Adiabatic Quantum Search Scheme With Atoms In a Cavity Driven by Lasers

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We propose an implementation of the quantum search algorithm of a marked item in an unsorted list of N items by adiabatic passage in a cavity-laser-atom system. We use an ensemble of N identical three-level atoms trapped in a single-mode cavity and driven by two lasers. In each atom, the same level represents a database entry. One of the atoms is marked by having an energy gap between its two ground states. Appropriate time delays between the two laser pulses allow one to populate the marked state starting from an initial entangled state within a decoherence-free adiabatic subspace. The time to achieve such a process is shown to exhibit the $\sqrt{N}$ Grover speedup.

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One typical problem of quantum computation concerns the search of a marked entry in an unsorted database by accessing it a minimum number of times. The Grover algorithm [1] achieves this task quadratically faster than any classical algorithm. It is formulated in terms of a series of quantum gates applied to a quantum register consisting of a collection of qubits encoding the database entries. An initial uniform superposition $|w\rangle$, independent of the searched state $|m\rangle$, is rotated step-by-step under the action of appropriate gates. The searched state is exhibited by an oracle function which checks if a proposed input is the searched state, returning for instance 1 in this case and 0 otherwise. The number of steps grows as $N^{1/2}$ with N the database size, whereas a classical algorithm requires on average $N/2$ calls. This quantum circuit algorithm has been tested experimentally for two qubits ($N = 4$) by several techniques resting on NMR [2,3], optics [4,5], and trapped ions [6]. There have also been proposals of experimental implementations using cavity QED where the quantum gate dynamics is provided by a cavity-assisted collision [7] or by a strong resonant classical field [8].

A time-continuous version of the Grover algorithm has been proposed by Farhi and Gutmann [9], who, instead of using an explicit oracle, mark the searched state with an energy E while the others are degenerate with energy 0, and use a driving Hamiltonian that continuously leads the initial state to the marked one. Choosing a Hamiltonian $V = E|w\rangle\langle w|$ to drive the free system $H_0 = E|m\rangle\langle m|$, they have shown that a Rabi-like half-cycle leads to the target marked state in a time growing as $N^{1/2}/E$. Note that the Grover speedup is quadratic, independent of any increase of E with N, which would simply amount to renormalizing the time. An experimental realization of this analog Grover algorithm has been performed by NMR [10] in a setting where a quadrupolar coupling makes a spin 3/2 nucleus a two-qubit system ($N = 4$).

Adiabatic versions of the time-continuous Grover algorithm have been proposed [11–13] mainly to take advantage of the robustness of adiabatic passage with respect to fluctuations of the external control fields as well as to the imperfect knowledge of the model. They have been formulated with a Hamiltonian that connects adiabatically the initial ground superposition $|w\rangle$ to the marked state $|m\rangle$ through an avoided crossing: $H = [1 - u(t)]H_i + u(t)H_f$, where $H_i = I - |w\rangle\langle w|$, $H_f = I - |m\rangle\langle m|$, and $u(t)$ is a function of time growing from 0 to 1. Roland and Cerf [13] have shown that only a specific speed of the dynamics controlled by $u(t)$ allows one to achieve the transfer to the marked state in a time growing as $N^{1/2}$.

In this Letter, we show an implementation of the adiabatic Grover algorithm based on a physical system, which is in principle scalable. This is, to our knowledge, the first proposed implementation of the adiabatic Grover algorithm. It is formulated with a Hamiltonian $H = H_0 + V(t)$, where H_0 is considered as an oracle, and given, while $V(t)$ is slowly varying in time such that there is an instantaneous eigenvector adiabatically connecting the initial superposition $|w\rangle$ to the marked state $|m\rangle$. We use an ensemble of N identical three-level atoms trapped in a single-mode cavity of coupling frequency G and driven by two lasers of Rabi frequencies Ω and Ω'. The atomic levels are in a Λ configuration with two ground states $|g\rangle$

