Bounds on the Speed of Information Propagation in Disordered Quantum Spin Chains

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We investigate the propagation of information through the disordered XY model. We find that all correlations, both classical and quantum, are exponentially suppressed outside of an effective light cone whose radius grows at most logarithmically with $|t|$.

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How fast can information propagate through a locally interacting system? For classical systems an essentially universal answer to this question is that the velocity of information propagation is bounded (often only approximately) by an effective *speed of light*. It is a more subtle issue to formulate equivalent velocity bounds for quantum systems because they can encode *quantum information* in the form of *qubits* and therefore might be able to exploit quantum interference to propagate information faster. However, for quantum spin networks this is not the case: the *Lieb-Robinson bound* limits the velocity at which correlations can propagate [1].

The Lieb-Robinson bound implies that there is an effective light cone for two-point dynamical correlations; i.e., apart from an exponentially suppressed tail, two-point correlations propagate no faster than the speed of light. Simplified and alternative proofs of the Lieb-Robinson bound have been subsequently discovered [2–5]. More recently, it has been realized that the Lieb-Robinson bound is strong enough to bound not only the propagation of two-point correlations but of any local encoding of information [6] (see also Refs. [7–10]).

There are many consequences of the Lieb-Robinson bound. Apart from the aforementioned bounds on the velocity of information propagation, it has been realized that the Lieb-Robinson bound can be used to provide a method to efficiently simulate the properties of low-dimensional spin networks [11–15]. Additionally, using the Lieb-Robinson bound, dynamical entropy area laws for quantum spin systems can be obtained [6,16].

While the Lieb-Robinson bound is extremely general—it relies only on the ultraviolet cutoff imposed by lattice structure—it is, as a consequence, relatively weak. Thus, it is extremely desirable to develop stronger bounds constraining the propagation of quantum information through systems where more is known about the structure of the interactions. One setting where one would expect stronger bounds to be available is when the system has disordered interactions. This is because they can exhibit the striking phenomenon of *Anderson localization* [17], which means that information is essentially frozen: a quantum particle placed anywhere within a localized system diffuses only slightly, even for extremely large times. Thus, exploiting the parallels between bounds on information propagation and Lieb-Robinson bounds, we are motivated to conjecture that interacting spin systems with disordered interactions satisfy stronger bounds on correlation propagation (see Fig. 1). More specifically, we conjecture that for quantum spin networks with disordered interactions all correlations, both quantum and classical, are suppressed outside of a light cone whose radius grows at most *logarithmically* in time. (Contrast this with the light cone supplied by the Lieb-Robinson bound: it has a radius which grows linearly with time.)

In this Letter we study a setting where the dynamics of a class of disordered interacting spin systems can be shown to satisfy our logarithmic light cone conjecture; we study the XY spin chain with disordered interactions in a disordered magnetic field and show that information, and hence correlations, are exponentially attenuated outside of a light cone whose radius grows logarithmically with time. The main result of this Letter, the *logarithmic light cone* for the disordered XY model, can by summarized by the following bound on the dynamic two-point correlation functions:

$$\|[A_j, e^{itH}B_k e^{-itH}]\| \leq cn^2|t|e^{-v|j-k|/l_{\max}}, \qquad (1)$$

where A_j and B_k are local operators acting nontrivially

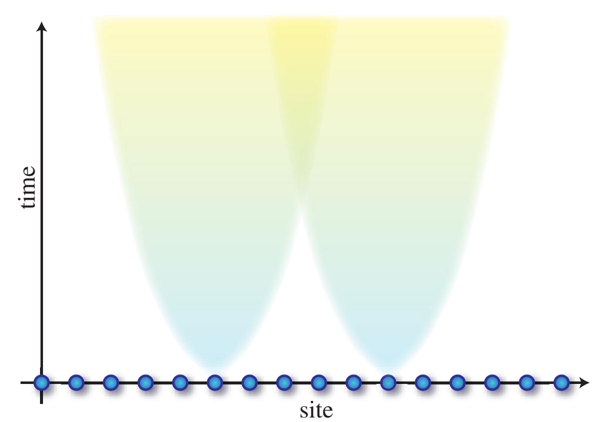

FIG. 1 (color online). Schematic illustration of the conjectured *logarithmic light cone* for disordered systems: as time progresses information is exponentially attenuated outside of a light cone whose radius grows at most logarithmically with time.

only on spins j and k respectively, $l_{\max}$ is the *localization length* of our system, and c and v are constants. We apply our new bound to study the structure of the propagator for large times and the scaling of the entropy of a block of spins in the evolving system. As a consequence, we prove the entropy saturation numerically observed by De Chiara *et al.* [18]. Our results also constitute a proof of a conjecture raised in Ref. [19]: namely, if two parties, Alice and Bob, have access to a bounded region at either end of the chain, respectively, then it is impossible for Alice to send any information to Bob, regardless of how Alice encodes the information in the single- and higher-excitation sectors.

