Universal Periods in Quantum Hall Droplets

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Using the hierarchy picture of the fractional quantum Hall effect, we study the ground-state periodicity of a finite size quantum Hall droplet in a quantum Hall fluid of a different filling factor. The droplet edge charge is periodically modulated with flux through the droplet and will lead to a periodic variation in the conductance of a nearby point contact, such as occurs in some quantum Hall interferometers. Our model is consistent with experiment and predicts that superperiods can be observed in geometries where no interfering trajectories occur. The model may also provide an experimentally feasible method of detecting elusive neutral modes and otherwise obtaining information about the microscopic edge structure in fractional quantum Hall states.

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With the recent surge of interest in quantum computing [1], quantum Hall systems [2] have received renewed attention due to their potential use in topologically protected qubits. In particular, the $\nu = 5/2$ and $\nu = 12/5$ quantum Hall states are believed to support non-Abelian excitations [3] which are crucial ingredients for topological quantum computation [1]. Here we will focus only on the Abelian quantum Hall states, but we will make use of their topological properties to reveal universal periodicities (as a function of magnetic flux through the droplet) in the ground-state energy and edge properties of a quantum Hall droplet inside a surrounding Hall fluid of a different filling factor. The universal periodicities in the ground-state properties can generically be used to probe the quantum Hall edge states in equilibrium settings.

Our work is motivated in part by a series of beautiful experiments done on quantum Hall interferometers where superperiods and fractional statistics have purportedly been observed [4]. Several theoretical studies have already addressed these experiments [5], but a complete picture, particularly in the fractional quantum Hall regime, is still lacking. In this Letter we study the universal properties of a finite size quantum Hall droplet inside a quantum Hall fluid of a different filling factor (Fig. 1). For most geometries and droplet filling factors we find that the ground-state energy of the system has a periodicity with magnetic flux through the inner droplet that is determined only by the two filling fractions in the limit that the charging energy of the surrounding fluid edge tends to zero. However, edges such as that of the $\nu = 2/3$ state (which have counterpropagating modes and disorder-influenced excitations [6]) require a special degree of consideration, as we discuss below.

Consider a droplet of filling factor ν_d surrounded by a fluid of filling factor ν_s, which itself may be inside an outer fluid of filling ν_o (Fig. 1). As magnetic flux is adiabatically threaded through the inner droplet, its ground-state energy and its radius oscillate with a universal periodicity. This periodicity reveals important information about the microscopic structure of the droplet edge itself, thus providing a mechanism by which theoretical edge models can be directly tested experimentally. We concentrate on two important special cases: (i) $\nu_d = 2/5$, $\nu_s = 1/3$, $\nu_o = 0$ and (ii) $\nu_d = 2/3$, $\nu_s = 0$, $\nu_o = 1$. General filling fractions with Abelian statistics follow one of the two cases above. While disorder does not play a central role in the physics of the edge in case (i), in case (ii) disorder determines the nature of the edge excitations [6], by causing counterpropagating modes of the droplet edge to "recombine" into charged and neutral modes. The precise way in which the radius changes with flux can directly probe whether this recombination occurs, or whether the edge structure is that of the clean system, as described in Ref. [7]. Therefore, the flux dependence of the conductance can reveal the presence or absence of elusive neutral modes, which to the

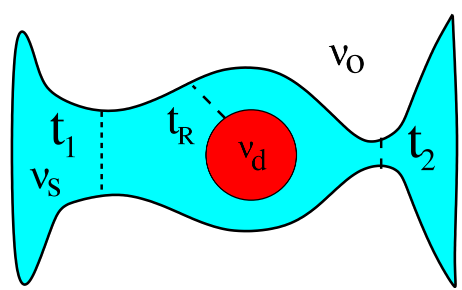

FIG. 1 (color online). Schematic of our setup. A quantum Hall droplet of filling factor ν_d is surrounded by a Hall fluid with $\nu_s \neq \nu_d$, itself surrounded by an outer fluid with ν_o. Tunneling between the ν_s fluid edges occurs at two point contacts with amplitudes t_1, t_2. Tunneling (with amplitude t_R) also occurs between the droplet edge and the surrounding fluid edge, which acts as a reservoir. Periodic charging of the droplet edge with flux will cause periodic modulations of the tunneling amplitudes t_1, t_2 implying conductance oscillations of the same period at the point contacts. The period will depend on ν_s, ν_d, and ν_o with periods greater than a flux quantum possible. The system could be contacted in the way described in Ref. [4].

best of our knowledge have not yet been experimentally detected.

