Efficient Generation of Large Number-Path Entanglement Using Only Linear Optics and Feed-Forward

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We show how an idealized measurement procedure can condense photons from two modes into one and how, by feeding forward the results of the measurement, it is possible to generate efficiently superposition states commonly called $N00N$ states. For the basic procedure sources of number states leak onto a beam splitter, and the output ports are monitored by photodetectors. We find that detecting a fixed fraction of the input at one output port suffices to direct the remainder to the same port, with high probability, however large the initial state. When instead photons are detected at both ports, macroscopic quantum superposition states are produced. We describe a linear-optical circuit for making the components of such a state orthogonal, and another to convert the output to a $N00N$ state. Our approach scales exponentially better than existing proposals. Important applications include quantum imaging and metrology.

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The fundamental limits to optical detection for metrology and imaging are quantum mechanical [1]. Of particular interest for reaching such quantum limits are path-entangled states of photons of the form $|N0\rangle + e^{i\phi}|0N\rangle$, in a basis of photon-number states, commonly referred to as $N00N$ states. A variety of interferometric applications have been suggested, including lithography and microscopy, as well as metrology with sensitivity surpassing the shot-noise limit [2–5]. However, building a source of $N00N$ states beyond two photons is challenging. Three- and four-photon experiments are reported in Refs. [6–8], and a six-photon experiment in Ref. [9], though in the latter experiment phase super-resolution was demonstrated without creating entangled states. In principle a source could be based on a nonlinear-optical interaction [10]. However, the required optical nonlinearity is not readily available. An alternative theoretical approach is based on linear optics, adopting techniques being developed in the field of quantum computation [11,12]. Here multiphoton light is propagated through a series of passive linear-optical elements, which combine different spatial or polarization modes, and is subjected to photodetection and feed-forward, for which components are actively switched according to earlier measurements. By making the output conditional on specific outcomes at photodetectors, $N00N$ states can be generated on the basis of a measurement-induced nonlinearity. A variety of schemes has been proposed [13–16]. However, none of these is truly scalable, in the sense that exponentially decreasing success probabilities outweigh the possible gains.

In this Letter we develop a linear-optics-based method for $N00N$-state generation which scales efficiently. There are two central ideas in our approach. First, we translate the problem into that of creating and manipulating correlations in the relative optical phase between modes. This is possible since a 50:50 beam splitter allows a $N00N$ state to be interconverted with a macroscopic superposition state characterized by components having a well-defined relative phase. A simple measurement-based procedure can generate single and multiple correlations in the relative phase, and is economical with respect to the required number of photodetections [17]. The correlated light at the output can then be manipulated independently of the total photon number using beam splitters and phase shifters. Existing proposals typically depend on engineering some precisely defined destructive interference (defined in a basis of Fock states) that occurs given some specific outcome at photodetectors. This approach is difficult to apply as the photon number grows or is uncertain. Second, we argue that an additional physical resource is required that is absent from all previous schemes—namely that of feed-forward. Feed-forward enables the basis for a measurement to depend on previous measurement results. Although demanding to implement experimentally, feed-forward based schemes are increasingly feasible and have been demonstrated in practice [18].

A measurement procedure for establishing simple two-mode correlations.—We now turn to the linear-optics-based measurement procedure depicted in Fig. 1, which we label as Circuit I. Initially we assume that the state at the source is a pure state, and specifically a dual Fock state $|N\rangle|N\rangle$. All modes are propagating, and the principal modes are labeled 1 and 2. Beam splitters of reflectance f couple modes 1 and 2 to ancillae modes 3 and 4, which are combined at a 50:50 beam splitter. They are then measured by number-resolving photodetectors labeled D_l and D_r, where on average a fraction f of the input photons are registered. In practice the number-resolving detectors can be implemented approximately using a large number of bucket detectors. This modification requires the components acting on modes 3 and 4 to be replaced with a series of similar units, coupling weakly to modes 1 and 2, and

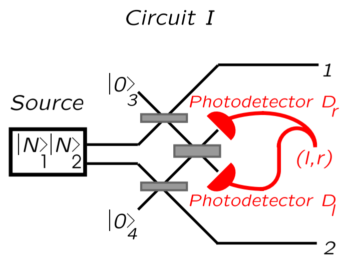

FIG. 1 (color online). Circuit I. Beam splitters couple some fraction f of the population from the principal modes 1 and 2 into ancillae modes 3 and 4, initially the vacuum. The ancillae are combined at a 50:50 beam splitter, and subjected to number-resolving photodetection.

configured so as to minimize the probability for detecting more than one photon at a time. Circuit I implements a measurement that is formally equivalent to one performed by cavity-based schemes, analyzed in the context of the debate over the existence of absolute phase coherence in many standard quantum optics experiments [19]. Here we ask different questions, arguing first for the scalability of the measurement procedure.

