Few-Optical-Cycle Solitons and Pulse Self-Compression in a Kerr Medium

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In a transparent medium with instantaneous Kerr nonlinearity we find a new class of few-optical-cycle solitons and prove them to be the fundamental structures in pulse propagation dynamics. We demonstrate numerically that in the asymptotic stage of pulse propagation the input pulse splits into isolated few-cycle solitons where the quantity and their parameters are determined by the initial pulse. We generalize the concept of the high-order Schrödinger solitons to the few-cycle regime and show how it can be used for efficient pulse compression down to the single cycle duration.

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Introduction.—The concept of optical solitons is widely and successfully used in modern optics due to both the fascinating physics involved and the potential applications ranging from nonlinear spatial confinement of light in the form of spatial solitons to ultrashort-pulse duration solitons in optical fibers (see, e.g., [1–3]). The greatest achievements this concept has attained is when the slow envelop approximation for the laser field is applicable [1]. There are only a few examples in optical physics where the wave equation for the real laser field is treated in the context of solitons, but most of them deal with the light propagation in resonant media [4]. However, laser science has recently progressed up to the realization of laser systems that generate pulses with a duration of their envelope comparable to the period of the electromagnetic field and with a spectrum spanning over one optical octave as well, wherein nonlinear effects, mainly due to the Kerr nonlinearity, lead to spectral broadening [2,3]. To study the nonlinear few-optical-cycle pulse propagation, two approaches have been considered recently. The first one is based on the nonlinear envelope equation extended to the few-cycle regime; it uses the Taylor expansion of the propagation constant around the central frequency [2]. Another approach developed in [5] assumes making calculations in Fourier space but using one way propagation constant $k(\omega)$. Both methods are essentially based on numerical calculations by using the split-step technique; they are very important for practical applications but cannot help to understand the key questions related to fundamental features. The purpose of the present work is to introduce (i) an exact solvable model of few-optical-cycle solitons and (ii) extend the concept of solitons to nonlinear pulse dynamics in the few-cycle regime. Furthermore, (iii) of particular interest is the investigation of the pulse self-compression down to single cycle duration.

Basic wave equation.—Let us start our consideration with the wave equation for isotropic media which generally in one dimensional case can be written as [1]

$$\partial_z^2 \mathbf{E} - \frac{1}{c^2}\partial_t^2 \int_{-\infty}^{t} \varepsilon(t-t')\mathbf{E}(t')dt' = \frac{4\pi}{c^2}\partial_t^2 \mathbf{P}_{\rm nl}, \qquad (1)$$

where $\partial_{i=z,t}$ stands for the respective derivatives, ε is the linear permittivity whose Fourier representation satisfies the fundamental Kramers-Kronig relation [6]

$$\varepsilon_r(\omega) = 1 + \frac{2}{\pi}\int_0^\infty \frac{x\varepsilon_i(x)dx}{x^2-\omega^2}, \qquad (2)$$

where ε_r and ε_i are the real and imaginary parts, respectively. It is interesting to note that for medium with ultrabroad spectral region of transparency ($\omega_1 \ll \omega \ll \omega_2$, e.g., in fused silica corresponding wavelengths of the nearest absorption lines are $\lambda_1 = 9.896\ \mu\mathrm{m} > 2\pi c/\omega_1$ and $\lambda_2 = 0.116\ \mu\mathrm{m} < 2\pi c/\omega_2$ [1]) the frequency dependence of ε_r and ε_i can be explicitly derived from Eq. (2). As noted in Ref. [6], if the frequency-dependent term of ε_r is comparable to ε_0 then it reads $\varepsilon_r(\omega) = \varepsilon_0 - a/\omega^2$, where $\varepsilon_0 = 1 + (2/\pi)\int_{\omega_2}^\infty (\varepsilon_i/x)dx$ is the static permittivity, $a = (2/\pi)\int_0^{\omega_1} x\varepsilon_i dx$. Thus, in the case of broad range of low absorption the dielectric permittivity can pass through zero. In fact, it comes from the anomalous character of group-velocity dispersion taken place for this case. However, typical situations in optics occur when the contributions of anomalous and normal character are equally important, which means that the higher term of the Taylor expansion in Eq. (2) (on small parameter $\omega^2/x^2 \ll 1$ at $x > \omega_2$) has to be taken into account. Then, ε_r reads

