Conceptual Explanation for the Algebra in the Noncommutative Approach to the Standard Model

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The purpose of this Letter is to remove the arbitrariness of the *ad hoc* choice of the algebra and its representation in the noncommutative approach to the standard model, which was *begging* for a conceptual explanation. We assume as before that space-time is the product of a four-dimensional manifold by a finite noncommmutative space F. The spectral action is the pure gravitational action for the product space. To remove the above arbitrariness, we classify the irreducible geometries F consistent with imposing reality and chiral conditions on spinors, to avoid the fermion doubling problem, which amounts to have total dimension 10 (in the K-theoretic sense). It gives, almost uniquely, the standard model with all its details, predicting the number of fermions per generation to be 16, their representations and the Higgs breaking mechanism, with very little input.

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In the phenomenological approach to determining the Lagrangian of the fundamental interactions all the present data are consistent with the standard model with neutrino mixing. The input that goes into the construction of the standard model is the following. First one needs the list of three families of 16 quarks and leptons and their representations under the gauge group $SU(3)_c \times SU(2)_w \times U(1)_Y$. For the first family, this is taken to be

$$\begin{pmatrix} u \\ d \end{pmatrix}_L = \left(3, 2, \frac{1}{3}\right)_L, \quad u_R = \left(3, 1, \frac{4}{3}\right)_R, \quad d_R = \left(3, 1, -\frac{2}{3}\right)_R$$

for the quarks and

$$\begin{pmatrix} \nu \\ e \end{pmatrix}_L = (1, 2, -1)_L, \quad e_R = (1, 1, -2)_R, \quad \nu_R = (1, 1, 0)_R$$

for the leptons. The gauge symmetry is then broken to $SU(3)_c \times U(1)_{\mathrm{em}}$ by employing a complex scalar field Higgs doublet H with representation (1, 2, 1). The Lagrangian is constructed by writing the most general renormalizable interactions consistent with the above symmetries. The freedom in the choice of the gauge group and the fermionic representations have led to many attempts to unify all the gauge interactions in one group and the fermions in one irreducible representation. The most notable among the unification schemes are models based on the $SO(10)$ gauge group and groups containing it such as E_6, E_7, and E_8. The most attractive feature of $SO(10)$ is that all the fermions in one family fit into the 16 spinor representation and the above delicate hypercharge assignments result naturally after the breakdown of symmetry. However, what is gained in the simplicity of the spinor representation and the unification of the three gauge coupling constants into one $SO(10)$ gauge coupling is lost in the complexity of the Higgs sector. To break the $SO(10)$ symmetry into $SU(3)_c \times U(1)_{\mathrm{em}}$ one needs to employ many Higgs fields in representations such as 10, 120, 126 [1]. The arbitrariness in the Higgs sector reduces the predictivity of all these models and introduces many arbitrary parameters, in addition to the unobserved proton decay.

The noncommutative geometric approach [2] to the unification of all fundamental interactions, including gravity, is based on the three ansatz [3,4]: (1) space-time is the product of an ordinary Riemannian manifold M by a finite noncommutative space F. (2) The K-theoretic dimension (defined below) of F is 6 modulo 8. (3) The physical action functional is given by the spectral action at unification scale.

The empirical data taken as input are as follows. (1) There are 16 chiral fermions in each of three generations. (2) The photon is massless. (3) There are Majorana mass terms for the neutrinos.

Furthermore, one makes the following "*ad hoc*" choice: the algebra of the finite space is taken to be $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, where $\mathbb{H}$ is the algebra of quaternions and $M_3(\mathbb{C})$ is the algebra of complex 3×3 matrices. One of the main purposes of this Letter is to show how this algebra arises.

With this input the basic data of noncommutative geometry are constructed, consisting of an involutive algebra $\mathcal{A}$ of operators in Hilbert space $\mathcal{H}$, which plays the role of the algebra of coordinates, and a self-adjoint operator D in $\mathcal{H}$ [2] which plays the role of the inverse of the line element. It was shown in [4] that the fermions lie in the desired representations and that the spectral action associated with this noncommutative space unifies gravitation with the standard model at the unification scale.

Although the emerging geometrical picture is very appealing and could be tested experimentally, the *ad hoc* choice of the algebra forces us to address the question of why singling this specific choice, and whether there are other possibilities, as in the case of grand unification. In addition, taking the number of fundamental fermions to be

16 as input prompts the question of whether there could exist additional fermions and whether there is a mathematical restriction on this number from the representations of the algebra. It is the purpose of this Letter to remove the choice of the algebra as input and to derive it by classifying the possible algebras compatible with the axioms of noncommutative geometry and minimal number of assumptions to be specified. We shall keep as physical input that there are three generations, the photon is massless, and that some of the fermions must acquire a Majorana mass. We shall prove that the number of fermions must be equal to the square of an even integer, and thus we are able to derive that there are 16 fermions per generation. We shall show that the axioms of noncommutative geometry essentially allow the choice of the algebra to be $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. The proof of these results is rather involved, and we shall only state the theorems, with the rigorous mathematical details given in [5].

