Critical Scaling in Standard Biased Random Walks

C. Anteneodo* and W. A. M. Morgado†

Departamento de Física, Pontifícia Universidade Católica do Rio de Janeiro, CP 38097, 22453-900, Rio de Janeiro, Brazil

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The spatial coverage produced by a single discrete-time random walk, with an asymmetric jump probability $p \neq 1/2$ and nonuniform steps, moving on an infinite one-dimensional lattice is investigated. Analytical calculations are complemented with Monte Carlo simulations. We show that, for appropriate step sizes, the model displays a critical phenomenon, at $p = p_c$. Its scaling properties as well as the main features of the fragmented coverage occurring in the vicinity of the critical point are shown. In particular, in the limit $p \rightarrow p_c$, the distribution of fragment lengths is scale-free, with nontrivial exponents. Moreover, the spatial distribution of cracks (unvisited sites) defines a fractal set over the spanned interval. Thus, from the perspective of the covered territory, a very rich critical phenomenology is revealed in a simple one-dimensional standard model.

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Since the beginning of the past century, random walk (RW) theory has allowed us to deal with a diversity of problems in a number of areas of physics, as well as in many other theoretical and applied fields, e.g., biology, chemistry, computer sciences, and finance [1]. The doubtless importance of RW models, with their wide range of distinct applications, stems from their simplicity and effectiveness in modeling systems experiencing disorder, noise or randomness, which are ubiquitous features of real systems. In particular, in physics, RWs can be seen as the "harmonic oscillator" of disordered and stochastic systems, serving as a starting point for more realistic models.

A fundamental quantity in any phenomenon where RWs are relevant is the number of distinct sites visited, since it furnishes the extent of the active territory. Indeed, it is crucial in processes ranging from reaction kinetics to population dynamics, and also in technical applications such as in search strategies [2,3]. As a consequence, analytical and numerical estimates of the covered territory are available for lattices of different geometry, dimensionality, and boundary conditions [4,5], for diverse statistics of jumps, symmetric or not [6], and other variants [7]. Time covering problems [8,9] and coverage by a large number of RWs [10] have been investigated too. The vast literature on coverage mainly deals with two dimensions, although there are also many works about the standard symmetric one-dimensional (1D) RW (e.g., [4,5,9]). Meanwhile, as far as we know, little or no attention has been paid to the asymmetric 1D case, despite of its importance in biased or anisotropic processes such as electrophoresis, polymer translocation through pores, and Brownian ratchets. However, as we will show, the asymmetric 1D problem presents its own peculiar features and nontrivial scaling properties.

In the present work, we investigate the coverage of an infinite 1D regular lattice by a single RW characterized by (i) asymmetry, that is, at each independent step there is a probability $p \neq 1/2$ to step, let us say, to the right, and additionally, (ii) distinct step sizes in opposite directions. Let us call l^+ and l^- the sizes of the steps in the positive and negative directions, respectively. They will be expressed as integer multiples of the arbitrary lattice parameter. In the symmetric case $l^+ = l^-$, only the positions that are multiple of l^+ are reachable. Moreover, the covered fraction of the interval spanned by the RW is $1/l^+$, independently of p. In particular, if $l^+ = l^- = 1$, complete coverage of the RW span occurs. However, for the asymmetric case $l^+ \neq l^-$, where the two anisotropic ingredients compete, a nontrivial changeover between different coverage regimes, dependent on p, may take place. In fact, we will show that a critical phenomenon occurs as the jump probability p reaches a critical value. Moreover, we will characterize the transition as well as the partially covered, fragmented, states, focusing on their scaling properties.

The general basic outlines to determine the number of distinct sites visited by a RW can be found, for instance, in Refs. [4,5]. In general, the average number of different sites visited at step n, S_n, can be expressed as $S_n = 1 + \sum_{s\neq 0}\sum_{i=1}^{n} F_i(s)$, where $F_i(s)$ is the probability that the walker arrives at site s for the first time at step i. Moreover, $F_i(s)$ and $P_j(s)$ (the probability that, at time step j, the walker is located at integer position s) are related through

$$P_n(s) = \sum_{i=1}^{n} F_i(s) P_{n-i}(0), \quad \text{for} \quad n \geq 1, \tag{1}$$

