G a_0^{-6}/(6\pi)$, $\rho_\Lambda = 3\Lambda/(8\pi G)$ is the energy density of the cosmological constant, σ is the constant comoving entropy density, and s is the physical entropy density. The right-hand side of the above equation contains three different terms, which correspond to, respectively, the cosmological constant, the stiff fluid, and the pressureless matter. However, the coefficients of these terms are not all independent. In fact, the current observational constraints tell us that the stiff fluid is negligibly small, for which we must assume $\rho_0 \ll 1$. This in turn implies that the pressureless matter given by the last term is also negligibly small. So the pressureless matter in the last term cannot account for dark matter. In other words, the dark matter must not be in equilibrium with the AH.

For the BD theory [25]

$$L = -\frac{\sqrt{-g}}{16\pi}\left[\phi R + \omega g^{\mu\nu}\frac{\partial_\mu\phi\partial_\nu\phi}{\phi}\right], \tag{6}$$

the BD scalar ϕ plays the role of the gravitational constant. The MS mass is [26]

$$\mathcal{M} = \frac{\phi\tilde{r}}{2}(1 - g^{ab}\tilde{r}_{,a}\tilde{r}_{,b}). \tag{7}$$

At the AH, $\mathcal{M} = \phi\tilde{r}_A/2$. The horizon entropy is $S_A = \pi\tilde{r}_A^2\phi$, so we get

$$T_A dS_A = \tfrac{1}{2}\tilde{r}_A d\phi + \phi d\tilde{r}_A. \tag{8}$$

On the other hand, we have

$$-dE = -k^a\nabla_a\mathcal{M}dt = -\tfrac{1}{2}\tilde{r}_A d\phi + \phi d\tilde{r}_A. \tag{9}$$

Comparing Eqs. (8) with (9), we find that the equilibrium thermodynamics $-dE = T_A dS_A$ fails to hold for the BD theory. Similarly, it can be shown that $-dE = T_A dS_A$ does not hold in the nonlinear and scalar-tensor theories of gravity. It is exactly because of this that it was argued nonequilibrium treatment might be needed.

As mentioned above, the mass, temperature, and entropy of the AH are all geometrical quantities, and the first law of thermodynamics of the AH can be regarded as a geometric relation. Now, the important question is whether a mass function exists that serves as the bridge between the Friedmann equation and the first law of thermodynamics of the AH without nonequilibrium correction. In the following, we show that the answer is affirmative. It has exactly the dimension of energy and is equal to the MS mass at the AH. To distinguish it with the MS mass, we call it the masslike function.

To show our above claim, let us write the $a - b$ components of the Einstein equation as

$$M_{,a} = -4\pi\tilde{r}^2(T_a^b - \delta_a^b T)\tilde{r}_{,b} + \tilde{r}_{,a}, \tag{10}$$

where the masslike function M is defined as

$$M \equiv \frac{\tilde{r}}{2G}(1 + g^{ab}\tilde{r}_{,a}\tilde{r}_{,b}). \tag{11}$$

At the AH, $g^{ab}\tilde{r}_{,a}\tilde{r}_{,b} = 0$ and the masslike function $M = \tilde{r}_A/2G$, which is equal to the MS mass. For the Robertson-Walker metric we have $g_{tt} = -1$, $g_{rr} = a^2/(1 - kr^2)$, and $\tilde{r} = ar$. Then, the mass formulas (10) yield the Friedmann equations

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho, \tag{12}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{13}$$

Combining Eqs. (12) and (13), we can derive the energy conservation law $\dot{\rho} + 3H(\rho + p) = 0$. Thus, the mass formulas (10) give rise to the full set of the cosmological equations.

At the AH, the masslike function $M = 4\pi\tilde{r}_A^3\rho/3$, which is the total energy inside the AH. The energy flow is

$$dE = k^a\nabla_a M dt = d(\tilde{r}_A)/G = T_A dS_A. \tag{14}$$

On the other hand, using the mass formulas (10), we get the energy flow through the AH

$$dE = k^a\nabla_a M dt = -4\pi\tilde{r}^2 T_a^b \tilde{r}_{,b}k^a dt = 4\pi\tilde{r}_A^3 H(\rho + p)dt. \tag{15}$$

Therefore, the Friedmann equation gives rise to the first law of thermodynamics $dE = T_A dS_A$ of the AH. While this result is the same as that obtained by using the MS mass, we show below that the equilibrium thermodynamics can be derived for BD and nonlinear gravities by using our newly defined masslike function, although it cannot be done by using the MS mass, as shown above.

