Negative Specific Heat in a Quasi-2D Generalized Vorticity Model

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Negative specific heat is a dramatic phenomenon where processes decrease in temperature when adding energy. It has been observed in gravo-thermal collapse of globular clusters. We now report finding this phenomenon in bundles of nearly parallel, periodic, single-sign generalized vortex filaments in the electron magnetohydrodynamic model for the unbounded plane under strong magnetic confinement. We derive the specific heat using a steepest-descent method and a mean-field property. Our derivations show that as temperature increases, the overall size of the system increases exponentially and the energy drops. The implication of negative specific heat is a runaway reaction, resulting in a collapsing inner core surrounded by an expanding halo of filaments.

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While Ref. [1] has proven that systems that are not isolated from the environment must have positive specific heat, the specific heat in isolated systems can be negative [2]. Negative specific heat is an unusual phenomenon first discovered in 1968 in microcanonical (isolated system) statistical equilibrium models of gravo-thermal collapse in globular clusters [3]. In gravo-thermal collapse, a disordered system of stars in isolation undergoes a process of core collapse with the following steps: (1) faster stars are lost to an outer halo where they slow down, (2) the loss of potential (gravitational) energy causes the core of stars to collapse inward some small amount, and (3) the resulting collapse causes the stars in the core to speed up. If one considered the "temperature" of the cluster to be the average speed of the stars, this process has negative specific heat because a loss of energy results in an increase in overall temperature.

In the intervening four decades, negative specific heat has been observed in few other places. In a magnetic fusion system or other thermally isolated plasma, should negative specific heat exist, the related runaway collapse could have profound implications for fusion where extreme confinement is critical to a sustained reaction.

Our results have general applicability to vortex systems. However, in this Letter, we address a plasma model known as the electron magnetohydrodynamic (EMH) model, where we report finding negative specific heat.

Typically, magnetohydrodynamic plasma models are two-fluid models, requiring equations governing the electron motion and equations governing the ion motion coupled together [4]. The EMH model bypasses the two-fluid model by representing the electron fluid and the magnetic field as a single, generalized fluid with a neutralizing ion background that is stationary on the time scale chosen.

The EMH model takes the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and the charged fluid vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ and combines them into a general vorticity field $\Omega = \nabla \times \mathbf{p}$ where the generalized momentum $\mathbf{p} = m\mathbf{v} - e\mathbf{A}$, m is the electron mass, $-e$ is the electron charge, $\mathbf{v}$ is the fluid velocity field, and $\mathbf{A}$ is the magnetic vector potential field. For a brief overview of the model, see Ref. [4]. A detailed model discussion can be found in Ref. [5].

Our goal is to find the specific heat of this vortex model in statistical equilibrium given an appropriate definition for energy and a microcanonical (isolated) probability distribution. Our approach is to describe the statistical behavior of a large number of discrete, interacting vortex structures and consider the limiting case.

The first step is to modify the continuous vorticity field Ω such that it describes a large number N of discrete vortex filaments. Therefore, we assume that Ω has a large number of periodic filaments that are nearly parallel to the z axis. The filaments are very straight because of strong magnetic confinement and angular momentum. The period L is assumed to be unity, $L = 1$, without loss of generality since all other distances and distance-dependent quantities can be scaled by L.

The vorticity field, which depends on space, $\mathbf{r} \in \mathbb{R}^3$, now looks like:

$$\Omega(\mathbf{r}) = \sum_{i=1}^{N} \int_0^1 d\tau \Gamma_i \delta(\mathbf{r} - \mathbf{r}_i(\tau)), \tag{1}$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{r}_i = (x_i, y_i, z_i)$. Periodicity requires that $\mathbf{r}_i(0) = \mathbf{r}_i(1)$. Therefore, we assume that the discretization of $\mathbf{r}$ into $\sum_i \mathbf{r}_i$ has this property in Eq. (1). Because of the nearly parallel constraint, the arclength τ has the property that $\tau \sim z_i \ \forall\ i$. For simplicity, we will represent $\mathbf{r}_i(\tau) = (x_i, y_i, \tau)$ as a complex number $\psi_i(\tau) = x_i(\tau) + iy_i(\tau)$.

In our analytical approach, it is easier to start with a finite number of filaments and take the limit $N \to \infty$ later, keeping total vorticity $\Lambda = \int_{\mathbb{R}^3} \Omega(\mathbf{r}) d\mathbf{r}$, constant by rescaling. This is known as a nonextensive thermodynamic limit approach because the overall vortex strength stays constant even as the number of vortex filaments increase towards infinity.