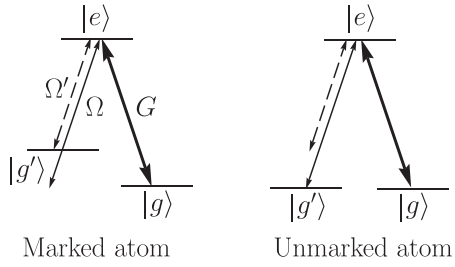

FIG. 1. Linkage pattern for the individual atoms. The unmarked atoms have two degenerate ground states $|g\rangle$ and $|g'\rangle$. One atom is marked with the state $|g'\rangle$ shifted in energy. The laser of Rabi frequency Ω' (Ω) is resonant with the g'-e transition for the marked (unmarked) atom(s). The cavity of Rabi frequency G is resonant with the g-e transition.

and $|g'\rangle$ coupled to the excited state $|e\rangle$ by, respectively, the cavity and the two lasers (see Fig. 1). This can be realized in practice by considering Zeeman states and laser and cavity fields of appropriate polarizations. The states $|g'\rangle$ of the N atoms are considered as the database entries. The energy of the state $|g'\rangle$ of the marked atom is shifted by an amount δ with respect to that of the unmarked atoms, which is set to zero. The states $|g\rangle$ allow for the coupling of all the atoms through the exchange of a single photon with the cavity [see Fig. 2(a)]. The initial state we shall start from is the entangled state $|w\rangle \equiv |g', 0\rangle \equiv (1/\sqrt{N}) \times \sum_{j=1}^{N} |g'_j, 0\rangle$, featuring a collective superposition of both types of atomic ground states. Note that the label of each atom is added as a subscript $1, \ldots, N$ and chosen so that the marked atom has tag N. Such a state can be prepared for instance before the marking of the atom using the stimulated Raman adiabatic passage (STIRAP) technique [14], exactly as shown in [15] to store single-photon quantum states.

The process we introduce here allows one to drive adiabatically the population from the entangled state $|g', 0\rangle$, which corresponds to the superposition of both the marked state $|m\rangle \equiv |g'_N, 0\rangle$ and an unmarked collective state $|g'_u, 0\rangle$ that we introduce below, to the single marked state $|g'_N, 0\rangle$ [see Fig. 2(b)]. This process will be referred to as an inverse fractional stimulated Raman adiabatic passage since it is a time inversion of the so-called fractional STIRAP, which transfers the population from a single state to a superposition of state [16]. This is implemented by first switching on Ω and Ω' together and next switching off Ω' before Ω. The characteristic time to achieve such a process will be shown to grow as $\sqrt{N}$ in order to satisfy adiabaticity.

The Hamiltonian describing the system of N atoms is

$$H_0 = \delta |g'_N\rangle\langle g'_N| + \omega \sum_{j=1}^{N} |e_j\rangle\langle e_j|. \tag{1}$$

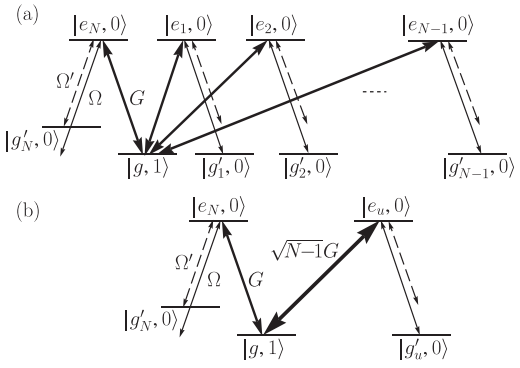

FIG. 2. (a) Coupling scheme in the cavity. The cavity G, laser Ω, and laser Ω' Rabi frequencies are featured by thick, thin, and dashed arrows, respectively. (b) Equivalent scheme where the states $|g'_i, 0\rangle$ ($|e_i, 0\rangle$), $i = 1$, $N-1$, of frame (a) form the collective unmarked ground state $|g'_u, 0\rangle$ (excited state $|e_u, 0\rangle$). The effective cavity Rabi frequency to the collective unmarked excited state is $\sqrt{N-1}G$.