The algebra $\mathcal{A}$ is a tensor product which geometrically corresponds to a product space. The spectral geometry of $\mathcal{A}$ is given by the product rule $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$ where the algebra $\mathcal{A}_F$ is finite dimensional, and

$$\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F, \qquad D = D_M \otimes 1 + \gamma_5 \otimes D_F,$$

where $L^2(M, S)$ is the Hilbert space of L^2 spinors, and D_M is the Dirac operator of the Levi-Civita spin connection on M. The Hilbert space $\mathcal{H}_F$ is taken to include the physical fermions. The chirality operator is $\gamma = \gamma_5 \otimes \gamma_F$. The real structure $J = J_M \otimes J_F$ is an antilinear isometry J: $\mathcal{H} \to \mathcal{H}$ with the property that

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \qquad J\gamma = \varepsilon'' \gamma J,$$

where $\varepsilon, \varepsilon', \varepsilon'' \in \{\pm 1\}^3$. There are 8 possible combinations for $\varepsilon, \varepsilon', \varepsilon''$ and this defines a K-theoretic dimension of the noncommutative space mod 8. These dimensions are identical to the dimensions of Euclidean spaces allowing the definitions for Majorana and Weyl spinors. In order to avoid the fermion doubling problem it was shown in [6,7] that the finite dimensional space must be taken to be of K-theoretic dimension 6 where in this case $(\varepsilon, \varepsilon', \varepsilon'') = (1, 1, -1)$. This makes the total K-theoretic dimension of the noncommutative space to be 10 and would allow to impose the reality (Majorana) condition and the Weyl condition simultaneously in the Minkowskian continued form, a situation very familiar in ten-dimensional supersymmetry. In the Euclidean version, the use of the J in the fermionic action would give for the chiral fermions in the path integral, a Pfaffian instead of determinant, and will thus cut the fermionic degrees of freedom by 2. In other words, to have the fermionic sector free of the fermionic doubling problem, we must make the choice

$$J_F^2 = 1, \qquad J_F D_F = D_F J_F, \qquad J_F \gamma_F = -\gamma_F J_F.$$

In what follows, we will restrict our attention to determination of the finite algebra, and we will omit the subscript F where F stands for finite.

There are two main constraints on the algebra from the axioms of noncommutative geometry. We first look for involutive algebras $\mathcal{A}$ of operators in $\mathcal{H}$ such that

$$[a, b^0] = 0, \qquad \forall\ a, b \in \mathcal{A},$$

where for any operator a in $\mathcal{H}$, $a^0 = Ja^* J^{-1}$. This is called the order zero condition. We shall assume that the following two conditions to hold. First, the action of $\mathcal{A}$ has a separating vector. Second, the representation of $\mathcal{A}$ and J in $\mathcal{H}$ is irreducible.

The strategy to determine the finite space F then involves the following steps. First, to classify the irreducible triplets $(\mathcal{A}, \mathcal{H}, J)$. Second, to impose the $\mathbb{Z}/2$ grading on $\mathcal{H}$. Third, to classify all the subalgebras $\mathcal{A}_F \subset \mathcal{A}$ which allows for an operator D that does not commute with the center of $\mathcal{A}$ but fulfills the order one condition

$$[[D, a], b^0] = 0, \qquad \forall\ a, b \in \mathcal{A}_F.$$

Starting with the classification of the order zero condition with the irreducible pair $(\mathcal{A}, J)$ one finds out that the solutions fall into two classes. Let $\mathcal{A}_{\mathbb{C}}$ be the complex linear space generated by $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$, the algebra of operators in $\mathcal{H}$. Then the two classes correspond to (1) the center $Z(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C}$; (2) the center $Z(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C} \oplus \mathbb{C}$.

The case $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}$.—In this case we can state the following theorem.

Theorem 1.—Let $\mathcal{H}$ be a Hilbert space of dimension n. Then an irreducible solution with $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}$ exists iff $n = k^2$ is a square. It is given by $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ acting by left multiplication on itself and antilinear involution

$$J(x) = x^*, \qquad \forall\ x \in M_k(\mathbb{C}).$$

This determines $\mathcal{A}_{\mathbb{C}}$ and its representations in $(\mathcal{A}, J)$ and allows only for three possibilities for $\mathcal{A}$. These are $\mathcal{A} = M_k(\mathbb{C})$, $M_k(\mathbb{R})$, and $M_a(\mathbb{H})$ for even $k = 2a$, where $\mathbb{H}$ is the field of quaternions. These correspond, respectively, to the unitary, orthogonal, and symplectic case.