We consider a one-dimensional chain of n spin-1/2 particles with XY-model-type interactions between nearest-neighboring spins in an additional transverse field (e.g., a magnetic B field). We allow the coefficients of the couplings and the transverse field strength to vary from site to site within the spin chain. Thus, we study the evolution of the chain under the Hamiltonian

$$H = \sum_{j=1}^{n-1} \mu_j(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y) + \sum_{j=1}^{n} \nu_j \sigma_j^Z, \tag{2}$$

where μ_j and ν_j are drawn from probability distributions $\mathbb{P}_\mu$ and $\mathbb{P}_\nu$, respectively, and where σ_j^α ($\alpha \in \{X, Y, Z\}$) is a Pauli spin operator acting on the spin at site j. Typically, $\mu_j = -J$ for all j; however, this is not necessary and we deal with the more general case here.

We solve this system using the *Jordan-Wigner transform* [20] (for an introduction to the Jordan-Wigner transform see Ref. [21]) which, when combined with some exact results from the theory of localization, allows us to bound the dynamics of our spin chain.

Let us start by applying the Jordan-Wigner transform, which maps a system of interacting qubits into a system of free fermions. The Jordan-Wigner transform defines annihilation operators $a_j = (\sigma_1^Z \cdots \sigma_{j-1}^Z)\sigma_j$ (where $\sigma_j = |0\rangle\langle 1|$ acts on site j) and the corresponding creation operators $a_j^\dagger$, which satisfy the canonical fermionic anticommutation relations. Using this we are able to rewrite the system Hamiltonian as $H = \sum_{j,k=1}^n M_{jk} a_j^\dagger a_k$, where the tridiagonal matrix M is defined via $M_{j,k} = 2\mu_k \delta_{j,k+1} + 2\mu_j \delta_{j,k-1} - 2\nu_j \delta_{j,k}$.

It is now possible (following the method described in Ref. [21]) to diagonalize H. After doing so we find the dynamics for the annihilation operators in the Heisenberg picture, with $a_j(t) = e^{itH} a_j e^{-itH}$:

$$a_j(t) = \sum_{k=1}^{n} v_{jk}(t) a_k, \tag{3}$$

where $v_{jk}(t) = (e^{-iMt})_{j,k}$. We now concentrate on bounding the $v_{jk}(t)$, which in turn bounds the dynamics of the system.

To proceed further, we must be more specific about our model: we suppose that $\mathbb{P}_\nu$ is a Cauchy distribution with parameter (or "width") δ

$$\nu_j \sim \mathbb{P}_\nu = \frac{1}{\pi} \frac{\delta}{(\nu_j - \nu)^2 + \delta^2}. \tag{4}$$

We note that, as pointed out in Ref. [19], the ν_j can have a different distribution without affecting the qualitative aspects of the following results (i.e., with a different distribution we still get localization of eigenstates), although of course the quantitative results will differ.

Under our assumption about the distribution of the ν_j, we know from the theory of localization [22] that the eigenstates of the matrix M are, with high probability, exponentially localized. [It can be shown that our bound Eq. (14) holds if and only if every eigenstate is localized.] The matrix M is an $n \times n$ matrix (it is the Hamiltonian for the single-particle sector). We write $\{|j\rangle\}$ for the basis induced by the notation $M_{jk} \equiv \langle j|M|k\rangle$. If $|E_\alpha\rangle$ is an eigenstate of M with eigenvalue E_α, then

$$|\langle j|E_\alpha\rangle| \le N_\alpha e^{-\lambda_\alpha |j_\alpha - j|} \tag{5}$$

where j_α is the site around which $|E_\alpha\rangle$ is localized, $N_\alpha = |\langle j_\alpha|E_\alpha\rangle|$ is a normalization constant, and $\lambda_\alpha = 1/l_\alpha$ is the inverse of the localization length of eigenstate $|E_\alpha\rangle$: small l_α means that $|E_\alpha\rangle$ is a highly localized eigenstate while large l_α means that $|E_\alpha\rangle$ is a weakly localized eigenstate.

