Correlators and Fractional Statistics in the Quantum Hall Bulk

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We derive one-particle and two-particle correlators of anyons in the lowest Landau level. We show that the two-particle correlator exhibits signatures of fractional statistics which can distinguish anyons from their fermionic and bosonic counterparts. These signatures include the zeros of the two-particle correlator and its exclusion behavior. We find that the one-particle correlator in finite geometries carries valuable information relevant to experiments in which quasiparticles on the edge of a quantum Hall system tunnel through its bulk.

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The fractional quantum Hall effect (FQHE) is believed to provide the paradigm of a many-body system that hosts quasiparticles that are anyons, i.e., that obey fractional statistics interpolating between the quantum statistics of fermions and bosons [1,2]. The fractional charge of the FQHE quasiparticles has been measured through shot noise experiments [3], but despite a variety of theoretical proposals [4–7] and experimental attempts [8], an unambiguous signal of their anticipated fractional statistics is still lacking. Most of the theoretical and experimental effort has been expended on investigating the effects of anyon statistics on the tunneling and correlations of the low energy edge quasiparticles [9]. The anyon bulk correlations [4] have received less attention—as has the problem of relating the bulk correlations to the QH edge. Given that anyons are intrinsically two dimensional, a study of their bulk properties is much called for. In this Letter, towards this end, we formulate and analyze anyon correlators in the lowest Landau level (LLL) for unbounded and bounded geometries [10].

A useful tool for investigating bulk statistics-induced effects is the two-particle kernel $K_2(\vec{r}_{1f}, \vec{r}_{2f}, \vec{r}_{1i}, \vec{r}_{2i})$. This is the amplitude for two quasiparticles to start at points $\vec{r}_{1i}$ and $\vec{r}_{2i}$ and end up at points $\vec{r}_{1f}$ and $\vec{r}_{2f}$ (see Fig. 1). It was first used in the 1950s by Hanbury Brown and Twiss [11,12] to measure stellar diameters and is used in modern optics experiments to reveal the "bunching" properties of photons. In this Letter we derive the form and properties of K_2 and $K_1(\vec{r}_f, \vec{r}_i)$ which is the amplitude for a single quasiparticle to propagate from an initial point $\vec{r}_i$ to a final point $\vec{r}_f$, for LLL anyons. In nuclear scattering, it is well known that the amplitude for two incoming fermions in vacuum to scatter at an angle of $\pi/2$ is zero in the absence of a magnetic field [11,13]. We use K_2 to show that anyon statistics gives rise to similar dramatic effects in LLL anyon-anyon correlations that qualitatively differ from the situation of no magnetic field and can be distinguished from their bosonic and fermionic counterparts.

We begin with a two-dimensional system of two anyons in a magnetic field; the wave function by definition picks up a phase of $e^{i\pi\alpha}$ ($e^{-i\pi\alpha}$) upon a anticlockwise (clockwise) exchange of the particles. The parameter α lies in the range $-1 < \alpha \leq 1$; $\alpha = 0$ and 1 correspond to bosons and fermions, respectively. Such LLL anyon models provide an effective description of quasihole excitations associated with the addition of vortices to the QH bulk [14–16]. In particular, for Laughlin states [15], quasiholes have fractional charge $q = -e/m$ and statistics $\alpha = 1/m$, where m is an odd integer [17–19]. As our interest is in quasihole propagation over short distances of the order of 300 nm [3], we neglect Coulomb interaction effects which only become significant over longer distances [20] and can in principle be treated perturbatively [4].

The Hamiltonian for two anyons in a perpendicular magnetic field $\vec{B} = B\hat{z}$ has the decoupled form

$$H = \frac{1}{4\mu}\left(P_x + \frac{qB}{c}Y\right)^2 + \frac{1}{4\mu}\left(P_y - \frac{qB}{c}X\right)^2 + \frac{1}{\mu}\left(p_x + \frac{qB}{4c}y\right)^2 + \frac{1}{\mu}\left(p_y - \frac{qB}{4c}x\right)^2, \quad (1)$$

in terms of center of mass and relative variables. We require that when the two particles are exchanged in a clockwise fashion their wave function gains a phase factor $e^{i\pi\alpha}$ [16]. Here the anyons are assumed to have mass μ (which is immaterial when states are projected to the LLL)

