Zonal Wind Driven by Inertial Modes

A. Tilgner
Institute of Geophysics, University of Göttingen, Friedrich-Hund-Platz 1, 37077 Göttingen, Germany
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Inertial modes are oscillatory modes in rotating fluids. Shear layers appear in inertial modes in spherical shells that become singularities in the inviscid limit. It is shown here that the nonlinearity in the shear layers drives a zonal flow whose amplitude diverges in the inviscid limit. These results are relevant for the dynamics of planets and stars in which inertial modes are excited by tidal forcing.

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Rotating bodies of fluid support oscillatory motion known as "inertial waves" or "inertial modes." The mathematical description of these modes starts from the linearized Navier-Stokes equation. Applications in geo- and astrophysics involve motions for which the dissipation term is small so that one is interested in the double limit of small amplitudes (which justifies linearization) and small viscosity. The linear inviscid problem is ill posed and smooth solutions exist only in special geometries such as a full sphere [1]. The spherical shell has been studied in detail because of its geo- and astrophysical relevance [2,3]. The ill posedness in this case leads to singularities in the form of shear layers on conical surfaces coaxial with the axis of rotation. The "critical latitudes" are those latitudes at which these conical surfaces are tangent to the boundaries. Shear zones are generated at critical latitudes, extend through the fluid volume, and reflect from the boundaries. Reflection conserves the angle formed by the normal to a shear layer with the rotation axis, and not the angle formed with the normal to the reflecting surface. Thanks to this peculiar reflection law, shear zones generally are focused on attractors after multiple reflection. A similar phenomenon exists for gravity waves in stably stratified fluids [4]. It has been emphasized in the past that the mathematical structure of this problem is related to quantum chaos and quantum billiards, and even to problems arising in general relativity [5].

In this Letter, the nonlinear term is included for the first time in numerical computations of inertial modes. It is shown that, in a spherical shell, the nonlinear interaction of an inertial mode with itself excites an axisymmetric flow whose amplitude diverges with decreasing Ekman number. This effect is due to the internal shear layers alluded to above. The results show that inertial modes excited by tides can drive significant zonal flows in planetary cores and in the atmosphere of gaseous planets. With the tidal application in mind, all the examples of eigenmodes studied here have an azimuthal wave number of 2.

Consider a spherical shell with gap size d filled with incompressible fluid of viscosity ν rotating about the z axis at angular velocity Ω. Using as units of time and length $1/\Omega$ and d, respectively, the adimensional equation of motion for the velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ becomes in the corotating frame of reference:

$$\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{\nabla} \times \boldsymbol{v}) \times \boldsymbol{v} + 2\hat{z} \times \boldsymbol{v} = -\boldsymbol{\nabla} p + \mathrm{Ek}\nabla^2 \boldsymbol{v}, \quad (1)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \quad (2)$$

where $\hat{z}$ is the unit vector in the z direction. All gradient terms (pressure, centrifugal force, and $\boldsymbol{\nabla}\boldsymbol{v}^2/2$) are collected in $\boldsymbol{\nabla} p$. The Ekman number Ek is defined by $\mathrm{Ek} = \nu/(\Omega d^2)$. The inner and outer boundaries of the shell have radii r_i and r_o with $r_o - r_i = 1$. On these boundaries, the radial component of the velocity must vanish and, throughout this Letter, the tangential stress is required to be zero as well.

Linearization can be useful only if the Rossby number Ro is small, which is defined as $\mathrm{Ro} = U/(d\Omega)$, where U is the typical dimensional velocity. If we expand $\boldsymbol{v}$ in a power series in Ro, $\boldsymbol{v} = \mathrm{Ro}\boldsymbol{v}_1 + \mathrm{Ro}^2\boldsymbol{v}_2 + \ldots$, and similarly for the variable p, the equation of motion becomes at the order Ro^{-1}:

$$\frac{\partial}{\partial t}\boldsymbol{v}_1 + 2\hat{z} \times \boldsymbol{v}_1 = -\boldsymbol{\nabla} p_1 + \mathrm{Ek}\nabla^2 \boldsymbol{v}_1, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 = 0. \quad (3)$$