Using the localization of the eigenstates of M we are able to bound the matrix elements $v_{jk}(t)$. First, notice that $v_{jk}(t) = \langle j|e^{-itH}|k\rangle = \langle j|k(t)\rangle = \sum_{\alpha=1}^n \langle E_\alpha|k\rangle e^{-itE_\alpha} \langle j|E_\alpha\rangle$. Using this observation, applying the Cauchy-Schwartz inequality, and Eq. (5) allow us to conclude that

$$|v_{jk}| \le \sum_{\alpha=1}^{n} e^{-|j-k|/l_{\max}} = n e^{-|j-k|/l_{\max}}, \tag{6}$$

where $l_{\max} = \sup_\alpha l_\alpha$ and we have used the fact that $N_\alpha = |\langle j_\alpha|E_\alpha\rangle| \le 1$. We have illustrated the time dependence of the absolute values of the matrix elements $|v_{jk}(t)|$ in Fig. 2. These matrix elements encode several time-dependent correlation functions which are suppressed by the disorder. We note that for small δ, $l_{\max} \sim \frac{\mu}{\delta}$, where μ is the geometric mean of the μ_j [19]. In particular, if $\mu_j = -J$ for all j, then $\mu = J$.

The inequality Eq. (6) is a quantitative statement of the result that the modulus of the diagonal matrix elements of e^{-iMt} are large, while the modulus of the off-diagonal matrix elements decay exponentially with distance from the diagonal. This means that $a_j(t)$ is effectively a linear combination of only a small number of a_k operators—namely those for which $|j - k|$ is small. It is this fact which inhibits the spread of operators on the chain, giving rise to the logarithmic light cone that we derive below.

We now turn to the proof of the improved Lieb-Robinson bound for our system. We begin by bipartitioning the spin chain into two sections, A and B, where we assume the boundary between partitions is between spins m and

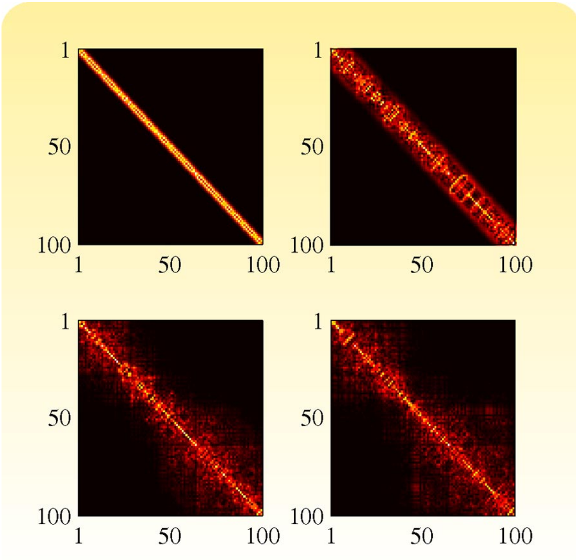

FIG. 2 (color online). Typical results for the magnitudes $|v_{j,k}(t)|$ of the matrix elements of e^{itM} for a 100-site chain at times $t = 1, 5, 25, 125$ (going from top left to bottom right). The darker the entry the smaller $|v_{j,k}(t)|$. The matrix elements $v_{j,k}(t)$ encode the time-dependent correlation functions $\langle\Omega|a_j(0)a_k^\dagger(t)|\Omega\rangle = \langle\Omega|\sigma_j^-(0)\sigma_k^+(t)|\Omega\rangle$. Notice the rapid decay of the correlation functions $v_{j,k}(t)$ for $j \gg k$ is roughly independent of time: all correlations are suppressed outside of the logarithmic light cone. Appropriate linear combinations of $v_{j,k}(t)$ yield similar results for the time-dependent correlation functions $\langle\Omega|\sigma_j^x(0)\sigma_k^x(t)|\Omega\rangle$ and $\langle\Omega|\sigma_j^y(0)\sigma_k^y(t)|\Omega\rangle$.