We use an approximate model for vortex behavior known as the local-induction approximation (LIA), useful for nearly parallel filaments, combined with a two-dimensional logarithmic interaction. Heuristically, this model comes from two separate results: One result, that of Ref. [6], shows that a discrete field of *perfectly* parallel vortices has a logarithmic interaction. Another result for a single filament in 3D, that of Ref. [4], is a first order approximation for the motion of fluid filaments, extended to the EMH model. This LIA approximation, derived from the Biot-Savart law of magnetic or velocity induction for filaments of magnetism or vorticity, depends on the binormal vector of the filament curve in 3-space causing Brownian variations.

The combination of the two results of Refs. [4,6] yields the familiar London free energy of type-II superconductors [7], equally valid for generalized vorticity [4]:

$$E_N = \alpha \int_0^1 d\tau \sum_{i=1}^{N} \Gamma_i/2|\partial\psi_i(\tau)/\partial\tau|^2 - \int_0^1 d\tau \sum_{i=1}^{N}\sum_{j>i}^{N} \Gamma_i\Gamma_j \log|\psi_i(\tau) - \psi_j(\tau)|, \quad (2)$$

where α is the vortex elasticity in units of energy/length. The generalized angular momentum is

$$M_N = \sum_{i=1}^{N} \Gamma_i \int_0^1 d\tau |\psi_i(\tau)|^2. \quad (3)$$

Γ_i is the vortex circulation. From this point on, we assume the vortex circulations are all scaled to unity, $\Gamma_i = 1 \ \forall\, i$.

Although we say the London free energy, we note that these vortex filaments are generalized and not purely flux lines. Furthermore, the plane is unbounded, not periodic. Other sources for this vorticity description as well as others can be found in Refs. [8–12]. We are the first to apply it to the EMH model to our knowledge.

For our isolated, classical system, the energy and angular momentum plus magnetic moment are conserved, giving rise to the following probability distribution for the filaments in equilibrium:

$$P(s) = Z^{-1}\delta(NH_0 - E_N - pM_N)\delta(NR^2 - M_N), \quad (4)$$

where H_0 is the total "enthalpy" per vortex per period of the plasma, s is the complete state of the system, and $Z = \int ds\,\delta(NH_0 - E_N - pM_N)\delta(NR^2 - M_N)$ is a normalizing factor called the partition function. Here E_N is the energy functional and M_N is the angular momentum. It is our intent to allow R^2, the size of the system,

$$R^2 = \lim_{N\to\infty}\left\langle N^{-1}\int_0^1 d\tau|\psi_i(\tau)|^2\right\rangle, \quad (5)$$

to be determined by other parameters in the system and keep enthalpy and pressure p fixed.

The size of the configuration space (partition function) Z cannot be found in closed form by any known analytical methods. Since our aim is an explicit expression for specific heat, we need to find a closed form approximation of Z.

To simplify the equations, we combine the large number of vortices into two average or "mean" vortices, which results in a mean vorticity field. This is the "mean-field" approach common in statistical mechanics. Our mean vortices are as follows: One mean vortex is a mean distance from the origin. The other is the statistical center of charge of all the filaments—a single, perfectly straight filament fixed at the origin with strength of the remaining vortices, $N - 1 \sim N$.

Given a filament i and a filament j, the mean-field approximation implies the following:

$$\langle|\psi_i - \psi_j|\rangle \to \sqrt{\int_0^1 d\tau|\psi_i(\tau)|^2} = ||\psi_i||, \quad (6)$$

where i is any filament index and the double bars, $||\cdot||$, indicate $\mathcal{L}_2$-norm on the interval [0, 1].

The energy function now reads as follows:

$$E'_N = \int_0^1 d\tau \sum_{i=1}^{N}\left[\frac{\alpha}{2}\left|\frac{\partial\psi_i}{\partial\tau}\right|^2 - \frac{N}{4}\log||\psi_i||^2\right]. \quad (7)$$

This assumption makes all vortices statistically independent, and the statistics of all the vortex structures can be found from those of one. We modify (4) and (3) appropriately and drop primes.

In statistical mechanics of isolated systems, all equilibrium statistics can be determined from maximizing the entropy. The entropy per filament S_N is defined by

$$e^{S_N} = \int D\psi\,\delta(NH_0 - E_N - pNR^2)\delta(NR^2 - M_N). \quad (8)$$

This definition implies

$$S_N = \log\left[\int D\psi\,\delta(NH_0 - E_N - pNR^2)\delta(NR^2 - M_N)\right]. \quad (9)$$

We have now set the stage to describe our derivation of negative specific heat.

Given the space available, we proceed to outline, rather than fully derive, our method of obtaining an explicit formula for the maximal entropy of this mean-field system in the nonextensive thermodynamic limit (as defined above) from which we obtain an explicit, closed form formula for the specific heat.