We consider a cavity mode of frequency ω and coupling strength G together with two lasers of frequencies ω, $\omega - \delta$ and pulse shapes $\Omega(t)$, $\Omega'(t)$ which do not grow with N. The resonant driving provided by the cavity-laser-atom system is described by

$$V = \omega a^\dagger a + Ga \sum_{j=1}^{N} |e_j\rangle\langle g_j| + [\Omega e^{i\omega t} + \Omega' e^{i(\omega-\delta)t}] \times \sum_{j=1}^{N} |g'_j\rangle\langle e_j| + \text{H.c.} \tag{2}$$

The full Hamiltonian $H = H_0 + V$ has a photonic block diagonal structure. Each block is labeled by the number k of photons in the cavity when the N atoms are in their ground state $|g\rangle$. The corresponding multipartite state $|g_1 \cdots g_N\rangle \otimes |k\rangle$ is denoted $|g, k\rangle$. All the states that are connected to $|g, k\rangle$ span a subspace whose projection operator is P_k. As each block is decoupled under H from the other ones, $H = \sum_{k=0} P_k H P_k$, we shall focus on the block $P_1 H P_1$ associated with a single photon in the cavity and show that it allows us to implement an adiabatic Grover search algorithm.

The multipartite state $|g, 1\rangle$ is connected by (2) to exactly two families of states as illustrated in Fig. 2(a). Upon absorption of the cavity photon, the excited state $|e\rangle$ of any of the N atoms, say atom j, can be reached while the other atoms remain in their ground state $|g\rangle$; the corresponding multipartite state is $|e_j, 0\rangle \equiv |g_1 \cdots g_{j-1} e_j g_{j+1} \cdots\rangle \otimes |0\rangle$. The state $|e_j, 0\rangle$ can also be reached by absorption of one laser photon when any of the atoms, say atom j, is in the ground state $|g'\rangle$ while the other atoms are in the ground state $|g\rangle$: $|g'_j, 0\rangle \equiv |g_1 \cdots g_{j-1} g'_j g_{j+1} \cdots g_N\rangle \otimes |0\rangle$. In order to remove the oscillatory time dependence introduced by the lasers, we consider atomic states that are dressed by laser and cavity photons and use the resonant transformation $R = e^{-i\delta t} |g'_N, 0\rangle\langle g'_N, 0| + e^{-i\omega t} |e_N, 0\rangle\langle e_N, 0| + e^{-i\omega t} |g, 1\rangle\langle g, 1| + \sum_{j=1}^{N-1} (|g'_j, 0\rangle\langle g'_j, 0| + e^{-i\omega t} |e_j, 0\rangle\langle e_j, 0|)$.

As we shall see, the states that are relevant for the Grover search are the $N-1$ states $|g'_j, 0\rangle$, which are unmarked (since they do not involve the state $|g'\rangle$ of atom N), and the state $|g'_N, 0\rangle$, which is marked. Among the unmarked atoms, none should play a privileged role. Hence we shall consider them collectively and label the corresponding state with a subscript u. We rewrite the Hamiltonian in a new basis which features the uniform superposition of the unmarked ground states

$$|g'_u, 0\rangle = \frac{1}{\sqrt{N-1}} \sum_{j=1}^{N-1} |g'_j, 0\rangle, \tag{3}$$

and the uniform superposition of the excited states associated with the unmarked atoms

$$|e_u, 0\rangle = \frac{1}{\sqrt{N-1}} \sum_{j=1}^{N-1} |e_j, 0\rangle. \tag{4}$$

In this basis, the part of the Hamiltonian restricted to the

subspace spanned by the states $|g'_u, 0\rangle, |g'_N, 0\rangle, |g, 1\rangle, |e_u, 0\rangle, |e_N, 0\rangle$ is decoupled from the rest and reads

$$H_1 = \begin{pmatrix} 0 & V_\ell \\ V_\ell^\dagger & V_c \end{pmatrix}, \tag{5}$$

with

$$V_\ell = \begin{pmatrix} 0 & \Sigma & 0 \\ 0 & 0 & \Sigma' \end{pmatrix},$$
$$V_c = \begin{pmatrix} 0 & \sqrt{N-1}G & G \\ \sqrt{N-1}G & 0 & 0 \\ G & 0 & 0 \end{pmatrix},$$

and $\Sigma = \Omega + e^{-i\delta t}\Omega'$, $\Sigma' = \Omega' + e^{i\delta t}\Omega$. This Hamiltonian, whose derivation is exact, is represented in Fig. 2(b).