$m + 1$. We then attempt to write e^{itH} as a product of e^{itH_A} and e^{itH_B}. Clearly this will not be exact and so we introduce an operator $V(t)$ which bridges the boundary between A and B, and which is designed to compensate for any errors introduced:

$$e^{itH} = e^{it(H_A+H_B)}V(t). \tag{7}$$

The operator $V(t)$ acts nontrivially on all spins in the chain; however, we now show that $V(t)$ can be well approximated by another operator, which we call $V^\Omega(t)$, which acts only on a small number $|\Omega|$ of spins. The reason we can do this is that $V(t)$ acts strongly on spins which are close to the boundary and progressively weaker on spins as we move away from the boundary. To prove this approximation is valid, we use the following differential equation for $V(t)$:

$$\frac{d}{dt}V(t) = iV(t)h_m(t), \tag{8}$$

where $h_m(t) = e^{-itH}h_m e^{itH}$ and h_m is the interaction term in the Hamiltonian which bridges the boundary. We let Ω denote a set of $|\Omega|$ spins centered on the boundary between the partitions A and B. We also define $h_m^\Omega(t) = e^{-itH_\Omega}h_m e^{itH_\Omega}$ where H_Ω contains only those interactions in H which act on sites in Ω. We then define $V^\Omega(t)$ via

$$\frac{d}{dt}V^\Omega(t) = iV^\Omega(t)h_m^\Omega(t). \tag{9}$$

Clearly the operator $V^\Omega(t)$ acts nontrivially only on Ω.

The error between $V(t)$ and $V^\Omega(t)$ is bounded by [23]

$$\|V(t) - V^\Omega(t)\| \le \int_0^{|t|}\|h_m(s) - h_m^\Omega(s)\|ds. \tag{10}$$

The error is small when $\|h_m(t) - h_m^\Omega(t)\|$ small. The physical intuition behind why this quantity should be small is as follows. The Jordan-Wigner decomposition allows us to express $h_m(t)$ and $h_m^\Omega(t)$ in terms of a *local* combination of the fermion creation and annihilation operators for the sites m and $m + 1$. Thanks to Eq. (3) and the result of Eq. (6) that the matrix elements $v_{j,k}(t)$ are very small when $|j - k|$ grows (as illustrated in Fig. 2) we can approximate the sum Eq. (3) by

$$a_j(t) \approx \sum_{k\in\Omega} v_{jk}(t)a_k. \tag{11}$$

Doing the summations and applying the triangle and Cauchy-Schwartz inequalities several times allows us to conclude that $\|h_m(t) - h_m^\Omega(t)\| \le cn^2e^{-|\Omega|/2l_{\max}}$ and that

$$\|V(t) - V^\Omega(t)\| \le c|t|n^2e^{-|\Omega|/2l_{\max}}, \tag{12}$$

for some constant c. In particular, given $\epsilon \ge 0$, choosing $|\Omega| \ge 2l_{\max}\log(c|t|n^2/\epsilon)$ ensures that $\|V(t) - V^\Omega(t)\| \le \epsilon$. That is, given any $\epsilon \ge 0$ we can choose Ω to be a large enough set such that $V^\Omega(t)$ approximates $V(t)$ to within ϵ. This enables us to write $e^{itH} = e^{it(H_A+H_B)}V^\Omega(t) + \mathcal{O}(\epsilon)$.

Following Ref. [11] we recursively apply the above partitioning procedure to find $e^{itH} = Q(t) + \mathcal{O}(\epsilon)$, where

$$Q(t) \equiv \left(\bigotimes_{j=1}^{n/|\Omega|} e^{itH_{\Omega_j}}\right)\left(\bigotimes_{k=0}^{n/|\Omega|} V^{\Omega'_k}(t)\right), \tag{13}$$

and where $\mathcal{P}_1 = \{\Omega_j\}$ is a partition of the chain into $\frac{n}{|\Omega|}$ blocks, each containing $|\Omega|$ spins and where $\mathcal{P}_2 = \{\Omega'_k\}$ is a partition of the chain obtained by shifting $\mathcal{P}_1$ along by $\frac{|\Omega|}{2}$ sites (note that Ω'_0 and $\Omega'_{n/|\Omega|}$ are half-size blocks of $\frac{|\Omega|}{2}$ sites each). This is our fundamental structure result for the dynamics of the disordered XY spin chain.