We consider first runs of Circuit I for which every photodetection occurs at one photodetector—specifically that r counts occur at detector D_r. We wish to determine the probability that if the remaining $2N - r$ photons propagate though a 50:50 beam splitter, *all* the photons are directed to the output port corresponding to the initial detections. This event may be termed a measurement-induced condensation, and the corresponding probability is denoted P_{cond}. The 50:50 beam splitter combining modes 3 and 4 in Circuit I acts to make the origin of photons from modes 1 and 2 indistinguishable. When a photon is registered at photodetector D_l (or D_r), the transformation is given by the Kraus operator $\hat{L} = (\hat{a} - \hat{b})/\sqrt{2}$ [or $\hat{R} = (\hat{a} + \hat{b})/\sqrt{2}$] (where $\hat{a}$ and $\hat{b}$ are the annihilation operators for modes 1 and 2, respectively). We denote the state of modes 1 and 2 at the output of Circuit I by $|\psi_{0r}\rangle$, and find that $|\psi_{0r}\rangle = \hat{R}^r|N\rangle|N\rangle/\sqrt{\langle N|\langle N|(\hat{R}^\dagger)^r(\hat{R})^r|N\rangle|N\rangle}$, normalizing to unity. The probability of a measurement-induced condensation is then given by $P_{\text{cond}} = \langle\psi_{0r}|(\hat{R}^\dagger)^S(\hat{R})^S|\psi_{0r}\rangle/S!$, where $S \equiv 2N - r$ denotes the total remaining photon number, and it is necessary to normalize by the total photon number prior to detection. Next we find $P_{\text{cond}} = ({}^{2N}C_N)^2/[2^S\sum_{k=0}^{r}({}^rC_k)^2({}^SC_{N-k})]$, where C denotes a binomial coefficient, and we assume that $r < N$. Evaluating P_{cond} numerically for initial states of increasing size, we find that its value is determined asymptotically by the proportion of the input that is measured. For example, setting r either as one-quarter or one-third of $2N$ suffices for $P_{\text{cond}} > 0.6$ or $P_{\text{cond}} > 0.7$, respectively. If this procedure were used to add the Fock-state inputs, P_{cond} has the same value as the fidelity of the larger Fock state at the end. We can reinterpret the results above following Refs. [17,19,20]. The principal effect of the initial r photodetections is to cause the relative phase of modes 1 and 2, which is completely undefined for the initial state, to localize strongly at the value zero. This enables the photons that remain to be directed into one mode using a 50:50 beam splitter. The procedure sometimes fails since the phase correlation which evolves after a finite number of detections is not ideal.

We now consider in detail the case for which Circuit I registers photons at both ports, leading to multiple correlations in the relative phase. The state at the output of Circuit I, after l photons are registered at D_l and r at D_r, was derived in Refs. [17,20]. We denote this state by $|\psi_{l,r}\rangle$. In the previous references, it was found convenient to work in an (overcomplete) basis of coherent states, which are of the form $|\alpha \equiv |\alpha|e^{i\theta}\rangle \propto \sum_{k=0}^{\infty}\sqrt{|\alpha|^{2k}/k!}\,e^{ik\theta}|k\rangle$ in a Fock basis. It was shown that $|\psi_{l,r}\rangle$ is proportional to

$$\int_0^{2\pi}\int_{-\pi}^{\pi} d\theta_{\text{av}}d\Delta e^{-iS\theta_{\text{av}}}[G(\Delta - \Delta_0) + e^{i\sigma}G(\Delta + \Delta_0)]|\alpha_1\rangle|\alpha_2\rangle,$$