The expansion and contraction of the inner droplet results from a charging and discharging of its edge. In the proposed setup, this affects the conductance of a quantum point contact near the droplet due to Coloumb interaction: the changing electric potential near the point contact affects the distance between the two edges that the point contact connects. Such an interaction-modulated conductance was used to detect charge states in a double quantum dot system and to coherently manipulate spin [8]. It might also be relevant for certain geometries of quantum Hall interferometers, such as those of Ref. [4], and may be the cause of the superperiods observed there, rather than interference. In fact, superperiods would also result from potential modulations at a *single* point contact via the physics outlined above. Whether the origin of observed superperiods in interferometers with two point contacts are the result of interaction-modulated tunneling, or of true interference, can be determined in experiment by suppressing tunneling at one of the point contacts (t_1 in Fig. 1, for example). Interference effects would disappear, but interaction-modulated tunneling effects would still produce oscillations. A similar suggestion was also made be Rosenow and Halperin in Ref. [5].

The ground-state periodicity for a quantum Hall droplet has been discussed before [9] using bulk fluid descriptions. Here we use an edge state description [10] to emphasize the physics that depends on the nature of the edge modes themselves. As we are interested in the universal properties of such a droplet in a surrounding Hall fluid, we use the theory describing the universal aspects of the fractional quantum Hall state [10,11]. The edge modes constitute a minimal model [12] described by the K-matrix formulation of Wen [10]. In this formulation, the action is

$$S_{\rm edge} = \int \frac{dtdx}{4\pi}[K_{ij}\partial_t\phi_i\partial_x\phi_j - V_{ij}\partial_x\phi_i\partial_x\phi_j - 2t_iA\partial_x\phi_i], \quad (1)$$

where K is a matrix determined by a choice of basis denoted by t, which determines the coupling to the vector potential A; the filling is $\nu = t^\dagger K^{-1} t$. The dimension of K reflects the filling ν of the quantum Hall state and equals the number of independent edge modes. V is a *nonuniversal* positive definite matrix determined by edge mode interactions and the confining potential, and $\phi_i(x, t)$ are bosonic fields parametrizing the edge modes.

The topological content of the quantum Hall state is encoded in K. For a droplet inside a surrounding fluid, the interface K-matrix is [13]

$$K = \begin{pmatrix} K_d & 0 \\ 0 & -K_s \end{pmatrix}, \quad (2)$$

where K_d and K_s describe the droplet and the surrounding liquid, respectively. If there exists an integer valued vector m such that $m^\dagger K^{-1} m = 0$ and $t^\dagger K^{-1} m = 0$, then K is topologically unstable [14] and may reconstruct by some edge modes "gapping" each other out, thus reducing the number of edge modes. If the droplet Hall state is a descendant of the surrounding state, this is always possible, and akin to low-level composite fermion Landau levels connecting adiabatically across the interface. The topological stability of the interface edge also depends on the nonuniversal V; here we assume instability, since it occurs for a wide and realistic range of V.

A crucial component for our setup is the finite size of the droplet. This implies a level quantization, which can be inferred using gauge invariance and quantized Hall conductance. Consider an edge described by the field ϕ, which is a linear combination of the ϕ_i that diagonalizes the matrices K and V and obeys $[\phi(x), \partial_x\phi(x')] = q\delta(x - x')$. The operator $\exp(i\phi)$ thus creates an edge excitation of charge q and upon flux h/e insertion at a point within the inner droplet, the creation operator must become $\exp(i\phi) \rightarrow \exp(i\phi + 2\pi i q \frac{x}{L})$, where L is the length of the edge. But from gauge invariance, the spectrum of the edge must remain unchanged. The charge in each of the orbitals must then be q to obtain quantized Hall conductance, $\sigma_{xy} = qe^2/h$. Thus, a finite edge loop can be described as a chiral Luttinger liquid which consists of discrete orbitals, each containing charge q.

