Nonlattice Simulation for Supersymmetric Gauge Theories in One Dimension

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Lattice simulation of supersymmetric gauge theories is not straightforward. In some cases the lack of manifest supersymmetry just necessitates cumbersome fine-tuning, but in the worse cases the chiral and/or Majorana nature of fermions makes it difficult to even formulate an appropriate lattice theory. We propose circumventing all these problems inherent in the lattice approach by adopting a nonlattice approach for one-dimensional supersymmetric gauge theories, which are important in the string or M theory context. In particular, our method can be used to investigate the gauge-gravity duality from first principles, and to simulate M theory based on the matrix theory conjecture.

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Introduction.—Lattice gauge theory, together with the developments of various simulation techniques, has provided us with a powerful nonperturbative method to study gauge theories such as QCD. However, when one tries to apply the method to *supersymmetric* gauge theories, which are interesting for many reasons, one has to face some practical and theoretical obstacles.

First of all, since the algebra of supersymmetry contains continuous translations, which are broken to discrete ones, it seems unavoidable to break it on the lattice. Then, one has to include all the relevant terms allowed by symmetries preserved on the lattice, and fine-tune the coupling constants to arrive at the desired supersymmetric fixed point in the continuum limit. Recent progress (See Ref. [1] and references therein) is that one can reduce the number of parameters to be fine-tuned (even to zero in some cases) by preserving some part of supersymmetry. In lower dimensions, one can alternatively take the advantage of super-renormalizability, and determine the appropriate counterterms by perturbative calculations to avoid fine-tuning. These two approaches can be nicely illustrated in one dimension by the example of a supersymmetric anharmonic oscillator [2,3].

The aim of this Letter is to point out that there exists an extremely simple and elegant nonlattice method to simulate supersymmetric gauge theories in one dimension, which are important in the string or M theory context.

Recent developments in string theory owe much to the discovery that the low energy behavior of open strings attached to a stack of N Dp branes in 10D is described by $(p+1)$-dimensional U(N) supersymmetric gauge theory. The gauge theory can be obtained by dimensionally reducing 10D $\mathcal{N}=1$ U(N) super Yang-Mills theory to $p+1$ dimensions. This led to the conjectured gauge-gravity duality, which states that the strong coupling limit of large-N gauge theories has a dual description in terms of classical supergravity. For instance, in the $p=0$ case, one obtains a 1D supersymmetric U(N) gauge theory with 9 adjoint scalars as a low energy effective theory of N D0 branes. In the large-N 't Hooft limit and in the strong coupling limit, this theory has a dual description in terms of a black D0-brane solution in 10D type IIA supergravity [4]. Using our method, one can confirm the duality by studying the strongly coupled dynamics of the gauge theory from first principles. Once this is done, one can turn around and investigate the quantum and/or stringy nature of the black hole in terms of gauge theory.

A different but closely related set of conjectures assert that nonperturbative formulations of superstring or M theory can be given by matrix models, which take the same form as the low energy effective theory of Dp branes. In particular, it is conjectured [5] that the aforementioned 1D supersymmetric U(N) gauge theory, in a different parameter region, actually describes M theory microscopically. We therefore expect that our method is also useful for simulating M theory. In that context the system with finite N corresponds to a sector of M theory compactified on a lightlike circle [6].

The bosonic version of the 1D gauge theory has been studied by Monte Carlo simulation using the lattice formulation [7] and the continuum quenched Eguchi-Kawai model [8]. As for the supersymmetric case, Ref. [9] proposes a lattice formulation, which preserves half of supersymmetry (SUSY) at the expense of breaking the SO(9) symmetry.

Let us recall that the importance of the lattice formulation lies in its manifest gauge invariance. In the present 1D case, however, the gauge dynamics is almost trivial. (We assume that the 1D direction is compact. The noncompact case would be easier since the gauge dynamics is completely trivial.) This gives us an opportunity to use a nonlattice formulation. More specifically, we first take the static diagonal gauge. Using the residual large gauge transformation, we can choose a gauge slice such that the diagonal elements of the constant gauge field lie within a minimum interval. Finally we expand the fields into Fourier modes, and keep only the modes up to some cutoff. The crucial point of our method is that the gauge symmetry is completely fixed (up to the global permutation group, which is kept intact) before introducing the cutoff. This is specific to one dimension, and the momentum cutoff regularization in higher dimensions generally breaks gauge invariance.

Supersymmetric anharmonic oscillator.—To gain some insight into our new approach, we first apply it to a nongauge supersymmetric theory, which is well studied by the lattice formulation. In particular, supersymmetry, which is broken by the cutoff in our formalism, is restored much faster than the continuum limit is achieved.