where $S \equiv 2N - l - r$ is the total photon number, $\alpha_j = |\alpha_j|e^{i\theta_j}$, $\theta_{\text{av}} = (\theta_1 + \theta_2)/2$, and $\Delta \equiv \theta_2 - \theta_1$ defines the relative-optical phase parameter. The scalar function $G(X)$ is given to good approximation by the Gaussian expression $\exp[-(l + r)X^2/4]$, assuming the total number of detections is not small, and in fact has a smaller spread than is the case when the detections are at D_r only. Focusing on the aspects of this result relevant to engineering $N00N$ states we note the following. There are sharply defined correlations in Δ at values plus and minus Δ_0, defining a macroscopic superposition state. This is because operators $\hat{L}$ and $\hat{R}$ are invariant under an exchange of the labeling of the modes, a symmetry which reverses the sign of the relative phase. The superposition phase σ takes the value $l\pi$, and hence the measurement record must be known exactly. This makes the scheme very sensitive to dark counts at the photodetectors, although these can be very low in experiments. For the source, the pure states assumed previously are not particularly feasible. However, we observe that Δ_0 is determined by the ratio of l to r and is independent of N, and standard linear-optical elements obey a superselection rule for the photon number. Hence the input state can in fact be a mixture of the form $\sum_N P_N|N\rangle|N\rangle\langle N|\langle N|$. Several two-mode squeezing processes strongly suppress relative number fluctuations, and hence might serve as practical sources. A proof-of-principle experiment could make use of an unseeded high-gain optical parametric amplifier, which serves as a source of the two-mode squeezed vacuum. However, the distribution P_N in this case is broad, and weighted toward the vacuum and low photon numbers. Looking to a source for which P_N is sharply peaked about some preferred photon number, we speculate that an optical parametric oscillator in certain configurations might be suitable, and point to recent theoretical and experimental developments [21–23].

Manipulating relative phase correlations and constructing N00N-state generators.—In the next stage of our analysis, we identify the precise linear-optical transformations required to convert the macroscopic superposition states generated by Circuit I to $N00N$ states. For the current purposes we can assume that a large number of detections have been performed and define $|\psi_\infty(\Delta_0)\rangle \propto \int_0^{2\pi} d\theta e^{-iS\theta}|\alpha\rangle|\alpha e^{i\Delta_0}\rangle$, where $\alpha = |\alpha|e^{i\theta}$, for a state with a total photon number S and a relative phase of Δ_0 (assumed to be normalized). It is instructive to identify these states as states for quantum reference frames—reference frames for a classically defined parameter composed of finite quantum resources. Quantum reference frames are subject to depletion and degradation as they are used, and are currently of interest for protocols in the field of quantum information, in which they are regarded as a resource [24]. This identification is helpful because we may understand linear-optical transformations of states of the form $|\psi_\infty\rangle$, which are algebraically involved, in terms of the corresponding manipulations of two classical fields. A single-mode classical field is represented by a complex number, with the square amplitude corresponding to the intensity, and the phase equal to the optical one.

We now consider a simple $N00N$-state generator that attempts to convert every macroscopic superposition state generated by Circuit I directly to a $N00N$ state. To relate these two types of states, we consider what happens when an S-photon Fock state is beaten against the vacuum at a 50:50 beam splitter. Denoting the beam splitter by $U_{\rm bs}$, we find that $U_{\rm bs}|S\rangle|0\rangle \propto |\psi_\infty(0)\rangle$, and $U_{\rm bs}|0\rangle|S\rangle \propto |\psi_\infty(\pi)\rangle$. Hence a beam splitter together with a phase shifter can convert a macroscopic superposition state generated by Circuit I to a $N00N$ state, whenever the relative phase correlations differ by π. This happens when the counts at detectors D_l and D_r in Circuit I are the same, and the values of the relative phase at the output are $\pm\Delta_0 = \pm\pi/2$. We label this process Circuit III (anticipating an intermediate process modifying the output from Circuit I). Our first $N00N$-state generator then is illustrated in Fig. 2(a), with the output of Circuit I fed directly into Circuit III, and we denote the state of the end product by $|\psi_{\rm output}\rangle$. We adopt the fidelity F to evaluate the output state, and define $F = \max_\phi |(\langle 0S| + \exp(-i\phi)\langle S0|)|\psi_{\rm output}\rangle|^2/2$, where S denotes the total photon number at the end. A full calculation shows that $F \sim \cos^{2S}[(\Delta_0 - \pi/2)/2]$. As with other proposals, this scheme in fact scales *exponentially poorly* whenever the relative phase correlations are less than π apart, as is typically the case. Inspecting the overlap for states $|\psi_\infty(\Delta_1)\rangle$ and $|\psi_\infty(\Delta_2)\rangle$, with total photon number S, we find that $|\langle\psi_\infty(\Delta_1)|\psi_\infty(\Delta_2)\rangle| = |\cos[(\Delta_2 - \Delta_1)/2]|^S$. Hence the poor scaling can be attributed to the nonorthogonality of the components of the macroscopic superposition states generated by Circuit I.

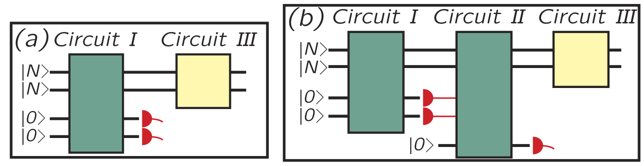

FIG. 2 (color online). (a) and (b) illustrate complete $N00N$-state generators in outline. Circuit I produces macroscopic superposition states nondeterministically. Circuit III consists of a $\pi/2$ phase shifter and a 50:50 beam splitter, and performs final conversion to a $N00N$ state. In (b) the Circuit I detection outcomes are feed-forward to Circuit II, which implements a correction step using a beam splitter of variable transmittance and an ancillary mode.