$$\varepsilon_r(\omega) = \varepsilon_o - \frac{a}{\omega^2} + b\omega^2, \qquad (3)$$

where $b = (2/\pi)\int_{\omega_2}^\infty (\varepsilon_i/x^3)dx$. A practically useful point is that in the spectral region where the frequency-dependent terms are equally important they are both very small compared to the first term, i.e., $a/\omega^2, b\omega^2 \ll \varepsilon_o$. In this case, the group-velocity dispersion (GVD) is characterized by the function $k''(\omega) = d^2(\omega\varepsilon_r^{1/2}/c)/d\omega^2 = (3b\omega - a/\omega^3)/(c\varepsilon_o^{1/2})$, in which the zero-dispersion frequency is $\omega_{\rm cr} = (a/3b)^{1/4}$ (in fused silica the corresponding wavelength is $\lambda_{\rm cr} = 1.27\ \mu\mathrm{m}$). Thus, we find that anomalous dispersion occurs for frequencies below this critical one, $\omega < \omega_{\rm cr}$, whereas the normal dispersion region is at $\omega > \omega_{\rm cr}$. The nonlinear polarization for a Kerr medium is characterized by the constitutive law given by $\mathbf{P}_{\mathrm{nl}} = \chi^{(3)}|\mathbf{E}|^2\mathbf{E}$ (where $\chi^{(3)}$ is the cubic nonlinear susceptibility) that assumes the instantaneous character of nonlinear atomic response (see, e.g., [1,2]).

With the use of dispersion Eq. (3), a nonlinear wave equation for the real electric field can be obtained by using the so-called slowly-evolving-wave approximation which neglects back reflection [2,5]. In dimensionless variables this equation reads (see also [7,8]).

$$\partial_{z\tau}^2\mathbf{u} + \mathbf{u} - \mu\partial_\tau^4\mathbf{u} + \partial_\tau^2[|\mathbf{u}|^2\mathbf{u}] = \mathbf{0}. \tag{4}$$

Here, $\mathbf{u} = \mathbf{E}(4\pi\chi^{(3)}/a)^{1/2}$, $z \rightarrow az/2c\varepsilon_o^{1/2}$, and $\tau = t - z\varepsilon_o^{1/2}/c$ is the retarded time. In fact, only one parameter $\mu = b/a = (3\omega_{\mathrm{cr}}^4)^{-1}$ accounts for the type of dispersion that actually (anomalous or normal) prevails for the main pulse spectrum. Thus, we emphasize that for few-cycle pulse propagation in transparent Kerr media Eq. (4) can play a fundamental role like the nonlinear Schrödinger equation for long pulses [1]. Equation (4) has the Hamiltonian $H = \int_{-\infty}^{\infty}[|\mathbf{u}|^4/2 + \mu|\mathbf{u}_\tau|^2 - |\int_{-\infty}^{\tau}\mathbf{u}d\tau|^2]d\tau$, whose value is calculated to control the precision of our simulations.

Few-optical-cycle solitons.—We restrict our consideration for circularly polarized pulses for which an exact analysis can be provided. We start with the case of $\mu = 0$ where solutions describe the most important case of stable propagation of the solitonlike field structures comprising a few optical cycles [9,10]. In fact, this describes the situation where the whole pulse spectrum lies in the anomalous dispersion region, as $\omega_{\mathrm{cr}} = \infty$. Assuming a solution of Eq. (4) of the form: $\mathbf{u} = a(z,\tau)\cos\varphi(z,\tau)\mathbf{e_x} + a(z,\tau)\sin\varphi(z,\tau)\mathbf{e_y}$, we can obtain a set of equations for the amplitude $a(z,\tau)$ and carrying phase $\varphi(z,\tau)$. For the stationary envelope propagation $a(z,\tau) = \gamma^{1/2}w(\xi)$, one solution of the phase is of the following form