Let us now treat the case of a $\nu_d = 2/5$ droplet in a $\nu_s = 1/3$ and $\nu_o = 0$ surrounding. For filling factor $\nu = n/(np + 1)$, the K-matrix in the symmetric basis is an n-dimensional matrix [6], $K_{ij} = \delta_{ij} + p$. Since the $2/5$ state is the daughter of the $1/3$ state, the K-matrix given by Eq. (2) is indeed unstable, and the resulting recombined edge is identical to that of a $\nu = 1/15$ Laughlin state. The $1/15$ effective edge within a $1/3$ edge leads to a $5\Phi_0$ periodicity of ground-state properties of the droplet with magnetic flux through it, a result in agreement with a bulk description [9] and experiment [4].

To see this, consider the gapped droplet state. Upon an adiabatic $\Phi_0 = h/e$ flux insertion at a point in the $\nu_d = 2/5$ droplet, a net charge of $2e/5$ is localized at the flux. This charge is sucked from the two edges: an $e/15$ orbital is vacated in the $2/5 - 1/3$ edge, and an additional $e/3$ orbital is vacated in the $1/3 - 0$ edge. Confirming our assertion above as to the edge structure, indeed $1/3 + 1/15 = 2/5$. Smearing the flux uniformly over the droplet yields the same result. Repeating the adiabatic flux insertion will progressively charge the droplet edge in units of $-e/15$ and the surrounding fluid edge in units of $-e/3$. Additional flux outside the droplet may create additional excitations of charge $-e/3$ on the outer edge, but it will not influence the edge charge of the droplet. Through the ubiquitous presence of disorder in quantum Hall systems, it is possible for quasiparticles to tunnel [15] between the $1/3 - 0$ edge and the $2/5 - 1/3$ edge and relax the energy of the system. The allowed charges are determined by K_s [10], and the most relevant operator in the present case is indeed $e^{(-i\phi_s + 5i\phi_d)}$, which tunnels charge $e/3$.

Assuming both edges are initially neutral, denote the number of additional filled edge states by n_d and n_s, for the droplet and surrounding fluid, respectively. The energy of the charged edges is $E_{d/s} = \frac{E_c^{d/s}}{2} n_{d/s}^2$ for the droplet or surrounding fluid. The energies E_c^d, E_c^s depend on the edge velocities and capacitances and are inversely proportional to the length of the edges. The total charge on the edges is $Q = -\frac{e}{15} n_d - \frac{e}{3} n_s$. The two distinct edge excitations have chemical potentials determined by $\mu \equiv \frac{\partial E}{\partial n}$, which gives $\mu_{d/s} = E_c^{d/s} n_{d/s}$. When the two edges are in equilibrium, $\mu_d \delta n_d + \mu_s \delta n_s = 0$, and from charge conservation, $\delta n_d = -5\delta n_s$. Thus $5\mu_d = \mu_s$ (i.e., the two edges have the same *voltage*). This also gives $E_c^s n_s = 5E_c^d n_d$ at edge equilibrium. Now, the edge occupations n_d and n_s depend on the flux threaded. Assuming all of the flux is through the droplet, we have $6\frac{\Phi}{\Phi_0} = n_d + 5n_s = (\eta + 5)n_s$, where $\eta \equiv \frac{E_c^s}{5E_c^d}$. Solving this for n_s gives $n_s = \text{round}[\frac{\Phi}{\Phi_0} + (\frac{1-\eta}{5+\eta})\frac{\Phi}{\Phi_0}]$.

The first term indicates that every flux insertion raises n_s by one. The second describes $e/3$ charge transfer between the two edges. In the limit $\eta \to 0$ (occurring when the length of the surrounding fluid edge is long compared to the droplet edge), every $5\Phi_0$ added increases n_s by one extra state due to the tunneling of an $e/3$ charge. This happens when the rounding function changes from rounding down to up. The one $e/3$ charge annihilates five $-e/15$ charges and returns the droplet edge to its initial state. In the opposite limit, $\eta \to \infty$, every $\nu = 1/3$ orbital vacated due to Φ_0 insertion immediately gets filled via charge transfer *from* the droplet edge. For general values of η (i.e., the ratio of edge "charging energies") the period is nonuniversal as shown in Fig. 2. Allowing flux insertion in both the droplet and the surrounding fluid changes the response of the 1/3 edge, but in the limit $\eta \to 0$, this will not affect the period with respect to droplet flux, as the "rate limiting" step is due to the finite compressibility of the droplet edge. Therefore, the droplet $5\Phi_0$ flux period emerges as the universal droplet edge charging result when $\eta \to 0$, independent of the area of the surrounding Hall fluid.