For the BD theory, the masslike function is defined as

$$M \equiv \frac{\phi\tilde{r}}{2}(1 + g^{ab}\tilde{r}_{,a}\tilde{r}_{,b}). \tag{16}$$

At the AH, it reduces to the MS mass, $M = \mathcal{M} = \phi\tilde{r}_A/2$. The $a - b$ components of the gravitational field equation become

$$\begin{aligned} M_{,a} = {} & -4\pi\tilde{r}^2(T_a^b - \delta_a^b T)\tilde{r}_{,b} + 2\pi\tilde{r}^3 T\frac{\phi_{,a}}{\phi} + (\phi\tilde{r})_{,a} \\ & - \frac{\omega + 2}{2\phi}\tilde{r}^2\phi_{,a}\phi_{,b}\tilde{r}^{;b} + \frac{\omega}{4\phi}\tilde{r}^2\tilde{r}_{,a}\phi_{,b}\phi^{;b} - \tilde{r}\tilde{r}_{,a}\tilde{r}_{,b}\phi^{;b} \\ & - \frac{1}{2}\tilde{r}^2\phi_{;ab}\tilde{r}^{;b} - \frac{1}{2}\tilde{r}^2\Box\tilde{r}\phi_{,a} - \frac{\tilde{r}^3}{4\phi}\phi_{,a}\Box\phi. \end{aligned} \tag{17}$$

Applying to the Robertson-Walker metric, the above equation gives the Friedmann equations

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3\phi}\rho + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi^2} - H\frac{\dot{\phi}}{\phi}, \tag{18}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3\phi}(\rho + 3p) - \frac{\omega}{3}\frac{\dot{\phi}^2}{\phi^2} - \frac{1}{2}H\frac{\dot{\phi}}{\phi} - \frac{1}{2}\frac{\ddot{\phi}}{\phi}. \tag{19}$$

The mass formulas (17) or Eqs. (18) and (19) are not sufficient to describe the full dynamics of the BD cosmology. In the BD cosmology, we also need the equation of motion of the BD scalar field ϕ in addition to Eqs. (18) and (19), which is given by

$$\ddot{\phi} + 3H\dot{\phi} = \frac{8\pi}{3 + 2\omega}(\rho - 3p). \tag{20}$$

From the definition of the masslike function (16), at the AH we find

$$dE = M_{,a}k^a dt = \tfrac{1}{2}\tilde{r}_A d\phi + \phi d\tilde{r}_A = T_A dS_A, \tag{21}$$

where the entropy now is $S_A = \pi\tilde{r}_A^2\phi$. Using the mass formulas (17), we get the energy flow through the AH

$$\begin{aligned} M_{,a}k^a = {} & \frac{8\pi}{3 + 2\omega}\tilde{r}_A^3 H[(\omega + 2)\rho + \omega p] + \frac{\omega}{2}\tilde{r}_A^3 H\frac{\dot{\phi}^2}{\phi} \\ & - 2\tilde{r}_A^3 H^2\dot{\phi} + \frac{1}{2}\tilde{r}_A\dot{\phi}, \end{aligned} \tag{22}$$

where we used Eq. (20) in deriving the above equation. From Eqs. (18)–(20), the right-hand side of Eq. (22) can be written as $\frac{1}{2}\tilde{r}_A\dot{\phi} + \phi\dot{\tilde{r}}_A$. Therefore, we see that in BD theory, the first law of thermodynamics of the AH $dE = T_A dS_A$ can be derived from the Friedmann equation.

The thermodynamic prescription can be easily extended to general scalar-tensor theory of gravity with the Lagrangian

$$L = f(\phi)R - \tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{23}$$

In this case, $f(\phi)$ plays the role of the gravitational constant, so now we can define the masslike function as

$$M \equiv \tfrac{1}{2}f(\phi)\tilde{r}(1 + g^{ab}\tilde{r}_{,a}\tilde{r}_{,b}) \tag{24}$$

and the horizon entropy as $S_A = \pi\tilde{r}_A^2 f(\phi)$. Then, using these definitions, we can show that $dE = M_{,a}k^a dt = T_A dS_A$.