Solutions to this equation are inertial modes. The equation is linear with coefficients independent of time and space so that solutions have the form

$$\boldsymbol{v}_1(\boldsymbol{r}, t) = \mathrm{Re}\{\boldsymbol{u}_1(r, \theta)e^{i(\omega t + m\varphi)}\} \quad (4)$$

in spherical polar coordinates (r, θ, φ). $\mathrm{Re}\{\ldots\}$ denotes the real part. The eigenmodes decouple according to their azimuthal wave number m. The frequency of the eigenmode is ω. In the following, we only consider the case $m = 2$.

The equation at order Ro^2 contains the inhomogeneous term $(\boldsymbol{\nabla} \times \boldsymbol{v}_1) \times \boldsymbol{v}_1$, which potentially has a component with $m = 0$ to drive a zonal flow in $\boldsymbol{v}_2$. Unless we specify initial conditions, the solution to that equation is not unique because an arbitrary solution of the homogeneous equation can be added to any particular solution of the full equation. In order to avoid the need to fix initial conditions, let us modify the problem as follows: Imagine (1) augmented by a forcing term (which might represent tidal forcing) chosen such that some particular eigenmode is maintained at constant amplitude. Such a forcing term exists for every mode. This effectively excites the solution (4) but with a frequency ω that is purely real. Any axisymmetric flow excited under these circumstances at order Ro^2 is necessarily time independent and governed by the equation

$$2\hat{z} \times \bar{\boldsymbol{v}}_2 = -\nabla \bar{p}_2 + \mathrm{Ek}\nabla^2 \bar{\boldsymbol{v}}_2 + \boldsymbol{F}, \qquad \nabla \cdot \bar{\boldsymbol{v}}_2 = 0, \quad (5)$$

where the overbar denotes the average over φ and

$$\boldsymbol{F} = -\overline{[\nabla \times \boldsymbol{v}_1(\boldsymbol{r}, t=0)] \times \boldsymbol{v}_1(\boldsymbol{r}, t=0)}. \qquad (6)$$

The numerical computations below obtain inertial modes in spherical shells from (3), calculate $\boldsymbol{F}$ according to (6) and finally find the axisymmetric response with (5), all subject to stress-free boundary conditions. The numerical method proceeds by decomposing $\boldsymbol{v}_1$ and $\bar{\boldsymbol{v}}_2$ into poloidal and toroidal scalars, which guarantees that the velocity field is divergence-free. The scalars are discretized spectrally using Chebyshev polynomials in radius and spherical harmonics in latitude. The eigenproblem (3) is solved by inverse iteration. The matrix inversions appearing in the inverse iteration and the solution of (5) are performed with a direct method for block diagonal matrices. In order to save computing resources, the matrices are set up for flows symmetric with respect to the equator. The spatial resolution used in the computations below reach up to 257 Chebyshev polynomials and spherical harmonics of degree 512. The method is nearly identical to the one used in [6].

The amplitude of $\boldsymbol{v}_1$ is arbitrary and set such that its kinetic energy is unity, $\int \boldsymbol{v}_1^2/2dV = 1$, where the integral extends over the spherical shell. The most interesting quantity to be extracted from $\bar{\boldsymbol{v}}_2$ is the kinetic energy contained in the differential rotation. This quantity has the most direct link to astronomical observations. If a simple rotation was detected in the zonal flow of a celestial body, it would be interpreted as rotation of the planet or the star as a whole and only differential rotation is a feature of atmosphere dynamics. For the stress-free boundary conditions used in this Letter, (5) allows arbitrary rotation about the z axis to be added to any solution $\bar{\boldsymbol{v}}_2$. Among all possible solutions, let us select the one that contains the least rotation in the sense that $\int \bar{\boldsymbol{v}}_2^2 dV \leq \int |\bar{\boldsymbol{v}}_2 - \zeta \hat{z} \times \boldsymbol{r}|^2 dV$ for all ζ. The kinetic energy in the differential rotation, E_{dr}, is then directly given by

$$E_{\mathrm{dr}} = \frac{1}{2} \int \bar{v}_{2\varphi}^2 dV. \qquad (7)$$

The energy of the meridional circulation, E_{mc}, will also appear below:

$$E_{\mathrm{mc}} = \frac{1}{2} \int (\bar{v}_{2r}^2 + \bar{v}_{2\theta}^2) dV. \qquad (8)$$