First, it is important to note that our approach relies heavily on the steepest-descent methods in Ref. [13] and spherical model approach in Refs. [14–17]. The works of Refs. [10,18] have preceded and inspired this work in their novel applications of the spherical model to barotropic vorticity models on the sphere. These works laid the ground for our derivation.

The steepest-descent and consequently spherical model methods convert Dirac-delta functions into their Fourierspace equivalent integral representations. The procedure for our derivation relies on two closely related facts: The integral representation of the Dirac-delta function in a microcanonical distribution, for example:

$$\delta(Nx - Nx_0) = \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} \frac{d\beta}{2\pi i} e^{N\beta(x - x_0)}, \tag{10}$$

and the steepest-descent limit, again only an example:

$$\lim_{N\to\infty} \frac{1}{N} \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} d\beta e^{N\beta(x-x_0)} = e^{\beta_0(x-x_0)}, \tag{11}$$

which allows quantities such as entropy to be determined, provided we can determine what β_0 is, a method for determining which Ref. [13] provides.

Using the first fact, we can convert the microcanonical problem into the canonical problem by converting delta functions into integrals and reordering the phase-space (ψ) and parameter-space (β) integrals to arrive at

$$e^{S_N} = \int \frac{d\beta}{2\pi i} e^{\beta N H_0} Z_{\rm bath}, \tag{12}$$

where

$$Z_{\rm bath} = \int D\psi e^{-\beta E_N - \mu M_N} \delta(NR^2 - M_N) \tag{13}$$

is a canonical-like partition function for the system in an external heat bath but with a microcanonical angular momentum constraint. In the infinite N limit, $Z_{\rm bath}$ approaches the canonical partition function [i.e., the same function but without the Dirac-delta factor in the integrand of Eq. (13)], making their use interchangeable in the limit. To convert the microcanonical problem to a canonical one, one has to prove that the integrals are finite, and indeed we can but do not show it here.

The maximal entropy per filament per period, having the form,

$$S_{\max}(H_0) = \lim_{N\to\infty} N^{-1} S_N, \tag{14}$$

is where the limit comes into play.

Coulomb interactions have a problem in that as the number of "particles" (in this case vortex filaments) grows infinitely large, the interaction energy grows with the square of the number of filaments. The solution is to rescale the temperature, which in turn rescales the interaction energy. Rescaling the temperature causes a chain of necessary scalings to restore the balance so that other quantities do not go to zero: $\beta' = \beta N$, $\alpha' = \alpha/N$, $p' = p/N$, and $H_0' = H_0/N$. These are reasonable because only the interaction energy needs rescaling.

With all these scalings there are no more mathematical obstructions, and the maximal entropy can be found by the standard mathematical procedures in Refs. [13,14]. We provide only the final formula obtained in view of space constraints, but the procedure is quite straightforward once the appropriate framework is set up:

$$S_{\max}(H_0) = \beta_0' H_0' + \frac{\beta_0'}{4}\log(R^2) - \frac{1}{2\alpha\beta_0 R^2} - \beta_0' p' R^2, \tag{15}$$

where

$$R^2 = \frac{\beta_0'^2 \alpha' + \sqrt{\beta_0'^4 \alpha'^2 + 32\alpha' \beta_0'^2 p'}}{8\alpha' \beta_0'^2 p'}, \tag{16}$$

where the mean temperature β_0 is as yet unknown. This entropy is exact within the mean-field assumption for $N \to \infty$.

By Ref. [13], we find the unknown multiplier β_0 by relating the enthalpy per filament parameter H_0 to the mean enthalpy, $NH_0 = \langle E_N + pM_N \rangle$, where $\langle\cdot\rangle$ denotes average against Eq. (4).

By Eq. (4) the average enthalpy is given by

$$\langle E_N + pM_N \rangle = \frac{\int D\psi (E_N + pM_N)\delta(NH_0 - E_N - pM_N)\delta(NR^2 - M_N)}{\int D\psi \delta(NH_0 - E_N - pM_N)\delta(NR^2 - M_N)}, \tag{17}$$

and, again going through some steepest-descent-based calculations given in Ref. [13], we find the formula,