$\mathbb{Z}/2$ *grading.*—In the setup of spectral triples one assumes that in the even case the Hilbert space $\mathcal{H}$ is $\mathbb{Z}/2$ graded, i.e., endowed with a grading operator $\gamma = \gamma^*$, $\gamma^2 = 1$ such that $\gamma \mathcal{A} \gamma^{-1} = \mathcal{A}$. In the $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}$ case, one can then show that it is not possible to have the finite space to be of K-theoretic dimension 6, with $J\gamma = -\gamma J$. We therefore can proceed directly to the second case.

The case $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$.—In this case we can state the theorem.

Theorem 2.—Let $\mathcal{H}$ be a Hilbert space of dimension n. Then an irreducible solution with $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$ exists iff $n = 2k^2$ is twice a square. It is given by $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ acting by left multiplication on itself and antilinear involution

$$J(x, y) = (y^*, x^*), \qquad \forall\ x, y \in M_k(\mathbb{C}).$$

With each of the $M_k(\mathbb{C})$ in $\mathcal{A}_{\mathbb{C}}$ we can have the three possibilities $M_k(\mathbb{C})$, $M_k(\mathbb{R})$, or $M_a(\mathbb{H})$, where $k = 2a$. At this point we make the *hypothesis* that we are in the "symplectic-unitary" case, thus restricting the algebra $\mathcal{A}$ to the form $\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C})$, $k = 2a$. The dimension of the Hilbert space $n = 2k^2$ then corresponds to k^2 fundamental fermions, where $k = 2a$ is an even number. The first possible value for k is 2 corresponding to a Hilbert space of four fermions and an algebra $\mathcal{A} = \mathbb{H} \oplus M_2(\mathbb{C})$. The existence of quarks rules out this possibility. The next possible value for k is 4 predicting the number of fermions to be 16.

$\mathbb{Z}/2$ *grading.*—In the above symplectic-unitary case, one can write the Hilbert space $\mathcal{H}$ as the sum of the spaces of $\mathbb{C}$-linear maps from V to W and from W to V where V is a four-dimensional vector space over $\mathbb{C}$ and W a two-dimensional right vector space over $\mathbb{H}$. There exists, up to equivalence, a unique $\mathbb{Z}/2$ grading of W and it induces uniquely a $\mathbb{Z}/2$ grading γ of $\mathcal{E}$. One then takes the grading γ of $\mathcal{H}$ so that the K-theoretic dimension of the finite space is 6, which means that $J\gamma = -\gamma J$. It is given by

$$\gamma(\zeta, \eta) = (\gamma\zeta, -\gamma\eta).$$

This grading breaks the algebra $\mathcal{A} = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$, which is nontrivially graded only for the $M_2(\mathbb{H})$ component, to its even part:

$$\mathcal{A}^{\mathrm{ev}} = \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C}).$$

The subalgebra and the order one condition.—From the previous analysis, it should be clear that the only relevant case to be subjected to the order one condition is $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$ and for $\mathcal{A} = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$. The center of the algebra $Z(\mathcal{A})$ is nontrivial, and thus the corresponding space is not connected. The Dirac operator must connect the two pieces nontrivially, and therefore it must satisfy

$$[D, Z(\mathcal{A})] \neq \{0\}.$$

The physical meaning of this constraint is to allow some of the fermions to acquire Majorana masses, realizing the seesaw mechanism, and thus connect the fermions to their conjugates. The main constraint on such Dirac operators arises from the order one condition. We have to look for subalgebras $\mathcal{A}_F \subset \mathcal{A}^{\mathrm{ev}}$, the even part of the algebra $\mathcal{A}$ for which $[[D, a], b^0] = 0$, $\forall\ a, b \in \mathcal{A}_F$. We can now state the main result which recovers the input of [4].

Theorem 3.—Up to an automorphism of $\mathcal{A}^{\mathrm{ev}}$, there exists a unique involutive subalgebra $\mathcal{A}_F \subset \mathcal{A}^{\mathrm{ev}}$ of maximal dimension admitting off-diagonal Dirac operators. It is given by

$$\mathcal{A}_F = \{\lambda \oplus q, \lambda \oplus m | \lambda \in \mathbb{C}, q \in \mathbb{H}, m \in M_3(\mathbb{C})\} \subset \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$$

using a field morphism $\mathbb{C} \to \mathbb{H}$. The involutive algebra $\mathcal{A}_F$ is isomorphic to $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ and together with its representation in $(\mathcal{H}, J, \gamma)$ gives the noncommutative space taken as input in [4].