$$\varphi(z,\tau) = \omega_o\tau + \gamma\omega_o z + \int_{-\infty}^{\xi}\frac{w^2(3-2w^2)}{2(1-w^2)^2}d\xi, \tag{5}$$

where $\xi = \omega_o(\tau - \gamma z)$ is the dimensionless retarded time in the soliton reference, ω_o is the soliton carrying frequency, γ^{-1} is the soliton group velocity. It should be noted that the carrying phase $\varphi(z,\tau)$ does not only contain the linear term with the carrying frequency but also the nonlinear term describing the spectral broadening in the few-cycle soliton. In fact, this term may also be responsible for the supercontinuum generation in photonic crystal fibers, as discussed in Ref. [5]. The equation for the amplitude $a(\xi)$ can be reduced to the following

$$\frac{dw}{d\xi} = \pm\frac{w}{(1-3w^2)}\sqrt{\delta^2 - \frac{3}{2}(\delta^2+1)w^2 + \frac{(4-5w^2)w^2}{4(1-w^2)^2}}, \tag{6}$$

where $\delta^2 = 1/\gamma\omega_o^2 - 1 > 0$. The solitonlike solutions of Eq. (6) exist at $\delta^2 \leq \delta_c^2 = 1/8$. This can be seen from analysis of Eq. (6) at a point where $w'(\xi) = 0$. The upper limit of δ_c defines the maximum value of the soliton amplitude $w_{s\max} = \sqrt{2/3}$ and its minimum duration $\xi_{s\min} = 2.31$ (FWHM). This limitation can be seen by taking into account that Eq. (4) contains the integral $\int_{-\infty}^{\infty}\mathbf{u}d\tau = 0$, which gives the constraint that the average of each field components must equal to zero. It is obvious that this constraint is not fulfilled for video pulses. The time profile of this limiting soliton and the dependence of the few-cycle soliton energy, calculated as $W_s = (\gamma/\omega_o)\int_{-\infty}^{\infty}w^2d\xi$, on the soliton duration are presented in Fig. 1. As is distinctly seen, a frequency in the central part of the pulse is essentially larger than that in the vicinity; i.e., the pulse spectrum is strongly broadened but all frequency components are locked. An attractive peculiarity of these solutions from the point of view of general nonlinear wave theory is that they can be considered as a continuation of the fundamental Schrödinger solitons for the envelope of quasimonochromatic wave ($\delta \ll 1$) to the few-optical-cycle regime (as δ approaches to δ_c). A significant difference between the few-cycle solitons and the Schrödinger ones, as seen in Fig. 1(b), takes place for the pulse durations less than 2–2.5 cycles.

The influence of the high-frequency dispersion, i.e., $\mu \neq 0$ is expressed by the ratio of the carrier frequency to the zero-dispersion frequency, $\omega_o/\omega_{\mathrm{cr}}$. It is obvious that for very small μ, corresponding to a case when the main part of pulse spectrum lies in the anomalous region, solitonlike structures given by Eqs. (5) and (6) will not be significantly changed; i.e., they will exist for a comparatively long time. Therefore the question is now what will happen if a perceptible part of pulse spectrum lies in the region of normal dispersion. To solve Eq. (4) we use the split-step Fourier method which provides conservation of the Hamiltonian with very good accuracy. In Fig. 2 the evolution of a soliton with the parameters $\delta = 0.32$ and $\omega_o/\omega_{\mathrm{cr}} = 0.667$ ($\mu = 0.0658$) taken as an initial field distribution (a) and the field spectrum (b) with the propagation distance is shown. As can be seen in Fig. 2(b), some part of the soliton spectrum (about 16% of energy) is initially located in the normal GVD region. As expected,

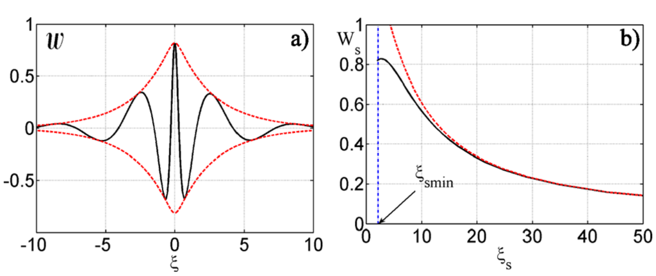

FIG. 1 (color online). (a) Temporal profile of the limiting soliton ($\delta = \delta_c$). The dashed line shows the envelope distribution. (b) Few-cycle soliton energy as a function of the pulse duration. The dashed line refers to the Schrödinger solitons.