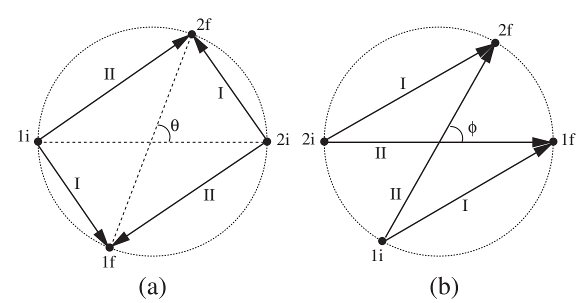

FIG. 1. Two representative configurations for anyons starting at points $\vec{r}_{1i}$ and $\vec{r}_{2i}$, and ending at points $\vec{r}_{1f}$ and $\vec{r}_{2f}$. As the particles are indistinguishable, it is not possible to determine which of two possible paths *I* and *II* each particle takes.

and charge q. We have assumed the symmetric gauge $\vec{A} = (B/2)(-y\hat{x} + x\hat{y})$. The center of mass coordinate and momentum are given by $\vec{R} = (\vec{r}_1 + \vec{r}_2)/2$ and $\vec{P} = \vec{p}_1 + \vec{p}_2$, while the relative coordinate and momentum are given by $\vec{r} = \vec{r}_1 - \vec{r}_2$ and $\vec{p} = (\vec{p}_1 - \vec{p}_2)/2$.

The eigenstates of Eq. (1) are products of eigenstates for the center of mass and relative coordinate systems. In the LLL, the center of mass eigenstates are given by

$$\psi_n(\vec{R}) = \frac{1}{l\sqrt{m\pi n!}}\left(\frac{Z}{\sqrt{m}}\right)^n \exp\left[-\frac{|Z|^2}{2m}\right], \tag{2}$$

where $n = 0, 1, 2, \ldots$. The complex parameter $Z = (X + iY)/l$ represent the components of $\vec{R}$, rescaled by the magnetic length $l = \sqrt{\hbar c/eB}$. The relative coordinate eigenstates are given by

$$\psi_p(\vec{r}) = \frac{(4\pi m)^{-1/2}}{l\sqrt{\Gamma(2p + \alpha + 1)}}\left(\frac{z}{2\sqrt{m}}\right)^{2p+\alpha} \exp\left[-\frac{|z|^2}{8m}\right], \tag{3}$$

where $p = 0, 1, 2, \ldots$, and $z = (x + iy)/l$ represents the rescaled coordinate $\vec{r}$. The eigenstates in Eq. (3) respect the anyon exchange property; we will set $\alpha = 1/m$ below.

We can now evaluate the one- and two-particle kernels, defined in imaginary time τ as

$$K_1(\vec{r}_f; \vec{r}_i) = \sum_n \psi_n(\vec{r}_f)\psi_n^*(\vec{r}_i)e^{-E_n\tau/\hbar},$$

$$K_2(\vec{r}_{1f}, \vec{r}_{2f}; \vec{r}_{1i}, \vec{r}_{2i}) = \sum_n \psi_n(\vec{R}_f)\psi_n^*(\vec{R}_i)e^{-E_n\tau/\hbar} \times \sum_p \psi_p(\vec{r}_f)\psi_p^*(\vec{r}_i)e^{-E_p\tau/\hbar}. \tag{4}$$

In the LLL, all the energies E_n are degenerate and can be set to zero. Thus the kernels have no time dependence. Using Eqs. (2) and (3), the one-particle kernel takes the form

$$K_1(\vec{r}_f; \vec{r}_i) = \frac{1}{2\pi m l^2} \exp\left[-\frac{1}{4m}(|z_f|^2 + |z_i|^2 - 2z_f z_i^*)\right], \tag{5}$$

and the two-particle kernel takes the form

$$K_2(\vec{r}_{1f}, \vec{r}_{2f}; \vec{r}_{1i}, \vec{r}_{2i}) = \frac{1}{(2\pi m l^2)^2} \exp\left[-\frac{1}{4m}(|z_{1f}|^2 + |z_{2f}|^2 + |z_{1i}|^2 + |z_{2i}|^2 - (z_{1f} + z_{2f})(z_{1i}^* + z_{2i}^*))\right] \times \sum_{p=0}^{\infty} \frac{[(z_{1f} - z_{2f})(z_{1i}^* - z_{2i}^*)/4m]^{2p+1/m}}{\Gamma(2p + 1/m + 1)}. \tag{6}$$

We remark here that a similar form for K_2 was presented by Laughlin in Ref. [15] using different reasoning.