Because of the convention used throughout this Letter that the linear modes $\boldsymbol{v}_1$ are normalized to unit energy, E_{dr} and E_{mc} really represent the energies of the axisymmetric flow divided by the energy of the linear response.

Figure 1 shows the zonal wind, $\bar{v}_{2\varphi}$, and the φ component of the inhomogeneous term, F_φ, driving that wind, for one particular eigenmode. F_φ is largest near the attractor pattern present in the eigenmode $\boldsymbol{v}_1$, which means near shear layers emanating from critical latitudes and their subsequent reflections off boundaries. This pattern also leaves its signature in the zonal flow $\bar{v}_{2\varphi}$, which otherwise is in geostrophic equilibrium and is constant along the z axis. By its definition (6), F_φ is symmetric with respect to the equator irrespective of the equatorial symmetry of $\boldsymbol{v}_1$.

Figure 2 shows E_{dr} for the same mode as in Fig. 1 and for different core sizes. Qualitatively identical pictures have been obtained for the 10 other modes that have been investigated in the course of this study. For the smallest core sizes included in the figure, E_{dr} is constant for Ekman numbers from 10^{-4} down to nearly 10^{-7}. An E_{dr} independent of Ek is the behavior expected for a full sphere. The limits $\mathrm{Ro} \to 0$ and $\mathrm{Ek} \to 0$ can be carried out independently and without difficulties in this case. Analytic ex-

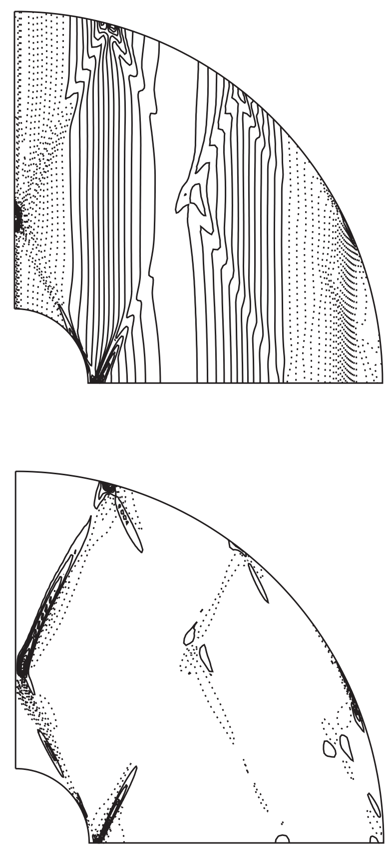

FIG. 1. Contour plots of $\bar{v}_{2\varphi}$ (upper panel) and F_φ (lower panel) for the mode with $m = 2$ and $\omega = 0.88$ at $\mathrm{Ek} = 10^{-6}$ in a shell with $r_i/r_o = 0.2$. Continuous (dashed) contour lines indicate positive (negative) values. The rotation axis is pointing upwards.