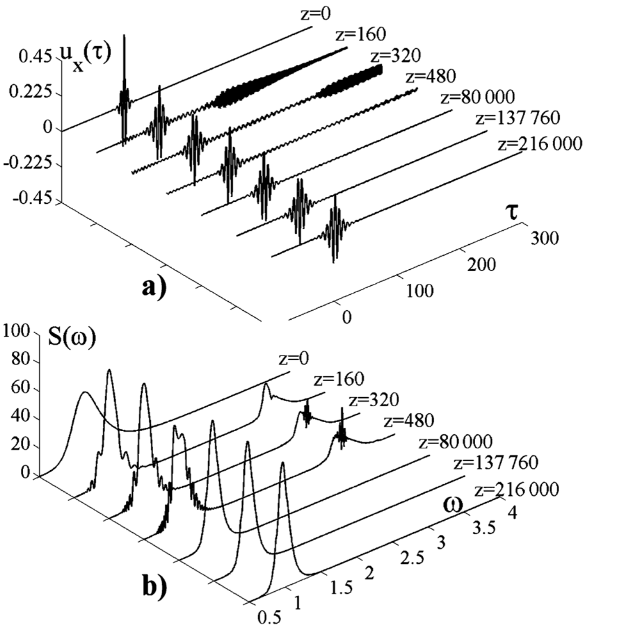

FIG. 2. Snapshots of field distribution (a) and spectrum (b) along the propagation distance. Input pulse is a few-cycle soliton with $\omega_o = \omega_o^{\text{in}} = 1$ and $\delta = \delta_{\text{in}} = 0.32$; $\omega_{\text{cr}} = 1.5$. For output few-cycle soliton: $\omega_o^{(n)} = 0.92$ and $\delta^{(n)} = 0.17$. Dispersion length for these parameters is $L_d \simeq 15$.

this leads to a splitting of the initial spectrum into two parts: the right part of the spectrum at $\omega > \omega_{\text{cr}}$ forms a spreading pulse, while the left part transforms into a new few-cycle solitonlike structure but with redshifted carrying frequency, which propagates further without changes as soon as it separates from the spreading background. It should be noted that the spectrum of this structure is entirely localized in the anomalous GVD region. To prove that this is actually a few-cycle soliton and to identify its physical parameters, ω_o and δ, we will use the approximate analytic solution of Eq. (6): $9/2\delta\sqrt{\delta^2 - w^2/2} - Arch(\sqrt{2}\delta/w) = \pm\delta\xi$ [10]. Now using this solution, we find the soliton amplitude a_m and its energy W_s

$$a_m = \sqrt{\frac{2}{\delta^2 + 1}\frac{\delta}{\omega_o}}, \qquad W_s = \frac{4\delta(1 - 3\delta^2)}{\omega_o^3(\delta^2 + 1)}. \qquad (7)$$

To select a soliton in the simulations, we use a filter window in τ space and then define its energy and its amplitude. Finally, solving Eqs. (7) for given a_m and W_s, we obtain the specific soliton carrier frequency $\omega_o = \omega_o^{(n)}$ and $\delta = \delta^{(n)}$ [or group velocity $\gamma^{-1} = (\delta^2 + 1)\omega_o^2$] that fully characterize a few-cycle soliton. Applying this procedure to the case of Fig. 2, we obtain the following parameters of a newly born soliton: $\omega_o^{(n)} = 0.92$ and $\delta^{(n)} = 0.17$, with energy of about 70% of the initial pulse energy.