In the case of fermions or bosons, the two-particle kernel can be separated into products of individual paths, i.e., $K_2(\vec{r}_{1f}, \vec{r}_{2f}; \vec{r}_{1i}, \vec{r}_{2i}) = K_1(\vec{r}_{1f}; \vec{r}_{1i})K_1(\vec{r}_{2f}, \vec{r}_{2i}) \mp K_1(\vec{r}_{2f}; \vec{r}_{1i})K_1(\vec{r}_{1f}, \vec{r}_{2i})$. Consequently, in the case of Fig. 1(a), K_2 can only vanish if the magnitudes of the kernels along paths of type I and II equal each other. This implies that $(z_{1f} - z_{2f})(z_{1i}^* - z_{2i}^*)$ is imaginary, i.e., that $\vec{r}_{1f} - \vec{r}_{2f}$ is perpendicular to $\vec{r}_{1i} - \vec{r}_{2i}$ and $\theta = \pi/2$. Furthermore, for the two separable paths to cancel one another, their phases must differ by π. This second condition requires that the quantity $(eB/hc)\hat{z}\cdot(\vec{r}_{1i} - \vec{r}_{2i}) \times (\vec{r}_{1f} - \vec{r}_{2f})$ must be an even (odd) integer for fermions (bosons). The geometric interpretation of this is that K_2 vanishes when the phase difference between paths of type I and II is π; this phase difference is given by the sum of the Aharonov-Bohm phase picked up by the loop in Fig. 1(a) and the phase $\pi/0$ due to an exchange of the two fermions or bosons. On setting $B = 0$, we recover the result that when there is no magnetic field, the two-particle kernel vanishes for fermions if $\theta = \pi/2$.

For the case of anyons, neither the two-particle wave function nor the two-particle kernel is of a separable form. However, a detailed analysis of Eq. (6) shows that the geometric arguments presented above still hold [21]. Thus, in the configuration of Fig. 1(a), the two-particle kernel vanishes for the same conditions stated for fermions and bosons, except that the statistical phase picked up by the anyons for a closed loop is $\pm\pi/m$ for a clockwise or anticlockwise loop. Hence, as shown in Fig. 2(a), the kernel vanishes if the rectangle in that figure is a square ($\theta = \pi/2$), and its area is quantized as $2r^2 = (hc/eB)m(n - 1/2 + 1/(2m))$. In general, if the charge of the quasiparticle is e^* and the statistics parameter is α, the kernel vanishes if the area of the square is $(hc/e^*B) \times (n - 1/2 + \alpha/2)$. (For incompressible fractional QH states with filling fraction different from $1/m$, e^*/e and α differ from each other [22].)

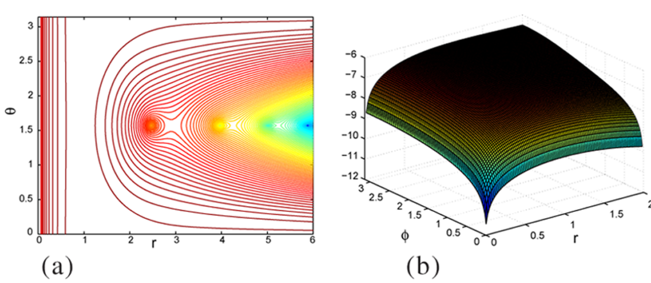

FIG. 2 (color online). Magnitude of the two-anyon kernel vs the radius r and the angle θ and ϕ for the configurations shown in Figs. 1(a) and 1(b), respectively, for $m = 3$. Figure 2(a) shows the first four zeros of the kernel lying on the line $\theta = \pi/2$, while Fig. 2(b) shows that the kernel vanishes as either r or ϕ approaches zero.