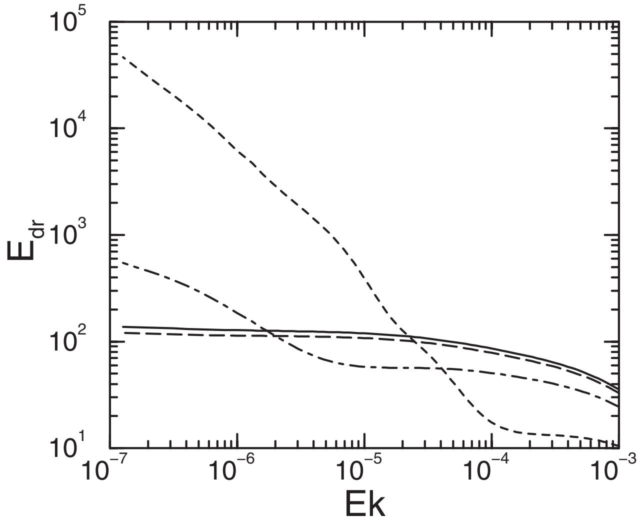

FIG. 2. E_{dr} for the mode with $m = 2$, $\omega = 0.88$, and different core sizes: $r_i/r_o = 0.01$ (continuous line), 0.03 (long dashed line), 0.1 (dot-dashed line), and 0.2 (short dashed line).

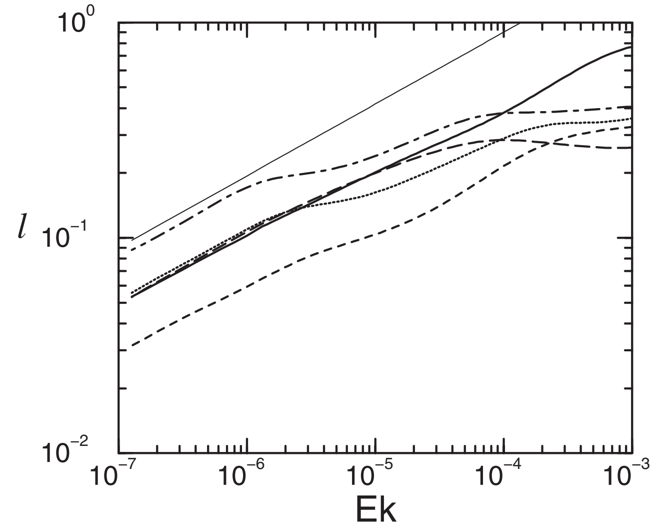

FIG. 3. The length l in a shell with $r_i/r_o = 0.2$ for modes with $m = 2$ and $\omega = -0.23$ (continuous line), -0.80 (long dashed line), 0.88 (short dashed line), -1.09 (dot-dashed line), and 1.64 (dotted line). The thin line indicates the power law $l \propto \mathrm{Ek}^{1/3}$.

pressions for inviscid eigenmodes are known for this geometry [7] because they are free of singularities. Their nonlinear interaction is known not to produce significant flows [8]. For $r_i/r_o = 0.1$, however, E_{dr} increases for decreasing Ek at small Ek, and this increase is already observed at larger Ek for larger cores. The same increase is expected for $r_i/r_o = 0.01$ and 0.03 at Ekman numbers not reached in Fig. 2. Figure 2 suggests that E_{dr} diverges for $\mathrm{Ek} \to 0$.

The connection between $\boldsymbol{F}$ and internal shear layers, already evident in Fig. 1, also manifests itself in a simple scaling relation. Let us look at $\nabla \times \boldsymbol{F}$ instead of $\boldsymbol{F}$ in order to eliminate the possibility that $\boldsymbol{F}$ is a gradient field that can be balanced by $\nabla \bar{p}_2$ in (5) and leaves $\bar{\boldsymbol{v}}_2 = 0$. Denote the thickness of the internal layers by l_1 and let v_1 be a typical velocity of the linear eigenmode within these layers. $\boldsymbol{v}_1$ is much larger inside than outside a layer, so that $\int \boldsymbol{v}_1^2 dV \propto v_1^2 l_1$ [2]. Likewise, $\boldsymbol{F}$ varies rapidly across the shear zones and slowly within the shear zones. The estimate for $\int |\nabla \times \boldsymbol{F}|^2 dV$ thus is

$$\int |\nabla \times [(\nabla \times \boldsymbol{v}_1) \times \boldsymbol{v}_1]|^2 dV \propto \left(\frac{1}{l_1} \frac{v_1}{l_1} v_1\right)^2 l_1 \propto \frac{1}{l_1^5} \left[\int \boldsymbol{v}_1^2 dV\right]^2. \quad (9)$$