while $P_o(s) = \delta_{s,0}$. Then, from Eq. (1), one obtains the following relation between generating functions: $P(s, z) = \delta_{s,0} + F(s, z)P(0, z)$, where $P(s, z) = \sum_{n\geq 0} P_n(s)z^n$ and $F(s, z) = \sum_{n\geq 1} F_n(s)z^n$. Assuming $|z| \leq 1$, one obtains

$$S(z) = [(1 - z)^2 P(0, z)]^{-1}, \tag{2}$$

where $S(z) \equiv \sum_{n\geq 0} S_n z^n$. For the present problem, it is easy to show that $P(0, z)$ explicitly is

$$P(0, z) = \sum_{k\geq 0} \binom{\frac{(l^+ + l^-)k}{l^-}}{k} \tilde{z}^{(l^+ + l^-)k/l^-}, \qquad (3)$$

with $\tilde{z} = zp^{l^-/(l^+ + l^-)}(1 - p)^{l^+/(l^+ + l^-)}$. From the definition of $S(z)$, the quantity S_n can be obtained as $1/n!$ times the nth derivative of $S(z)$, evaluated at $z = 0$.

If l^+ and l^- have common factors, a mapping exists into the corresponding case of reduced (mutually prime) lengths. Therefore, we will restrict our study to asymmetric coprime couples of step lengths. Within the latter class of RWs, one has the subclass where one of the lengths is unitary. Let us consider as representative of this subclass, the case $(l^+, l^-) = (2, 1)$ that admits an exact solution. In this case, the sum in Eq. (3) becomes

$$P(0, z) = \sum_{k\geq 0} \binom{3k}{k} p^k (1 - p)^{2k} z^{3k},$$

that can be reduced to

$$P(0, z) = \Re(iy + \sqrt{1 - y^2})^{1/3}/\sqrt{1 - y^2}, \qquad (4)$$

for $|y| \leq 1$, where $y^2 = (27/4)p(1 - p)^2 z^3$. Tauberian methods can be applied to evaluate S_n [4]. Alternatively, the nth derivative of $S(z)$ can be calculated through the Cauchy integral formula over a suitable contour encircling the origin. Since $S(z)$ given by Eq. (2) has one single pole in the complex plane, at $z = 1$, then, in the limit of large n, one gets (after conveniently deforming the integration path) $S_n = -d[z^{n+1}P(0, z)]^{-1}/dz|_{z=1} = (n + 1)/P(0, 1) + c_0$, where c_0 is a constant of order 1. It is noteworthy that this is the same asymptotic law found for the standard RW, with unbiased symmetric jumps to nearest neighbors (hence $l^+ = l^- = 1$), but in 3D regular lattices [4]. The fraction of different sites visited (measured over the average length of the RW) is $f_{v,n} \equiv S_n/L_n$, where L_n is the average total displacement. In the large n limit, the length of the RW, for $p \neq 1/3$, is $L_n \sim |\langle s\rangle_n| = |3p - 1|n$. Thus, asymptotically, $f_{v,n}$ becomes $f_v = [|3p - 1|P(0, 1)]^{-1}$; hence, the fraction of unvisited sites is

$$f_u = 1 - f_v = 1 - [|3p - 1|P(0, 1)]^{-1}, \qquad (5)$$

where $P(0, 1)$ is given by Eq. (4).

Figure 1(a) exhibits f_u as a function of p, for $(l^+, l^-) = (2, 1)$. A transition occurs at $p_c = 1/3$, where f_u vanishes as $f_u = \Delta + \mathcal{O}(\Delta^2)$, with $\Delta \equiv p - p_c$, that can be derived exactly from Eq. (5). For $p \leq p_c$ all sites are eventually visited at least once, as expected, because, as soon as $\langle s\rangle_n = 3\Delta < 0$, the walker is biased towards the direction of unitary steps, which in turn implies full coverage of the RW length. Meanwhile, for $p > p_c$, sequences of adjacent visited sites (fragments) are interrupted by unvisited ones. Therefore, the RW undergoes a transition from a fully covered state to a fragmented one. For other instances of $(l^+, 1)$, the transition occurs at the critical probability $p_c = 1/(l^+ + 1)$, where $\langle s\rangle_n = (pl^+ + p - 1)n$ changes sign (driftless diffusion). The case $(l^+, l^-) = (3, 1)$, obtained by means of Monte Carlo (MC) simulations up to $n \approx 10^7$ time steps, is also displayed in Fig. 1(a), exhibiting similar features. In both cases, $f_u(\Delta)$ vanishes with unitary exponent [see inset of Fig. 1(a)].