For the nonlinear theory of gravity $f(R)$, we can define the masslike function as

$$M \equiv \tfrac{1}{2}f'(R)\tilde{r}(1 + g^{ab}\tilde{r}_{,a}\tilde{r}_{,b}), \tag{25}$$

and the horizon entropy $S_A = \pi\tilde{r}_A^2 f'(R)$, where $f'(R) = df/dR$. Again, it is easy to show that $dE = M_{,a}k^a dt = T_A dS_A$. Therefore, the thermodynamics of the AH holds for both the general scalar-tensor theory of gravity and the nonlinear theory of gravity.

Now we show how to derive the first law of thermodynamics of the AH from the Friedmann equation in the Lovelock gravity. The Lovelock Lagrangian is $L = \sum_{n=0}^m c_n L_n$ [27], where

$$L_n = 2^{-n}\delta^{\mu_1\nu_1\cdots\mu_n\nu_n}_{\alpha_1\beta_1\cdots\alpha_n\beta_n}R^{\alpha_1\beta_1}_{\mu_1\nu_1}\cdots R^{\alpha_n\beta_n}_{\mu_n\nu_n}.$$

Using the Robertson-Walker metric, we obtain the Friedmann equations in $N + 1$ dimensional space-time

$$\sum_{i=1}^m \hat{c}_i\left(H^2 + \frac{k}{a^2}\right)^i = \frac{16\pi G}{N(N-1)}\rho, \tag{26}$$

and

$$\sum_{i=1}^{m}\hat{c}_i i\left(H^2+\frac{k}{a^2}\right)^{i-1}\left(\dot{H}-\frac{k}{a^2}\right)=-\frac{8\pi G}{N-1}(\rho+p), \quad (27)$$

where $\hat{c}_0 = c_0/[N(N-1)]$, $\hat{c}_1 = 1$, and $\hat{c}_i = c_i \prod_{j=3}^{2m}(N+1-j)$ for $i > 1$. The masslike function can now be defined as

$$M \equiv \frac{N(N-1)\Omega_N \tilde{r}^N}{16\pi G}\sum_{i=1}^{m}\hat{c}_i\left[2\tilde{r}^{-2i}-\left(H^2+\frac{k}{a^2}\right)^i\right] = \Omega_N \tilde{r}_A^N \rho, \quad (28)$$

where Ω_N is the volume of unit N-dimensional sphere and the last equality is evaluated at the AH. Note that although the geometric form is different, the masslike function at the AH has the same value as that in Einstein theory of gravity, which is the total energy inside the AH. The entropy of the AH is

$$S_A = \frac{N\Omega_N}{4G}\sum_{i=1}^{m}\frac{i(N-1)}{N-2i+1}\hat{c}_i \tilde{r}_A^{N+1-2i}. \quad (29)$$

From Eqs. (28) and (29), we can easily check that $dE = M_{,a}k^a dt = T_A dS_A$ holds with the horizon temperature $T_A = 1/(2\pi\tilde{r}_A)$. Using the Friedmann Eqs. (26) and (27), we find the energy flow through the AH is $dE = N\Omega_N H \tilde{r}_A^N(\rho + p)$, which is the same as that in Einstein's gravity.

By properly defining the masslike function in each theory of gravity, we find that the corresponding Friedmann equations can be written in the form $dE = T_A dS_A$ of the first law of thermodynamics at the AH. In other words, the thermodynamic description of the gravitational dynamics is manifest through the mass formulas. Therefore, the gravitational dynamics can be considered as the thermodynamic identity $dE = T_A dS_A$. This is true for a variety of theories of gravity, including the Einstein, Lovelock, nonlinear, and scalar-tensor theories. This nontrivial connection between the thermodynamics of the AH and the Friedmann equation may represent a generic connection, and it suggests the unique role that the AH can play in the thermodynamics of cosmology. Such a thermodynamic description of the AH can also be used to probe other physical systems and properties, such as the nature of dark energy and the thermodynamics of black holes in each of these theories.

Finally, we would like to note that, although the newly defined masslike function reduces to the MS mass at the AH, the corresponding energy flows passing through the horizon are different. This explains why our masslike function gives rise to the first law of thermodynamics in various theories of gravity, while the MS mass does not. Because of the masslike function, the energy-momentum tensor includes the contribution of gravitational fields such as BD scalars, or curvature scalars in nonlinear theory of gravity, in addition to the matter fields. This treatment allows a reinterpretation of the nonequilibrium correction introduced in [7]. The studies of other properties of the newly defined masslike function, including the physical and geometrical difference between the MS mass and it, are important and should be reported somewhere else.

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