Keeping in mind that we use eigenvectors normalized to unit kinetic energy, we find $l_1 \propto [\int |\nabla \times \boldsymbol{F}|^2 dV]^{-1/5}$. Motivated by this estimate, let us define a length l as

$$l = \left[\int |\nabla \times \boldsymbol{F}|^2 dV\right]^{-1/5}. \quad (10)$$

This length l is shown in Fig. 3.

Visualizations of shear layers in eigenmodes in spherical shells have revealed that the thickness of most layers scales in $\mathrm{Ek}^{1/4}$ [2]. However, analytical models of time dependent free shear layers in rotating flows yield layer thicknesses in $\mathrm{Ek}^{1/3}$ [1,5,9]. This leads to the assumption that we are, in fact, dealing with nested layers [1,5]. The thinnest layer should produce the largest contribution to the nonlinear term so that we expect $l \propto \mathrm{Ek}^{1/3}$. This is indeed verified in Fig. 3, which incidentally is the first demonstration of layers scaling like $\mathrm{Ek}^{1/3}$ in spherical shell eigenmodes.

The previous figures show that the driving term $\boldsymbol{F}$ is connected in a straightforward manner with the internal shear layers. Irrespective of the exact scaling of their thickness with Ek, it is clear that they must become thinner with decreasing Ek so that singularities appear in the eigenmode in the inviscid limit. Because $\int |\nabla \times \boldsymbol{F}|^2 dV = l^{-5}$, it is concluded that $\nabla \times \boldsymbol{F}$ diverges for $\mathrm{Ek} \to 0$. This implies a divergence of E_{dr} and E_{mc}.

Note that for decaying eigenmodes, the values of l and $\nabla \times \boldsymbol{F}$ are exactly the same as above if the instantaneous energy of the linear mode is normalized to one. As explained earlier, the only reason for studying stationary or forced modes is that E_{dr} and E_{mc} acquire well defined values, independent of additional parameters such as initial conditions.

The next step is to relate the magnitudes of the zonal flow and of $\boldsymbol{F}$. The numerical data show that at small Ek, $E_{mc} \ll E_{dr}$, which is expected because the meridional circulation can never be in geostrophic equilibrium. For an estimate of $\bar{v}_{2\varphi}$, we can therefore assume $\bar{v}_{2\theta} = \bar{v}_{2r} = 0$. Integrating the φ component of (5) over z, one then finds for an eigenmode with unit kinetic energy: $\int \hat{\boldsymbol{\varphi}} \nabla^2 \bar{\boldsymbol{v}}_2 dz \propto \mathrm{Ek}^{-1} l^{-1}$. In order to estimate the integral, we need to know

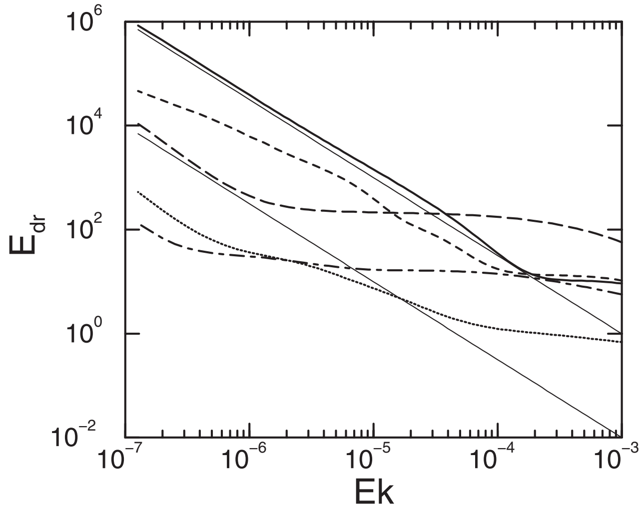

FIG. 4. E_{dr} for the same shell and the same modes as in Fig. 3. The thin continuous lines indicate the power law $E_{dr} \propto \mathrm{Ek}^{-3/2}$.