A finite tunneling amplitude between the two edges foils the exact quantization of the expectation value of the droplet edge charge [16], as shown in the inset of Fig. 2 in the limit $\eta \to 0$. With the smooth oscillation of edge charge, there is a flux dependent oscillation of the electrical potential that will affect the conductance of a nearby point contact, as in Fig. 1. Modeling the droplet as an outer ring of charge Q_{ring} and a uniformly charged inner disk of net charge Q_{disk}, the potential from a droplet of radius R at a distance $d > R$ from the center is $V = V_{\text{ring}} + V_{\text{disk}}$, where $V_{\text{ring}} = \frac{Q_{\text{ring}}}{\epsilon\pi}(\frac{K[-4Rd/(d-R)^2]}{d-R} + \frac{K[4Rd/(d+R)^2]}{d+R})$ and $V_{\text{disk}} = \frac{2Q_{\text{disk}}}{\epsilon\pi R^2}\int_0^R rdr(\frac{K[-4rd/(d-r)^2]}{d-r} + \frac{K[4rd/(d+r)^2]}{d+r})$. Here ϵ is the dielectric constant and $K(x)$ is the complete elliptic integral of the first kind. The potential fluctuations are plotted in Fig. 3 and should be observable. A metallic gate placed 100–200 nm above the droplet will reduce the potential modulations by no more than 30%.

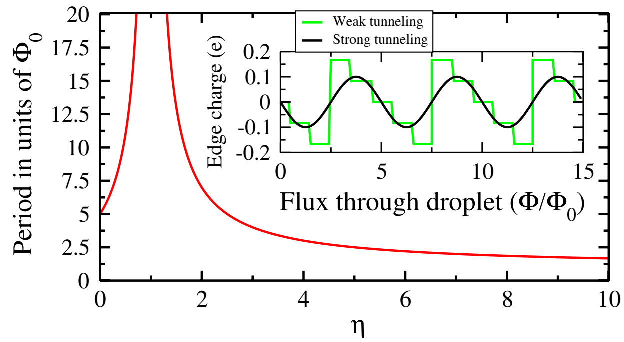

FIG. 2 (color online). Periodicity of ground-state structure vs flux through a $\nu_d = 2/5$ droplet in a $\nu_s = 1/3$ surrounding Hall fluid. $\eta = \frac{E_c^s}{5E_c^d}$ is described in the text. Note the period is nonuniversal unless one edge is very long compared to the other. When $\eta \to 0$ the periodicity is independent of the flux through the surrounding Hall fluid and is equal to a universal value, $5\Phi_0$. Inset: Droplet edge charge in the universal limit vs flux for weak and strong tunneling t_R.

Let us now focus on a case where the droplet edge has counterpropagating modes: $\nu_d = 2/3$, $\nu_s = 0$, and $\nu_o = 1$. The $\nu_d = 2/3$ to zero edge itself has counterpropagating modes which leads to a disorder-dependent edge structure. The clean 2/3 edge has an outer $\nu = 1$ mode and an inner (counterpropagating) $\nu = -1/3$ mode [7,17]. But as Ref. [6] predicts, in the disorder-dominated phase, the effective low-energy degrees of freedom are a charge mode with $q = 2e/3$, which gives a quantized Hall conductance, and a counterpropagating neutral mode localized when $T \neq 0$.