In simple terms, this means that the off-diagonal elements of the Dirac operator, connecting the 16 spinors to their conjugates, break $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C}) \to \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. We have thus recovered the main input used in deriving the standard model with the minimal empirical data. These are the masslessness of the photon and the existence of mixing terms for fermions and their conjugates. The main mathematical inputs are that the representations of $(\mathcal{A}, J)$ are irreducible, and there is an antilinear isometry with nontrivial grading on one of the algebras. Having made contact with the starting point of [4] we summarize the results in that work.

Let M be a Riemannian spin 4-manifold and F the finite noncommutative geometry of K-theoretic dimension 6 but with multiplicity 3. Let $M \times F$ be endowed with the product metric. The unimodular subgroup of the unitary group acting by the adjoint representation $\mathrm{Ad}(u)$ in $\mathcal{H}$ is the group of gauge transformations $SU(2)_w \times U(1)_Y \times SU(3)_c$. The unimodular inner fluctuations of the metric give the gauge bosons of the standard model. The full standard model (with neutrino mixing and seesaw mechanism) coupled to gravity is given in Euclidean form by the action functional [3,4]

$$S = \mathrm{Tr}\left[f\left(\frac{D_A}{\Lambda}\right)\right] + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle, \qquad \tilde{\xi} \in \mathcal{H}_{\mathrm{cl}},$$

where D_A is the Dirac operator with inner fluctuations. To explain the role of the spectral action principle, we note that one of the virtues of the axioms of noncommutative geometry is that it allows for a shift of point of view, similar to Fourier transform in which the usual emphasis on the points $x \in M$ of a geometric space is replaced with the spectrum $\Sigma \subset \mathbb{R}$ of the operator D. The hypothesis, which is stronger than diffeomorphism invariance, is that "the physical action only depends upon Σ".

We conclude that our approach predicts a unique fermionic representation of dimension 16, with gauge couplings unification. These properties are only shared with the $SO(10)$ grand unified theory. The main advantage of our approach over the grand unification approach is that the reduction to the standard model gauge group is not due to plethora of Higgs fields, but is naturally obtained from the order one condition, which is one of the axioms of noncommutative geometry. There is also no proton decay because there are no additional vector particles linking the lepton and quark sectors. The spectral action is the pure gravitational sector of the noncommutative space.

This is similar in spirit to the Kaluza-Klein approach, but with the advantage of having a finite spectrum and not the infinite tower of states. Thus the noncommutative geometric approach manages to combine the advantages of both grand unification and Kaluza-Klein without paying the price of introducing many unwanted states. We still have few delicate points which require further understanding. The first is to understand the need for the restriction to the symplectic-unitary case which is playing an important role in the construction. The second is to determine the number of generations. From the physics point, because of CP violation, we know that we need to take $N \geq 3$, but there is no corresponding convincing mathematical principle.

We would like to stress that the spectral action of the standard model comes out almost uniquely, predicting the number of fermions, their representations, and the Higgs breaking mechanism, with very little input. The geometrical model is valid at the unification scale and relates the gauge coupling constants to each other and to the Higgs coupling. When these relations are taken as boundary conditions valid at the unification scale in the renormalization group (RG) equations, one gets a prediction of the Higgs boson mass to be around 170 ± 10 GeV, the error being due to our ignorance of the physics at unification scale. In addition, there is one relation between the sum of the square of fermion masses and the W particle mass square

$$\sum_{\text{generations}} (m_e^2 + m_\nu^2 + 3m_u^2 + 3m_d^2) = 8M_W^2,$$

which enables us to predict the top quark mass compatible with the measured experimental value.

We note that general studies of the Higgs sector in the standard model [8] show that when the Higgs and top quark masses are comparable, as in our case, then the Higgs mass will be stable under the renormalization group equations, up to the Planck scale.

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[1] H. Georgi, in *Particles and Fields*, edited by C. E. Carlson (AIP, New York, 1975).
[2] A. Connes, *Noncommutative Geometry* (Academic, New York, 1994).
[3] Ali H. Chamseddine and Alain Connes, Phys. Rev. Lett. **77**, 4868 (1996); Commun. Math. Phys. **186**, 731 (1997).
[4] Ali H. Chamseddine, Adv. Theor. Math. Phys. **11**, 991 (2007).
[5] Ali H. Chamseddine and Alain Connes, arXiv:hep-th0706.3688 [J. Geom. Phys. (to be published)].
[6] Alain Connes, J. High Energy Phys. 11 (2006) 081.
[7] John Barrett, J. Math. Phys. (N.Y.)**48**, 012303 (2007).
[8] Thomas Hambye and Kurt Riesselmann, Phys. Rev. D **55**, 7255 (1997).