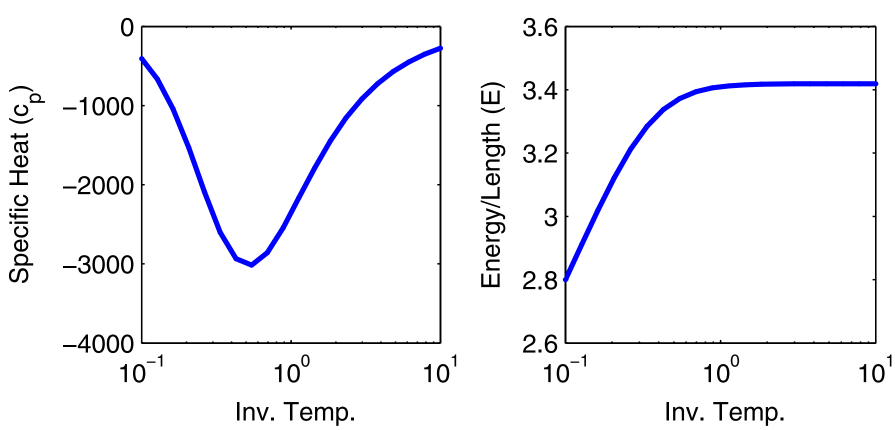

FIG. 1 (color online). The specific heat at constant pressure [Eq. (20)] for the thermally isolated system is negative, meaning that the constant pressure enthalpy per length [Eq. (18)] decreases with increasing temperature. (Here $\alpha' = 5 \times 10^5$ and $p' = 8 \times 10^4$.)

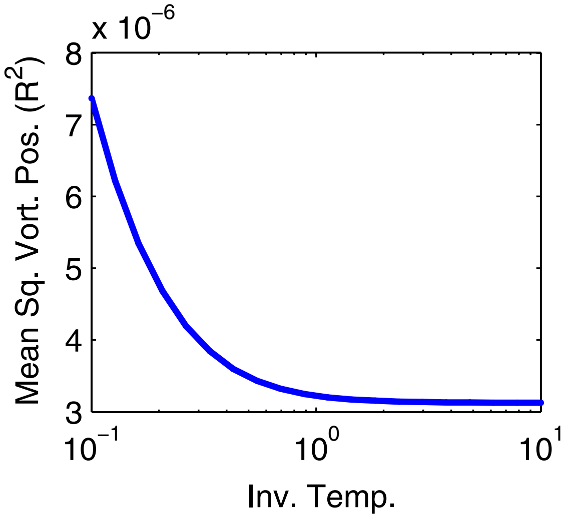

FIG. 2 (color online). The mean square vortex position [Eq. (16)] increases exponentially at high temperature, while it is nearly constant at low temperature. (Here $\alpha = 5 \times 10^5$ and $p = 8 \times 10^4$.)

$$H_0' = \frac{\partial}{\partial \beta_0'}\left(-\frac{\beta_0'}{4}\log R^2 + \frac{1}{2\alpha'\beta_0' R^2} + \beta_0' p' R^2\right) \tag{18}$$

exactly. We cannot give an explicit expression for β_0 because it is a root of a transcendental equation, but such is unnecessary for the following negative specific heat result.

We define specific heat at constant generalized pressure p,

$$c_p = -\beta_0^2 \frac{\partial H_0}{\partial \beta_0}, \tag{19}$$

and after evaluating with Eq. (18) and simplifying (dropping primes and 0 subscripts)

$$c_p = \frac{\beta}{4}\left(\frac{\alpha\beta^2}{\sqrt{\alpha\beta^2(\alpha\beta^2 + 32p)}} - 1\right). \tag{20}$$

Equation (20) is significant. It indicates that the specific heat is not only negative for this system, but *strictly* negative if parameters are nonzero (Fig. 1). In the low-temperature (large β) case, for constant field strength, R^2 does not change significantly with temperature, indicating that filaments are in a stable configuration for a large range of low temperatures. Because the filaments do not move relative to one another at low temperatures and the self-induction is negligible, the enthalpy does not change. As the temperature rises, the increase in internal entropy causes a massive expansion in the overall size of the system (Fig. 2). The strong magnetic field absorbs this energy, but, since it is assumed to be an infinitely massive reservoir able to maintain the enthalpy at H_0, the confinement remains constant.

Only a local-induction approximation is necessary for this analysis, even though 2D point vortices have no negative specific heat in equilibrium. The negative specific heat here can be explained as a process: (1) a vortex's Brownian motion causes it or part of it to move away from the center, (2) potential energy decreases, and (3) the vortices in the center can move closer together and temperature increases.

As mentioned in the preceding paragraph, the negative specific heat indicates a runaway reaction (i.e., the fixed energy, fixed angular momentum equilibrium point is metastable). Considering its similarity to gravo-thermal collapse: We hypothesize that the metastable point could have a collapse similar to globular clusters in which an outer halo of columns separates from an inner core that collapses in on itself, possibly resulting in nuclear fusion. Further research will focus on answering this question, but clearly 3D effects, even only a LIA, are crucial.

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