A Lieb-Robinson bound is an upper bound on quantities such as $\|[A, B(t)]\|$. We now show how the above structure result implies a version of the Lieb-Robinson bound which is substantially stronger than the original. Define $\tilde{B}(t)$ to be the operator which arises when we evolve B according to the approximation $Q(t)$ of e^{itH}, namely, $\tilde{B}(t) = Q(t)BQ^\dagger(t)$. This enables us to write $B(t) = \tilde{B}(t) + \mathcal{O}(\epsilon)$. Note that $\tilde{B}(t)$ acts trivially on all sites which are a distance greater than $3|\Omega|/2$ away from those sites on which B acts. Thus, if $d(A, B) \ge 3|\Omega|/2$, where $d(A, B)$ is the distance between A and B, then $[A, \tilde{B}(t)] = 0$, and so for a given $|\Omega|$:

$$\|[A, B(t)]\| = \|[A, \tilde{B}(t)]\| + \mathcal{O}(\epsilon) \leq cn^2|t|e^{-kd(A,B)/l_{\max}}, \quad (14)$$

where c and k are constants. This is the logarithmic light cone for the two-point dynamical correlation functions. Compare this to the original Lieb-Robinson bound, which reads $\|[A, B(t)]\| \leq ce^{k_1|t|}e^{-k_2 d(A,B)}$.

We now mention two consequences of the logarithmic light cone for the disordered XY model. The first is a proof of the conjecture that two parties, Alice and Bob, with access to only bounded regions A and B at either end of the chain, respectively, cannot use the dynamics of the disordered model to send information from Alice to Bob. We follow the argument of Ref. [6], appropriately modified to take account of our stronger bound.

Let $C = L \setminus (A \cup B)$, where L is the chain, be the region that Alice and Bob cannot access. The most general way Alice can encode her message is via a set of unitary operators $\{U_A^k | k = 1, 2, \ldots, m\}$ on her system, where k is varied according to the message she wants to send. After a time t has elapsed the system has evolved from an initial state ρ_0 to $\rho(t) = e^{-iHt}\rho_0 e^{iHt}$. We interpret this as a quantum channel with input $\rho_{ABC}^k = U_A^k \rho_0 U_A^{k\dagger}$ and output $\rho_B^k(t) = \mathrm{tr}_{AC}[U_A^k(t)\rho_0 U_A^{k\dagger}(t)]$. As argued in Ref. [6], the output states are all very close together, as measured in trace norm:

$$\|\rho_B^k(t) - \rho_B(t)\|_1 \leq cn^2|t|e^{-vd(A,B)/l_{\max}},$$

where $\rho_B(t) = \mathrm{tr}_{AC}(e^{-iHt}\rho_0 e^{iHt})$.

If Alice applies the unitaries $\{U_A^k\}$ according to the probability distribution $\{p_k\}$, the amount of information that is sent through the channel is given by the Holevo capacity:

$$\chi(t) = S\left(\sum_{k=1}^{m} p_k \rho_B^k(t)\right) - \sum_{k=1}^{m} p_k S[\rho_B^k(t)],$$

where $S[\cdot]$ is the von Neumann entropy. Applying Fannes inequality [24] we find that

$$\chi(t) \leq 2\epsilon[|B| - \log_2(\epsilon)],$$

where $\epsilon = cn^2|t|e^{-vd(A,B)/l_{\max}}$. That is, Bob has to wait an exponentially long time [in $d(A,B)$] before a nontrivial amount of information can arrive. The optimal encoding for Alice to adopt was investigated in Refs. [25,26] and completely solved in the single-use case in Ref. [27].

The second consequence of the logarithmic light cone bound is that the entropy of any contiguous block B of spins in a dynamically evolving product state $|\psi(t)\rangle = e^{itH}|00\cdots 0\rangle$ is bounded. Indeed, applying the argument of Refs. [16,28], we find that $S[\rho_B(t)] \leq c_1 + c_2\log_2(n|t|)$ as $|B| \to \infty$, where c_1 and c_2 are constants. This provides a theoretical explanation for the phenomenon numerically observed by De Chiara *et al.* [18].

In this Letter we have proposed that disorder in interacting systems should result in a logarithmic light cone for *all* time-dependent correlators of local observables. We have verified this conjecture for the XY model. We have also shown that such a logarithmic light cone, if true, implies that the block entropy can only grow sublinearly with time and also that *all* information, both classical and quantum, will take an exponential time to leak from bounded regions.

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