the typical length scale over which $\bar{v}_{2\varphi}$ varies. However, the variation of $\bar{v}_{2\varphi}$ is not characterized by a single length scale as Fig. 1 shows: For the mode shown there, strong gradients appear in the same regions where F_φ is large, but also near the axis and near a cylindrical surface of radius $0.9r_o$. The investigation of these shear zones is beyond the scope of this Letter. We obviously find an upper bound for E_{dr} if we assume that $\bar{v}_{2\varphi}$ varies only on an $\mathcal{O}(1)$ length scale, which leads to $E_{dr} < \mathrm{const} \cdot \mathrm{Ek}^{-2} l^{-2}$. All investigated modes verify this bound. If one insists on fitting power laws to E_{dr}, one finds a dependence close to $E_{dr} \propto \mathrm{Ek}^{-3/2}$ at the lowest numerically accessible Ek for most modes (Fig. 4).

Let us now investigate with an order of magnitude estimate whether tides can drive significant zonal flows in planets. For this purpose, it will be assumed that the velocity in the semidiurnal equilibrium tide is representative of the amplitude of tidally excited inertial modes even though future work will have to decide whether this assumption is justified for arbitrary singular inertial modes. The typical (dimensional) velocity in a semidiurnal tide of amplitude η_0 is $\eta_0 \Omega/\pi$, so that the assumed energy of the inertial mode is $\int \boldsymbol{v}_1^2/2dV \approx (\eta_0\Omega/\pi)^2/(\Omega d)^2 V_s/2$ where V_s is the volume of the shell. Using the relationship $E_{dr} = (\mathrm{Ek}/10^{-3})^{-3/2}(\int \boldsymbol{v}_1^2/2dV)^2$ (see Fig. 4), one can estimate the typical zonal velocity as $\Omega d(2E_{dr}/V_s)^{1/2}$. For the tides raised by Io on Jupiter and assuming $\mathrm{Ek} = 10^{-15}$, one finds in this way a zonal velocity of 15 m/s, which is close to the observed value. These estimates, of course, neglect higher orders in Ro, the detailed structure of the planets, the compressibility of the fluid, and the exact form of the tidal response. But they show that tides plausibly generate significant zonal flows and that thermal convection need not be the only mechanism one has to invoke in order to explain, for example, the surface winds of Jupiter.

In summary, it has been shown that inertial modes drive zonal flows in spherical shells. Inertial modes decouple in azimuthal wave number m in every axisymmetric container. The question is, by which mechanism can a mode excited at $m \neq 0$ drive an axisymmetric flow with $m = 0$? Nonlinear interactions in no slip boundaries are one possibility, which has been demonstrated by Busse [10] for the particular case of the "spin over mode" (the eigenmode excited by precession). Inertial modes are marked by internal shear layers in all but the simplest geometries, such as full spheres. If these shear layers become unstable, they transport angular momentum in such a way that a mean flow is maintained [11]. In unstable flows, triad resonances [12] can also drive axisymmetric flows starting from modes with $m \neq 0$. The mechanism described in this Letter is more fundamental in the sense that it needs neither instability nor a special type of boundaries (F_φ in Fig. 1 is not confined to a boundary layer at a critical latitude). Instead, it is shown that in a spherical shell, the inviscid limit is not only peculiar because singularities appear in linear eigenmodes but also because nonlinear interactions in shear layers drive an axisymmetric flow whose amplitude (divided by the amplitude of the linear mode) diverges in the limit of the vanishing Ekman number. Because this mechanism is related to shear zones in inertial modes, it is expected to operate in all containers other than those for which the inviscid inertial modes are smooth. An obvious excitation mechanism for inertial modes in celestial bodies is tidal forcing. The zonal circulation driven by tidal flows through the mechanism described in this Letter may be important for the dynamics of planets and stars.

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