By means of a unitary transformation U, we diagonalize the block V_c, which admits the eigenvalues $\gamma_0 = 0$ and $\gamma_\pm = \pm\sqrt{N}G$, whose associated eigenvectors read

$$|\gamma_0\rangle = \sqrt{1 - \frac{1}{N}}|e_N, 0\rangle - \frac{1}{\sqrt{N}}|e_u, 0\rangle$$
$$|\gamma_\pm\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{N}}|e_N, 0\rangle + \sqrt{1 - \frac{1}{N}}|e_u, 0\rangle \pm |g, 1\rangle\right). \tag{6}$$

Notice that for large N, $|\gamma_0\rangle$ is essentially the collective state $|e_N, 0\rangle$ with the marked atom in the excited state $|e\rangle$ and all the others in the ground state $|g'\rangle$, whereas $|\gamma_\pm\rangle$ are mainly balanced superpositions of $|e_u, 0\rangle$ and $|g, 1\rangle$. The new Hamiltonian reads, in the basis $|g'_u, 0\rangle, |g'_N, 0\rangle, |\gamma_0\rangle, |\gamma_+\rangle, |\gamma_-\rangle$,

$$U^\dagger H_1 U = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}, \tag{7}$$

with

$$A = \frac{1}{\sqrt{N}}\begin{pmatrix} 0 & 0 & -\Sigma \\ 0 & 0 & \sqrt{N-1}\Sigma' \\ -\Sigma^* & \sqrt{N-1}\Sigma'^* & 0 \end{pmatrix}$$
$$B = \frac{1}{\sqrt{2N}}\begin{pmatrix} \sqrt{N-1}\Sigma & \sqrt{N-1}\Sigma \\ \Sigma' & \Sigma' \\ 0 & 0 \end{pmatrix} \tag{8}$$
$$C = \sqrt{N}\begin{pmatrix} G & 0 \\ 0 & -G \end{pmatrix}.$$

The time evolution of the nonresonant components in Σ and Σ' is much faster than the evolution of Ω and Ω', which occurs over a time scale $\mathcal{T} \gg \delta^{-1}$. Hence it is justified to replace these contributions with their vanishing average values over times $\delta^{-1} \ll \tau \ll \mathcal{T}$: $\bar{\Sigma} \simeq \Omega$ and $\bar{\Sigma}' \simeq \Omega'$, where $\bar{f}(t) = \frac{1}{\tau}\int_t^{t+\tau} du f(u)$. This is the resonant approximation. Similarly, the unitary evolution of C is much faster than that of A if $\Omega_{\text{peak}}/NG \ll 1$, since the respective eigenvalues scale as $\sqrt{N}G$ and $\Omega_{\text{peak}}/\sqrt{N}$, where Ω_{peak} is the peak amplitude of the pulse Ω. Upon performing an adiabatic elimination we thus obtain an effective Hamiltonian $H_{\text{eff}} = \bar{A} - \bar{B}C^{-1}\bar{B}^\dagger$ that contains all the contributions up to order $(\Omega_{\text{peak}}/NG)^4$. As the columns of B are identical and C is traceless, there is no correction of second order: $H_{\text{eff}} = \bar{A}$.