The above suggests an experiment for measuring the charge and statistics of the quasiparticles. Consider a configuration in which there are two sources and two detectors of anyons, corresponding to the positions z_{1i}, z_{2i} and z_{1f}, z_{2f}, respectively, which are arranged as a square. Holding the area of the square and the filling fraction ν fixed, one can gradually change the magnetic field and determine some successive values of the field where a simultaneous observation of the anyon tunneling current in the two detectors gives a null result. By the above arguments, a plot of the field B versus the number n will be a straight line whose slope will give the charge e^* and whose intercept will give the value of α. We should point out here that the concept of fractional statistics is valid only when the separation between two anyons is greater than about $10l$ [22]; for $\nu = 1/3$, l is about 10 nm.

The two-particle kernel also exhibits features which reflect the exclusion statistics of anyons [23]. For the case shown in Fig. 1(b), the kernel exhibits the power-law dependence $K_2 \sim |\phi|^{2/m}$ as $\phi \to 0$. Physically, the amplitude for two anyons to start at nearby points and to have a small scattering angle vanishes as the angle becomes small. Similarly, the probability that two anyons are a distance r apart is related to the two-particle kernel with $\vec{r}_{1i} = \vec{r}_{1f} = 0$ and $\vec{r}_{2i} = \vec{r}_{2f} = \vec{r}$. The kernel goes as $r^{2/m}$ as $r \to 0$. For the case $m = 1$, we reproduce the result that the probability that a fermion is at a given distance r away from another fermion is proportional to r^2. On the other hand, as $m \to \infty$, one particle does not experience the existence of another; this is indeed the situation for condensed bosons. For intermediate values of m, the power-law behavior shows that the presence of one particle excludes that of another (thus the Laughlin quasiparticles are fermionlike); the antibunching property becomes more pronounced for smaller values of m.

Towards understanding how the bulk features described above can affect properties of the boundary, we now turn to finite-size geometries which are relevant to the physical setting of the Hall bar. As a first step, we will analyze one-particle properties here. [An analysis of two-particle properties in bounded systems is needed to completely study the signatures of fractional statistics in edge-state quasihole tunneling through the bulk, but this is beyond the scope of this Letter.] In a geometry such as the one shown in Fig. 3, we will provide a simple quantum mechanical derivation of one-particle correlations along the edge, thereby addressing the assumptions made on phenomenological grounds for one-particle tunneling events in previous treatments. The system is confined in the y direction via a potential $V(y)$. The Landau gauge $\vec{A} = -By\hat{x}$ proves to be convenient here. The corresponding one-particle eigenstates are of the form $\psi_{k,n}(x, y) = e^{ikx} f_{k,n}(y)$, where the function $f_{k,n}$ depends on the confining potential and the momentum $k = 2\pi p/L_x$ along the x direction, where p is an integer and L_x is the length of the strip.

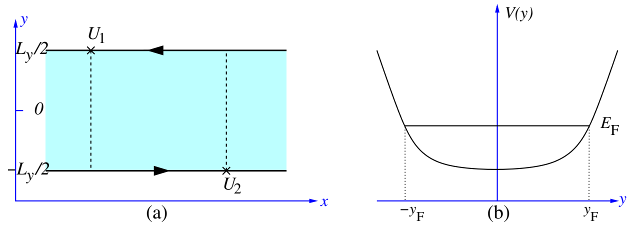

FIG. 3 (color online). Quantum Hall state in a strip geometry of width L_y in the presence of a confining potential $V(y)$. States are filled up to a Fermi energy E_F. Tunneling across the strip can take place via impurities denoted by U_i.

We first consider a simple example with no external potential $V(y)$ except for hard boundaries confining the strip to a width L_y centered at $y = 0$. In the LLL ($n = 0$), the eigenfunctions $f_{0,k}$ are Gaussians proportional to $\exp[-(y + kl^2)^2/(2l^2)]$, where k ranges from $-L_y/(2l^2)$ to $L_y/(2l^2)$. The one-particle kernel can be evaluated using Eq. (4); its salient features are as follows. For large width $L_y \gg l$, the magnitude of the kernel goes as $|K_1| \sim \exp[-(z_i - z_f)^2/(4l^2)]$ when the points lie well within the bulk or on opposite edges $y_i = -y_f = L_y/2$. If the two points lie on the same edge, $y_i = y_f = L_y/2$, the kernel obeys the power law $|K_1| \sim 1/|x_i - x_f|$ in the limit $x_i - x_f \gg l$. This is consistent with the power-law decay obtained from the edge-state picture for integer filling $\nu = 1$. However, in the interesting case of the width becoming *comparable* to the magnetic length, $|K_1|$ shows oscillations with a wavelength of about $2\pi l^2/L_y$ which become more pronounced when the two points lie on opposite edges. This oscillatory behavior suggests that finite-size effects in realistic situations could cause significant deviations from predictions for extended systems.