If the energy of the spectrum located in the normal GVD region is sufficiently large (which is the case of $\omega_o/\omega_{\text{cr}} \gtrsim 1$) then it provides an effective generation of a spectral supercontinuum with a temporal spreading in the time domain (see [5,8]). However, a long duration soliton could also be created in this case as soon as a part of the initial spectrum is localized below the critical frequency that is consistent with the recent experiment [11]. Figure 3(b) summarizes the simulation results and shows the efficiency of soliton generation depending on $\omega_{\text{cr}}/\omega_o$. We would like to emphasize that this picture reflects both the stable single few-cycle soliton generation and the energetic efficiency of such process. Thus, few-cycle solitons could easily be excited in a broad range of conditions and therefore should be considered as basic structures which play a fundamental role in the dynamics of extremely short pulses.

Pulse self-compression and soliton formation.—Another point that is of interest for applications is pulse self-compression in the subcarrier frequency regime. As was shown numerically in Ref. [9], a combined medium with instantaneous nonlinear response and plasma dispersion can provide compression of an initially quasimonochromatic pulse below its carrier period. Based on few-cycle solitons as the fundamental structures, we propose a strategy for pulse self-compression of any short pulse down to the single cycle duration. Taking into account that the few-cycle solitons given by Eqs. (5) and (6) are a continuation of the Schrödinger solitons, for the compression strategy we will follow a way which is known within the frame of the nonlinear Schrödinger equation or its modifications [1]. The basis of this idea is not only the existence of few-cycle solitons, but rather their stability against collisions [10]. In that way the compression is determined by a soliton number N contained in a high-order Schrödinger-like soliton which was taken as an initial distribution [1,12]. Analogously, we introduce this number by taking an input pulse in the form of a high-order few-cycle soliton, i.e., $\mathbf{u}(\xi) = N\gamma^{1/2}w(\xi)[\mathbf{e}_x\cos\varphi(\xi) + \mathbf{e}_y\sin\varphi(\xi)]$, where $\varphi(\xi)$ and $w(\xi)$ obey Eqs. (5) and (6). Then this pulse splits into few-cycle solitons with the parameters $\delta^{(n)} \simeq (2n - 1)\delta_{\text{in}}$, where $n = 1, \ldots, [N]$ is the sequence of integers, as for the Schrödinger solitons since for the most few-cycle solitons $\delta^2 \ll 1$ is satisfied. The number of solitons that actually interact with each other during the pulse propagation is an

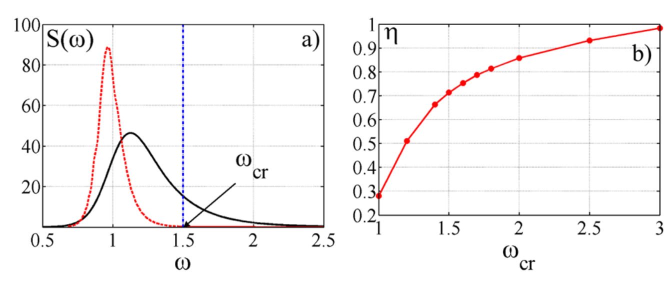

FIG. 3 (color online). (a) The input (solid line) and output (dashed line) soliton spectra corresponding to Fig. 2. (b) Dependence of the ratio of the energy of generated soliton to the input pulse energy as a function of ω_{cr}. Input pulse as in Fig. 2.