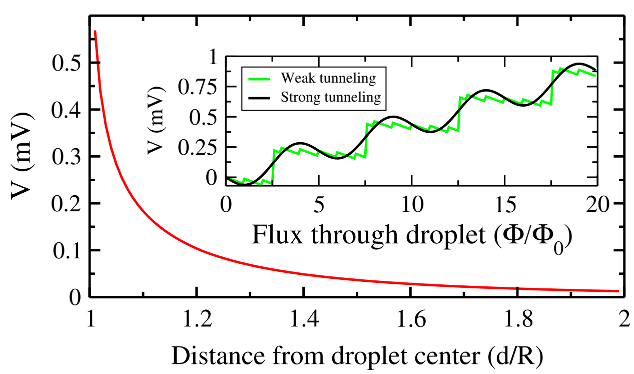

FIG. 3 (color online). Potential created by a disk and a ring of opposite charge equal to that of one electron. The dielectric constant is assumed to be that for GaAs, $\epsilon = 12$. Inset: Different form of the periodic voltage modulations depending on tunneling t_R between the droplet edge and the surrounding fluid edge with $d/R = 1.1$ for a $\nu_d = 2/5$ droplet in a $\nu_s = 1/3$ surrounding fluid. Subtracting a smooth background will lead to oscillations like those in the inset of Fig. 2.

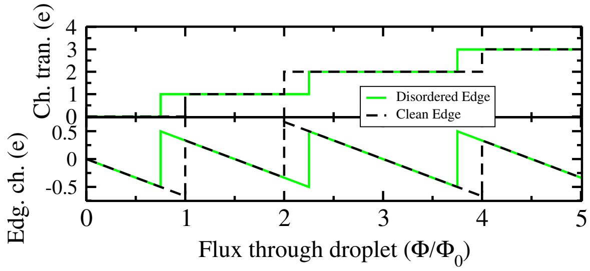

FIG. 4 (color online). Comparison of two different phases of the $\nu_d = 2/3$ edge in the $\nu_s = 0$, $\nu_o = 1$. Top: Electron transfer vs flux through the droplet. Bottom: Droplet edge charge vs flux through the droplet. The microscopic edge structure determines the flux dependence of the charge transfer and electrical potential created at a nearby point contact, as in Fig. 1.

The "surrounded droplet" setup allows an equilibrium verification of the charge or neutral recombination scenario. When flux is threaded through the $\nu_d = 2/3$ droplet, the edge charge relaxes through *electron* tunneling across the vacuum. For the disorder-dominated edge, the edge charge orbitals effectively consist of a single $\nu = 2/3$ mode as the neutral modes do not respond to flux. (This requires the droplet edge to be long compared to the recombination length of the neutral/charge modes, which is sample dependent and finite even at $T = 0$.) Here, after every $3\Phi_0/2$ flux threading, the edge loses one electron and can then discharge by an electron tunneling from the $\nu_0 = 1$ fluid. But for the clean phase, the flux dependence of the edge charge is different. When Φ_0 is threaded through the droplet the clean $2/3$ edge accumulates a $-e$ charge on the outer $\nu = 1$ mode and a $e/3$ charge on the inner mode. In the limit of large $\nu_o = 1$ edge length, an electron from the outer edge tunnels in to lower the energy. This continues when a second Φ_0 is threaded, but when the third Φ_0 is threaded, the edge instead relaxes to its original state by three $e/3$ inner mode excitations canceling one $-e$ outer mode excitation. This sequence is shown in Fig. 4. Fourier transforming the signal should allow for a clear identification of each case; one charge mode on the droplet edge leads to one periodicity appearing, whereas the two independent charge modes of the clean edge should exhibit two periodicities. Other edges with counterpropagating modes (such as $\nu_d = 3/5$) are amenable to similar considerations.

In this Letter we propose the surrounded droplet model near a point contact to investigate universal properties of composite edges. The periodic change of the droplet size with flux is measured by its effect on the conductance on a nearby point contact. We propose to use this effect to explore the nature of the $\nu = 2/3$ edge, i.e., whether the edge recombines into neutral and charged modes. This is the first proposal that may be able to do so in equilibrium. We assumed that the interior of all Hall droplets is gapped and that the only compressible areas are at the boundaries, neglecting the possibility that Hall droplets may break down into incompressible and compressible regions [18]. With sufficient disorder, all quasiparticle states in the interior compressible regions are localized, keeping our analysis intact. A back gate close to the sample, however, will avoid this complication altogether, with the relevant length scale being of order 200 nm, i.e., comparable to the distance between the (in-plane) front gate in Ref. [18] and the outer edge of the Hall droplet. This will impose a rather uniform chemical potential on the electronic fluid and hamper the creation of compressible domains, but should still allow sufficient potential modulation at the point contact to observe the predicted effects.

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