To summarize, our previous $N00N$-state generator is effective when Circuit I generates macroscopic superposition states with components that are orthogonal. However, this occurs with low probability. Hence we now devise a circuit, labeled Circuit II, to orthogonalize the components of the states generated by Circuit I, using additional processes of measurement and feed-forward, as illustrated in Fig. 2(b). To identify a suitable circuit, we examine the transformation of two general classical fields at a 50:50 beam splitter. If the beam splitter is configured so as not to cause additional phase shifts to the modes, it outputs two classical fields described by the sum and difference of the values for the inputs. Both the phases and the square amplitudes for the classical fields are changed. For example, if the input has a relative phase of 0 or π, and equal intensities for each mode, the population is transferred entirely into a single mode. On the other hand, if the input has a relative phase of plus or minus $\pi/2$, and equal intensities in each mode, the relative phase and intensities are preserved. The action of the beam splitter on quantum fields in the state $|\psi_\infty\rangle$, having a relative phase of Δ_0, a total photon number S, and an intensity $S/2$ for each mode, is similar. In detail, $U_{\rm bs}|\psi_\infty\rangle \propto \int_0^{2\pi} d\theta e^{-iS\theta}|\sqrt{I_1}e^{i\theta}\rangle|\sqrt{I_2}e^{i(\theta\pm\pi/2)}\rangle$. The final state has an intensity $SI_1/(I_1 + I_2) = S[1 - \cos(\Delta_0)]/2$ in mode 1 and $SI_2/(I_1 + I_2) = S[1 + \cos(\Delta_0)]/2$ in mode 2, and a relative phase of plus $\pi/2$ when $0 < \Delta_0 \le \pi/2$ and of minus $\pi/2$ when $-\pi/2 \le \Delta_0 < 0$. We consider cases for which the intensity is increased in favor of mode 2. Hence a beam splitter acting on modes 1 and 2 at the output of Circuit I can change the difference of the relative phase variables to π, but creates a difference in the intensities between the modes in doing so. We propose that Circuit II then beats mode 2 against the vacuum, so as to move the difference of the intensities to an ancillary mode, which can be removed by photodetection. The cost of this correction is a decrease in the total photon number, which varies nondeterministically. A fraction of $\cos(\Delta_0)$ of the photons are lost on average. Our proposal depends critically on feeding forward the result of the detections at D_l and D_r in Circuit I, so that the correct intensity correction may be applied using a beam splitter of variable reflectance. The latter might be implemented using 50:50 beam splitters and controlled phase shifters, as well as delay loops after Circuit I.

Overall Circuits I through III constitute a complete $N00N$-state generator. Runs for which Circuit I fails to generate a macroscopic superposition state, or too many photons are lost in the detection process, are discarded. We have found by analytical and numerical methods, that the fidelities at the output are, on average, 0.87, 0.94, or 0.98, when a fraction of one-third, one-half, or two-thirds, respectively, of the input photons are detected by Circuit I. Higher fidelities are possible when the photon number at the input is small. If allowance is made for sufficient input photons to be detected by Circuit I, and a further half to be detected in Circuit II, the probability of failure is not too large. We anticipate the main sources of experimental error as being due to decorrelation of the source, and loss due to inefficiencies for the photodetectors and the delay loops. It is beyond the scope of this Letter to quantify these effects. To verify the correct operation of our method, we propose following Circuits I and II with a further measurement unit similar to Circuit I (omitting Circuit III), so that the generation of macroscopic superposition states with relative phase components π apart can be demonstrated by the ratio of photodetections at the end. We note that the photodetector in Circuit II determines only the final photon number, and can be omitted. Although we have focused here on optical experiments, our results are also relevant to many controlled bosonic systems, for which techniques for coherent manipulation have been demonstrated (for example, Bose-Einstein condensates and singly-trapped bosonic atoms [25]). Finally, our analysis has broader interest. The scaling we derive for our measurement-induced condensation is a new result on interfering light from independent sources and localizing relative-optical phase. These phenomena have analogues in a variety of physical systems [26]. We have left as an open question the extent to which this scaling can be attributed to Bose statistics, as is the case for some dynamical processes of condensation. Our study of special macroscopic superposition states may also have application to quantum computing. Here it has been proposed that qubits be encoded using macroscopic superposition states defined in one mode only [27].

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