Our aim is to transfer adiabatically the population from the initial state $|g', 0\rangle$, which gives no privileged role to any of the N states $|g'_j, 0\rangle$, to a final state that coincides with the marked state $|g'_N, 0\rangle$ in a time that scales as $\sqrt{N}$. The population transfer mechanism is most easily revealed in the basis of the instantaneous eigenstates of $H_{\text{eff}}(t)$:

$$|0\rangle(t) = \cos\theta(t)|g'_N, 0\rangle - \sin\theta(t)|g'_u, 0\rangle$$
$$|\pm\rangle(t) = \frac{1}{\sqrt{2}}(\sin\theta(t)|g'_N, 0\rangle + \cos\theta(t)|g'_u, 0\rangle \pm |\gamma_0\rangle), \tag{9}$$

pertaining to the eigenvalues 0 and $\pm\Lambda(t)$, where

$$\Lambda(t) = \frac{1}{\sqrt{N}}\sqrt{(N-1)\Omega'^2(t) + \Omega^2(t)}. \tag{10}$$

Note that $|0\rangle$ has no component on the collective excited states $|e_N, 0\rangle$ or $|e_u, 0\rangle$ and is therefore a so-called dark state, which is immune to loss by spontaneous emission (in contrast to the states $|\pm\rangle$). The instantaneous angle $\theta(t)$ is defined through the relation

$$\tan\theta(t) = -\sqrt{N-1}\frac{\Omega'(t)}{\Omega(t)}. \tag{11}$$

Requiring the instantaneous eigenstate $|0\rangle$ to coincide at the initial time with the uniform superposition $|g', 0\rangle = \frac{1}{\sqrt{N}}|g'_N, 0\rangle + \sqrt{1 - \frac{1}{N}}|g'_u, 0\rangle$ and at the final time with the marked state $|g'_N, 0\rangle$ entails that

$$\tan\theta(t_i) = -\sqrt{N-1}, \qquad \tan\theta(t_f) = 0. \tag{12}$$

This implies that the two pulses must be switched on simultaneously, $\Omega'(t_i) = \Omega(t_i)$, and that the pulse Ω' is to be turned off before Ω. In the adiabatic representation (9), the effective Hamiltonian reads

$$H_{\text{eff}}^{\text{ad}} = \begin{pmatrix} \Lambda & \frac{i}{\sqrt{2}}\dot{\theta} & 0 \\ -\frac{i}{\sqrt{2}}\dot{\theta} & 0 & -\frac{i}{\sqrt{2}}\dot{\theta} \\ 0 & \frac{i}{\sqrt{2}}\dot{\theta} & -\Lambda \end{pmatrix}, \tag{13}$$

where $\dot{\theta} = \frac{1}{1+\tan^2\theta}\frac{d}{dt}\tan\theta$. In the adiabatic regime, the transitions between instantaneous eigenstates are negligible. This will be achieved if the Hamiltonian varies sufficiently slowly in time so as to keep $\dot{\theta} \ll \Lambda$. On the other hand, we wish to control the process duration and, in particular, to prevent it from becoming arbitrarily large. For that purpose, as proposed in [13], we choose to require $\dot{\theta}$ and Λ to be in a constant (small) ratio ε at all times, independent of N:

$$\dot{\theta} = \varepsilon\Lambda. \tag{14}$$

Given a laser pulse Ω, this will allow us to determine the pulse Ω' that is needed to remain in the instantaneous eigenstate $|0\rangle(t)$ with a probability around $1 - \varepsilon^2$ through-