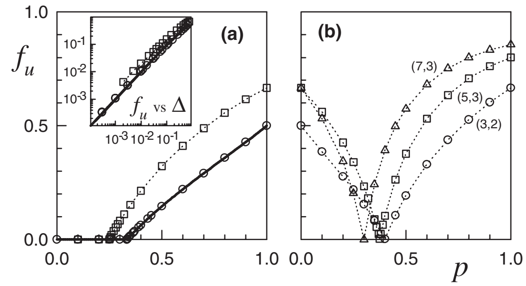

FIG. 1. Fraction f_u of sites left unvisited as a function of p. In all cases, symbols correspond to MC simulations. and dotted lines are guides to the eye. (a) $(l^+, l^-) = (2, 1)$ (circles) and (3,1) (squares). The full line corresponds to the theoretical prediction given by Eq. (5). Inset: f_u vs $\Delta \equiv p - p_c$ in log-log scale for the same data of the main frame. (b) (l^+, l^-) takes diverse coprime values indicated on the figure.

For nonunitary coprime step lengths [see Fig. 1(b)] a more general scenario arises. Full coverage occurs only at the critical point $p_c = l^-/(l^+ + l^-)$, where $\langle s\rangle_n = [pl^+ + (p - 1)l^-]n$ is strictly null. Fragmented states are found both below and above p_c, with maximal unvisited fractions, $f_u^- = 1 - 1/l^-$ and $f_u^+ = 1 - 1/l^+$, respectively. Thus, the cases $(l^+, 1)$, with $l^+ > 1$, constitute special instances where one of the states is fully covered, in accordance with the fact that the corresponding maximal unvisited fraction f_u^- vanishes. Although we are not dealing with symmetric steps, notice that in the symmetric case (1,1), $f_u^- = f_u^+ = 0$ and the full curve $f_u(p)$ collapses to zero, in agreement with the facts that there is no transition in such case and that full coverage occurs for any p.

As a paradigmatic example, we will analyze the analytically soluble case $(l^+, l^-) = (2, 1)$, in the vicinity of the critical point, i.e., in the limit $\Delta \to 0^+$. In order to quantitatively characterize fragment sizes, the usual computed quantities are [11]

$$\tilde{n}_\ell = \sum_{\ell\geq 1} n_\ell, \qquad \langle\ell\rangle = \sum_{\ell\geq 1} n_\ell \ell^2 \Big/ \sum_{\ell\geq 1} n_\ell \ell, \qquad (6)$$

where n_ℓ is the mean number of fragments of size ℓ, normalized per site. Since two contiguous fragments are separated, in the (2,1) case, by one single unvisited site, then $\tilde{n}_\ell \approx f_u$, that vanishes as $\sim\Delta$ [see Fig. 1(a)]. Also, straightforwardly, $\sum_{\ell\geq 1} n_\ell \ell = 1 - f_u$, that approaches 1 in the critical limit. Noticing that $n_\ell \ell$ is the probability that a given site belongs to a fragment of size ℓ, then, $\tilde{\ell} \approx \sum_{\ell\geq 1} n_\ell \ell^2$ defines the mean size of the fragments. In order to compute $\langle\ell\rangle$, the distribution of sizes of covered clusters (or fragments), n_ℓ, was numerically built from MC simulations run up to $n \approx 10^6/\Delta$ steps and averaged over at least 10^2 different realizations. The distributions for different values of Δ are displayed in Fig. 2. For very large ℓ, the decay is exponential: $\sim \exp(-\ell/\lambda)$. Parameter λ, together with $\langle \ell \rangle$, are plotted as a function of Δ in the upper inset of Fig. 2 (being $\lambda \approx \langle \ell \rangle/2 \sim \Delta^{-\gamma}$, with $\gamma \approx 1.15$). Meanwhile, $n_1 \sim \Delta$, representing a finite fraction of f_u. In the lower inset of Fig. 2, the same distributions of the main frame are scaled. Let us employ the standard *ansatz* for cluster size distributions [11], defined through

$$n_\ell(\Delta) \propto \Delta^\omega \phi(\Delta^{1/\sigma}\ell)/(\Delta^{1/\sigma}\ell)^\tau, \tag{7}$$

where $\phi(x)$ goes to a constant value for small x and decays exponentially in the opposite limit of large x. The power-law decay, with exponent $\tau \approx 1.15$, that emerges in the limit of vanishing Δ is characteristic of a critical behavior and signals the coexistence of fragments of all sizes in that limit.