As an application of the one-particle correlator to several proposals and experiments, we now consider tunneling between integer quantum Hall edges caused by scattering off localized impurities which we model as

$$U(x, y) = \sum_n U_n \delta(x - x_n)\delta(y - y_n). \qquad (7)$$

In the absence of a confining potential, all k states are degenerate and the scatterers cause mixing between all states. In reality, as shown in Fig. 3(b), the confining potential breaks the degeneracy, and electrons fill states up to a Fermi momentum k_F and an associated width $L_y = 2y_F$. The confining potential produces an effective electric field along the edge, $\mathcal{E} = -(dV/dy)_{y=y_F}$. Electrons experience a drift velocity given by $v_F = c|\mathcal{E}/B|$ and they move in opposite directions along the top and bottom edges. Within the first-order Born approximation and assuming a linearized potential close to each edge (and thus a linearized dispersion about the Fermi energy), we find that the scatterers couple each k state to the corresponding $\pm k$ states [21]. The reflection amplitude for a right-moving edge state $k \approx k_F$ to scatter to a left-moving state $k \approx -k_F$ is given by

$$r = -\frac{i}{\hbar v_F} \sum_n U_n e^{i2k_F x_n} \exp[-y_n^2/l^2] \left(\frac{1}{\pi l^2}\right)^{1/2} \times \exp[-y_F^2/l^2]. \quad (8)$$

This amplitude is directly related to the matrix element for particles to tunnel between edge states; it describes one-particle propagation from one edge to another. Our derivation of Eq. (8) is simple enough that it can go beyond the strip geometry to any smooth confining potential and to any configuration of the tunneling sites.

The form of Eq. (8) has several noteworthy features. As expected, the tunneling matrix element for each impurity decays exponentially over a magnetic length. For an impurity localized on an edge at a point x_n, tunneling to the other edge occurs along the *shortest* path. The treatment here is valid for electrons with charge e. In general, consistent with derivations of tunneling matrix elements which explicitly use the Laughlin wave function [24], we expect a similar form for any particle having charge e^* with this charge replacing e; hence the decay of the *bare* tunneling matrix element is enhanced or suppressed by a factor of e^*/e in the exponent. For the case of more than one impurity, the reflection coefficient is sensitive to interference effects arising from multiple paths. In the case of two impurities of equal strength lying on either edge at points x_1 and x_2, reflection amplitudes off the two impurities have a phase difference of $2k_F(x_2 - x_1) = 2y_F(x_2 - x_1)/l^2$. Thus, as phenomenologically described in Ref. [7], we explicitly see Aharonov-Bohm interference arising from the particle traversing two different paths enclosing a rectangular area of length $|x_1 - x_2|$ and width $2y_F$.

In conclusion, we have analyzed two ubiquitous entities—the one- and two-particle kernels—in the physically motivated situation of charged particles in a strong magnetic field. The two-particle kernel in the quantum Hall bulk contains information on statistics which is strikingly manifest in the zeros of the kernel. The one-particle kernel in a finite geometry is shown to provide an understanding of certain features of bulk mediated tunneling between edges, such as the tunneling amplitude and Aharonov-Bohm physics in a system with two tunneling centers. In principle, some of our predictions for the two-particle kernel can be tested in realistic gate-defined Hall geometries. Our studies show that a complete explanation of experiments that measure two-particle properties, whether of bulk or edge-state quasiparticles, will need to take into account correlations and exclusion effects in the bulk. More spectacularly, our studies indicate that it may be possible to perform experiments in quantum Hall geometries, perhaps involving multiedge tunneling, wherein correlations show signatures of fractional statistics in angular dependences similar to those observed for fermions and bosons in scattering experiments.

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