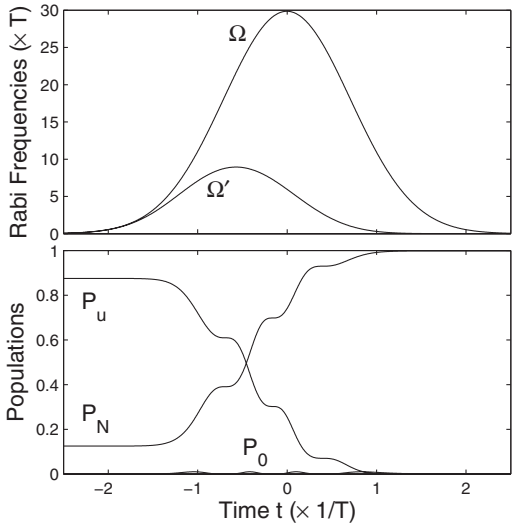

FIG. 3. Dynamics of the effective Hamiltonian for $N = 8$, $\varepsilon = 0.05$ and a Rabi frequency with a Gaussian profile $\Omega(t) = \Omega_{\text{peak}} e^{-(t/T)^2}$, with $\Omega_{\text{peak}} T = \sqrt{N-1}/(\varepsilon\sqrt{\pi})$. $\Omega'(t)$ is determined from (15). Upper panel: Rabi frequencies $\Omega(t)$ and $\Omega'(t)$. Lower panel: Populations $P_u(t) \equiv |\langle g'_u, 0|\phi\rangle(t)|^2$ of the collective unmarked state, $P_N(t) \equiv |\langle g'_N, 0|\phi\rangle(t)|^2$ of the marked state, and $P_0(t) \equiv |\langle \gamma_0, 0|\phi\rangle(t)|^2$ of the superposition of excited states, where $|\phi\rangle(t)$ is the dynamical state vector.

out the process, starting from the uniform superposition and ending up in the marked state. Indeed, rewriting (10) with (11) as $\Lambda = \frac{1}{\sqrt{N}}\sqrt{1+\tan^2\theta}\,\Omega$, we obtain from (14) a differential equation for $\tan\theta$, i. e., for the ratio Ω'/Ω. Its solution satisfying the initial condition (12) reads

$$\frac{\Omega'(t)}{\Omega(t)} = \frac{1 - \frac{\varepsilon \mathcal{A}(t)}{\sqrt{N-1}}}{\sqrt{1 + \varepsilon \mathcal{A}(t)\{2\sqrt{N-1} - \varepsilon \mathcal{A}(t)\}}}, \qquad (15)$$

where $\mathcal{A}(t) \equiv \int_{t_i}^{t} du\,\Omega(u)$. Expressing the total area $\mathcal{A}$ of the pulse Ω as the product of its peak amplitude Ω_{peak} and a characteristic duration $\mathcal{T}$, we arrive at

$$\Omega_{\text{av}} \mathcal{T} = \frac{\sqrt{N-1}}{\varepsilon}. \qquad (16)$$

This shows that, for a peak amplitude independent of N, the duration scales as $\sqrt{N}$. Note that we can equivalently increase the peak amplitude as $\sqrt{N}$ for a constant duration.

Figure 3 displays the pulses and the population dynamics resulting from (15) with $N = 8$, $\varepsilon = 0.05$, and a Gaussian pulse Ω of characteristic duration $\mathcal{T} (= \sqrt{\pi} T)$. As predicted, the transfer to the marked state is very efficient. Notice also that the transient population in the excited states is very low. These results stem from the fact that the dynamics remains in the instantaneous decoherence-free eigenstate $|0\rangle(t)$ in the adiabatic limit.

An experimental implementation of the robust processes proposed here requires to trap atoms in a cavity, for instance using a standing-wave dipole-force trap [17]. As a realistic Λ atomic scheme, we can consider the typical $2\,^3S_1$–$2\,^3P_0$ transition in metastable helium which is of linewidth $\Gamma \sim 10^7\ \text{s}^{-1}$ and Rabi frequency $\Omega \sim 10^8\sqrt{I}\ \text{s}^{-1}$ (with the intensity I in W/cm^2). To neglect spontaneous emission, as assumed in (2), we require the condition $(\Omega_{\text{peak}} T)^2 \gg \Gamma T$. It is fulfilled when $\Omega_{\text{peak}} T \gg 1$ and $\Omega_{\text{peak}} \gg \Gamma$, which are satisfied in practice, e.g., for $I_{\text{peak}} \sim 10^4\ \text{W/cm}^2$ and $T \sim 10$ ns. The feasibility of fixing the ratio of two pulses (here essentially required at early times) has been shown in [18] using acousto-optical modulation of a cw laser in such a nanosecond regime.

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