By means of integral approximations to the sums in Eqs. (6) and employing Eq. (7), one gets the following relations amongst critical exponents. First, $1 \approx \sum_{\ell\geq 1} n_\ell \ell \approx \int_1^\infty n_\ell \ell d\ell \sim \Delta^{\omega-2/\sigma}$, implying $\omega = 2/\sigma$. Second, $\Delta^{-\gamma} \sim \langle \ell \rangle \approx \int_1^\infty n_\ell \ell^2 d\ell \sim \Delta^{\omega-3/\sigma}$, hence $\omega = 3/\sigma - \gamma$, that, together with the preceding relation, implies $\gamma = 1/\sigma$ and $\omega = 2/\sigma$. The latter equality is in good accord with the behavior of the envelope of the distributions that has slope -2 (Fig. 2). Excellent data collapse is obtained for $\omega = 2/\sigma$, with $\gamma \approx 1.15$. Additionally, since $\tilde{n}_\ell \sim \Delta$, then, from $\tilde{n}_\ell \approx \int_1^\infty n_\ell d\ell \sim \Delta^{\omega-\tau\gamma}$, it must be $\tau = 2 - \sigma = 2 - 1/\gamma$. From the scaled histograms, we obtained $\tau \approx 1.15 \pm 0.05$, consistent with the theoretical prediction within error bars.

At this point, it is worth comparing our results with those for another 1D critical phenomenon, namely, 1D percolation (1DP) with bonds connecting nearest neighbors [12], to which many important 1D models are related (e.g., Ref. [13]). On one hand, for 1DP, $\tilde{n}_\ell \sim \Delta^{2-\alpha_p}$, with $\alpha_p = 1$, as in the present problem. On the other hand, $\langle \ell \rangle \sim \Delta^{-\gamma_p}$, with $\gamma_p = 1$ and $\omega_p = 2\gamma_p = 2$, values that are close but different from those found for the present problem. Moreover, the distribution of fragment sizes is a power-law, in contrast with the pure Poissonian one for 1DP. Then, we may conclude that the present model does not belong to the 1DP universality class. Indeed, by identifying visited sites with occupied ones, the occupation probability in our problem is f_v, that tends to one in the critical limit. However, different from the standard percolation problem, in the present case, unvisited sites are not independently located, e.g., if $(l^+, l^-) = (2, 1)$, a sequence of two or more adjacent unvisited sites has associated a strictly null probability of occurrence. Therefore, occupation correlations arise which are absent in the standard percolation problem.

Concerning unvisited sites, their spatial distribution was investigated through a box-counting procedure [11,14]. From the history of a single RW, a segment of length $L = 2^{20} \approx 10^6$ was divided into boxes of length 2^k, with $k \geq 0$. For each $\varepsilon = 2^k/L$, the number of boxes containing unvisited sites, $N(\varepsilon)$, was computed. Outcomes, accumulated over 10^2 realizations, are displayed in Fig. 3. The neat behavior $N(\varepsilon) \sim \varepsilon^{-d_f}$, for small ε, means that the spatial distribution of unvisited sites constitutes a fractal set, with dimension d_f. Moreover, the fractal exponent is in good accord with the exact scaling relation $d_f = \tau - 1$.

In summary, we have investigated the spatial coverage of single discrete-time anisotropic RWs, moving on an infinite one-dimensional lattice. Anisotropy manifests both in the length (l^+, l^-), as well as in the probabilities $(p, 1-p)$, of jumps in opposite directions. We revealed the existence of a critical phenomenon in 1D that may