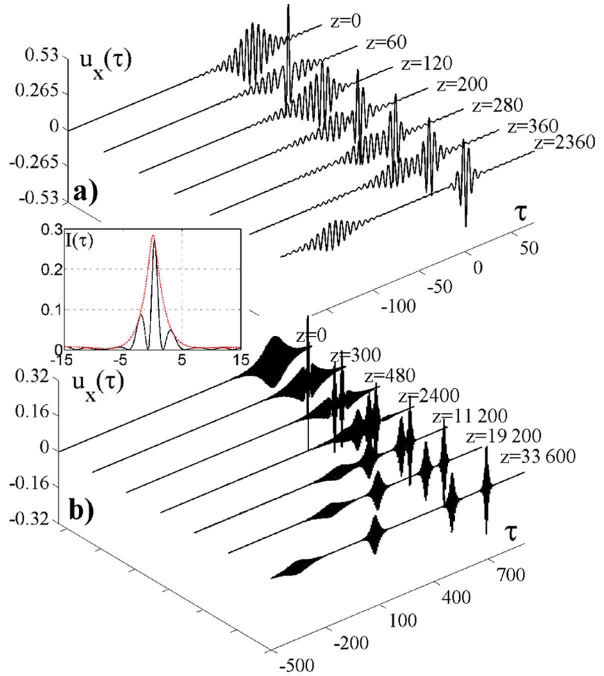

FIG. 4 (color online). Snapshots of field distribution along the propagation distance for the case of $N = 2.04$ and $N = 4.02$. Intensity shape of the compressed pulse at $z = 60$ for the case $N = 2.04$ is shown in the inset.

integer of $[N]$; however, it is valid when the inequality $\delta^{([N])} < \delta_c$ is fulfilled, i.e.,

$$(2[N] - 1)\delta_{\text{in}} < 1/\sqrt{8}. \tag{8}$$

In all other cases, pulse dynamics is more complex, but eventually a number of solitons with $\delta^{(n)} < \delta_c$ will be produced.

Figure 4 shows the results of a simulation of pulse propagation for the case of $N = 2.05$ and $N = 4.02$ but for $\delta = 0.08$ and $\delta = 0.02$ soliton shapes, respectively. As can be seen, the intensity dynamics demonstrates clearly the compression effect at distances $z \sim 60$ and $z \sim 300$ in Figs. 4(a) and 4(b) [the corresponding linear dispersion lengths are $L_d \simeq 60$ (a) and $L_d \simeq 1000$ (b)], where very short spikes are formed with durations 2.88 (a) and 5.86 (b) at FWHM. The energy within the spikes is about 49% and 40% of the total pulse energy, respectively. The established parameters of the well-isolated solitons, which are formed in the asymptotic stage, are: $\omega_o^{(1)} = 0.92$ and $\delta^{(1)} = 0.24$, $\omega_o^{(2)} = 0.97$ and $\delta^{(2)} = 0.08$ in Fig. 4(a); $\omega_o^{(1)} = 0.97$ and $\delta^{(1)} = 0.14$, $\omega_o^{(2)} = 0.98$ and $\delta^{(2)} = 0.1$, $\omega_o^{(3)} = 0.99$ and $\delta^{(3)} = 0.06$, $\omega_o^{(4)} = 1$ and $\delta^{(4)} = 0.02$ in Fig. 4(b). Note that the parameters $\delta^{(n)}$ of the newly born solitons are in good agreement with Eq. (8). In the few-optical-cycle regime, the compression factor does not only depend on N but on the initial pulse duration, τ_p, as well. It is well fitted by the power law $\tau_p/\tau_{\min} \propto N^{\alpha}$ where α actually depends on the initial pulse duration and increases with decreasing τ_p: $\alpha = 1.54$; 1.6; 1.85; 2.1 for $\tau_p = 60\pi$; 45π; 30π; 20π (the carrier period is 2π), respectively.

After self-compression the pulse starts to broaden, then it splits into several subpulses depending on the parameter N, and subsequently few-cycle solitons, which are well separated in space, are formed. In Figs. 4(a) and 4(b), as an outcome, two and four solitons are observed in agreement with the soliton number in initial pulse. To identify each soliton, we used the procedure described above. A remarkable feature of Fig. 4(b) is that, starting with a longer pulse but higher N, we can generate more short few-cycle solitons, for example, as short as one cycle duration for the shortest one.

In conclusion, we have found analytically a new class of wave soliton structures which describe a few-cycle optical solitons in transparent media with instantaneous Kerr nonlinearity and have shown that these structures play a key role in the evolution of laser pulses. In the asymptotic stage, the input pulse splits into spatially isolated few-cycle solitons with quantity and parameters determined by the initial pulse. We have generalized a concept of the high-order Schrödinger solitons to few-cycle regime and demonstrated how it can be used for efficient pulse compression down to single cycle duration.

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