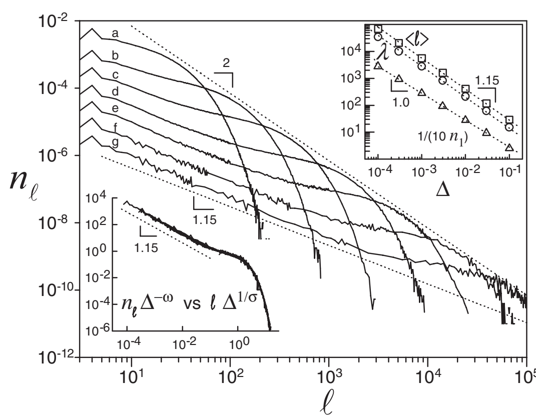

FIG. 2. Distribution of the sizes of covered fragments (n_1 was omitted), for $(l^+, l^-) = (2, 1)$ and different values of $\Delta = 10^{-1}, 3 \times 10^{-2}, \ldots, 10^{-4}$, from a to g, respectively. Upper inset: mean size of fragments $\langle \ell \rangle$ (squares), inverse exponential rate λ (circles), and $[10n_1]^{-1}$ (triangles) as a function of Δ. Lower inset: Scaling plot of all the distributions represented in the main frame, with $\omega = 2/\sigma$ and $1/\sigma = 1.15 \pm 0.05$. Dashed lines are drawn for comparison and their slopes indicated on the figure.

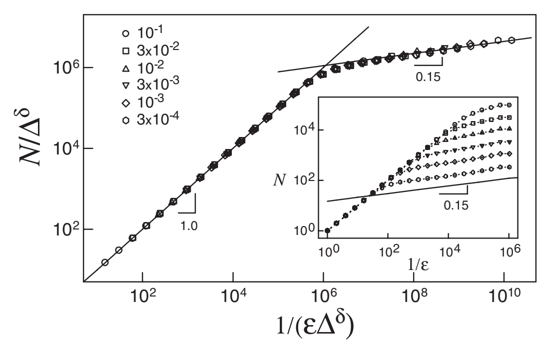

FIG. 3. Scaling plot of the number of boxes N containing unvisited sites as a function of ε (box size in units of L, where $L = 2^{20}$) for $(l^+, l^-) = (2, 1)$ and different values of Δ indicated on the figure. The scaling exponent is $\delta = 1/(1 - d_f)$, where $d_f \simeq 0.15$. Inset: original plots of the data scaled in the main frame. All solid lines are drawn for comparison and their slopes indicated on the figure.

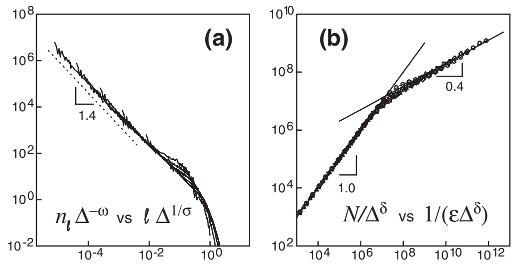

FIG. 4. Critical behavior for $(l^+, l^-) = (5, 3)$. Results correspond to the limit $\Delta \to 0^+$, but the same exponents are found in the limit $\Delta \to 0^-$. (a) Scaling plot of the distribution of the sizes of covered fragments, values of Δ as in Fig. 2, with $\omega = 2/\sigma$ and $1/\sigma = 1.5 \pm 0.2$, $\tau \simeq 1.4$. (b) Scaling plot of the number of boxes N containing unvisited sites as in Fig. 3. In this case $d_f \simeq 0.4$. The dotted line in (a) and solid lines in (b) are drawn for comparison and their slopes indicated on the figure.

result from the competition between the opposite trends provided by the two anisotropic ingredients (step size and step probability). We illustrated our findings with the particular case in which the steps are $(l^+, l^-) = (2, 1)$, which undergoes a transition from fully to partially covered states as the jump probability p overcomes a critical value. The power-law distribution of sizes of covered segments, occurring in the limit $p \to p_c^+$, indicates the coexistence of fragments of all lengths, with no characteristic length scale. Moreover, the spatial distribution of scission points (unvisited sites) determines a fractal set, in contrast with other models where the deposition of cracks has common statistics (e.g., 1D percolation [12] and scission model [15]). It is pertinent to remark that similar features have been observed in one-dimensional reaction-diffusion [16,17], q-state Potts spin flipping [17], and fragmentation dynamics [18], although criticality is attained as time evolves and critical exponents are different. A possible connection remains to be investigated.

Other asymmetric instances with steps $(l^+, 1)$, whose critical curves are illustrated in Fig. 1(a), display a qualitatively similar picture to the case (2,1). Meanwhile, if both steps take nonunitary coprime values [Fig. 1(b)], the same critical phenomenology is observed in both limits $\Delta \to 0^\pm$. As a further example, scaling plots are also displayed, in Fig. 4, for the case $(l^+, l^-) = (5, 3)$ in the limit $\Delta \to 0^+$. In general, critical exponents related to the fractal dimension are not universal but depend on the step lengths, since distinct site occupation correlations take place.

On one hand, the asymmetric RW, seen from the present perspective, may bear interest *per se* because of the nontrivial criticality contained in a simple model. On the other hand, it may constitute a useful statistical paradigm for the formation of domains or fragments by a nonequilibrium process driven by biased signal propagation. Additionally, the current coverage problem may be potentially useful in technical applications, e.g., in search strategies such as for cache hit/miss ratio optimization [3].

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*celia@fis.puc-rio.br
†welles@fis.puc-rio.br

[1] G. H. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, New York, 1994); B. D. Hughes, *Random Walks and Random Environments, Random Walks* (Clarendon Press, Oxford, 1996), Vol. 1; P. G. Doyle and J. L. Snell, *Random Walks and Electric Networks* (Math. Assoc. Amer., Washington, DC, 1999).
[2] G. M. Viswanathan, S. V. Buldyrev, S. Havlin, M. G. E. da Luz, E. P. Raposo, and H. E. Stanley, Nature (London) **401**, 911 (1999); E. P. Raposo, S. V. Buldyrev, M. G. E. da Luz, M. C. Santos, H. E. Stanley, and G. M. Viswanathan, Phys. Rev. Lett. **91**, 240601 (2003).
[3] D. Thiébaut, IEEE Trans. Comput. **38**, 1012 (1989); I. Gluhovsky, D. Vengerov, and B. O'Krafka, ACM Trans. Comp. Syst. **25**, 2 (2007).
[4] E. W. Montroll and G. H. Weiss, J. Math. Phys. (N.Y.) **6**, 167 (1965).
[5] J. Vineyard, J. Math. Phys. (N.Y.) **4**, 1191 (1963); H. Larralde and G. H. Weiss, J. Phys. A **28**, 5217 (1995).
[6] M. Ferraro and L. Zaninetti, Phys. Rev. E **64**, 056107 (2001).
[7] J. E. Gillis and G. H. Weiss, J. Math. Phys. (N.Y.) **11**, 1307 (1970); P. Molinàs-Mata, M. A. Muñoz, D. O. Martínez, and A. L. Barabási, Phys. Rev. E **54**, 968 (1996); F. vanWijland and H. J. Hilhorst, J. Stat. Phys. **89**, 119 (1997); F. vanWijland, S. Caser, and H. J. Hilhorst, J. Phys. A **30**, 507 (1997); S.-Y. Huang, X.-W. Zou, W.-B. Zhang, and Z.-Z. Jin, Phys. Rev. Lett. **88**, 056102 (2002); E. Almaas, R. V. Kulkarni, and D. Stroud, Phys. Rev. E **68**, 056105 (2003).
[8] K. R. Coutinho, M. D. Coutinho, M. A. F. Gomes, and A. M. Nemirovsky, Phys. Rev. Lett. **72**, 3745 (1994).
[9] M. S. Nascimento, M. D. Coutinho, and C. S. O. Yokoi, Phys. Rev. E **63**, 066125 (2001).
[10] H. Larralde, P. Trunfio, S. Havlin, H. E. Stanley, and G. H. Weiss, Phys. Rev. A **45**, 7128 (1992); Nature (London) **355**, 423 (1992).
[11] J. Feder, *Fractals* (Plenum, New York, 1988).
[12] P. J. Reynolds, H. E. Stanley, and W. Klein, J. Phys. A **10**, L203 (1977); W. Klein, H. E. Stanley, S. Redner, and P. J. Reynolds, J. Phys. A **11**, L17 (1978); D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1992).
[13] K. Christensen, H. Flyvbjerg, and Z. Olami, Phys. Rev. Lett. **71**, 2737 (1993).
[14] T. Tél, Z. Naturforsch. A **43**, 1154 (1988).
[15] E. Ben-Naim and P. L. Krapivsky, Physica (Amsterdam) **107D**, 156 (1997), and references therein.
[16] G. Manoj and P. Ray, J. Phys. A **33**, 5489 (2000).
[17] B. Derrida, V. Hakim, and V. Pasquier, Phys. Rev. Lett. **75**, 751 (1995).
[18] Z. Cheng and S. Redner, Phys. Rev. Lett. **60**, 2450 (1988).