While the manuscript was being prepared, we received a preprint [10], in which the same model is studied on the lattice using various methods. As far as nongauge theories are concerned, our approach is almost equivalent to the method from the nonlocal SLAC derivative [11]. The only difference is the identification of the modes at the boundary of the Brillouin zone in the lattice case. As a consequence, our results shown in Fig. 1 agree with the corresponding results in Ref. [10].

The model is defined by the action

$$S=\int_0^\beta dt\left[\frac{1}{2}\{(\partial_t\phi)^2+h'(\phi)^2\}+\bar{\psi}\{\partial_t+h''(\phi)\}\psi\right], \quad (1)$$

where ϕ is a real scalar field, and ψ is a one-component Dirac field, both in 1D, obeying periodic boundary conditions. This model has two supercharges for arbitrary

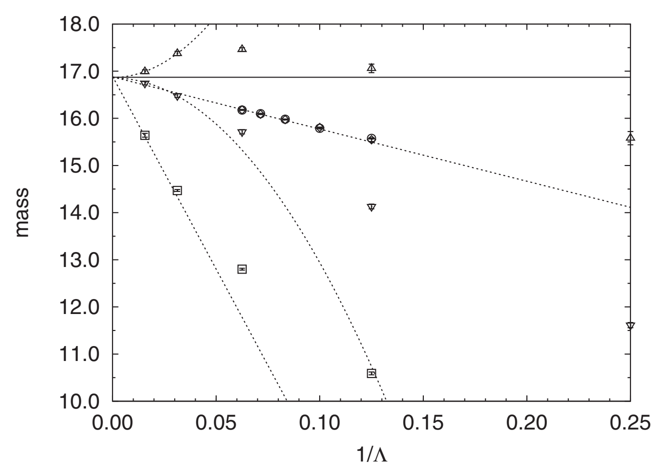

FIG. 1. The circles (diamonds) are the mass for the boson (fermion) obtained by our method for $\Lambda = 8, 10, 12, 14, 16$. The squares are the results obtained by Catterall and Gregory [2] with the lattice action preserving half of SUSY, hence degenerate. The triangles (inverted triangles) are the mass for the boson (fermion) obtained by Giedt *et al.* [3] with the $O(a)$-improved lattice action. The horizontal line represents the exact result, and the dotted lines represent the expected behaviors at large Λ.

function $h(\phi)$, but here we take $h(\phi)=\frac{1}{2}m\phi^2+\frac{1}{4}g\phi^4$. In our approach we make a Fourier expansion

$$\phi(t)=\sum_{n=-\Lambda}^{\Lambda}\tilde{\phi}_n e^{i\omega nt}; \qquad \omega\equiv\frac{2\pi}{\beta}, \quad (2)$$

and similarly for the fermionic fields, where n takes integer values, and Λ is the UV cutoff. In terms of the Fourier modes, the action can be written as $S=S_B+S_F$, where

$$S_B=\beta\left[\sum_{n=-\Lambda}^{\Lambda}\frac{1}{2}\{(n\omega)^2+m^2\}\tilde{\phi}_n\tilde{\phi}_{-n} + mg(\tilde{\phi}^4)_0+\frac{1}{2}g^2(\tilde{\phi}^6)_0\right], \quad (3)$$

$$S_F=\sum_{nk}\tilde{\bar{\psi}}_n\mathcal{M}_{nk}\tilde{\psi}_k, \qquad \mathcal{M}_{nk}=\beta[(in\omega+m)\delta_{nk}+3g(\tilde{\phi}^2)_l\delta_{n,k+l}]. \quad (4)$$

We have introduced a shorthand notation

$$(f^{(1)}\cdots f^{(p)})_n\equiv\sum_{k_1+\cdots+k_p=n}f^{(1)}_{k_1}\cdots f^{(p)}_{k_p}. \quad (5)$$

Integrating out the fermions first, we obtain the effective action for the bosons as $S_{\rm eff}=S_B-\ln\det\mathcal{M}$, where $\det\mathcal{M}$ is real positive for positive m and g.

As an efficient algorithm to simulate the model, we use the idea of the hybrid Monte Carlo (HMC) algorithm [12]. Let us introduce the auxiliary real field $\Pi(t)$, whose Fourier components are denoted as $\tilde{\Pi}_n$, and consider the action $S_{\rm HMC}=S_{\rm eff}+\sum_{n=-\Lambda}^{\Lambda}\frac{1}{2}\tilde{\Pi}_n\tilde{\Pi}_{-n}$. Integrating out the auxiliary field first, we retrieve the original action. In order to update the fields, we solve the equations

$$\frac{d\tilde{\phi}_n(\tau)}{d\tau}=\alpha_n\frac{\partial S_{\rm HMC}}{\partial\tilde{\Pi}_n}=\alpha_n\tilde{\Pi}_{-n}, \quad (6)$$

$$\frac{d\tilde{\Pi}_n(\tau)}{d\tau}=-\alpha_n\frac{\partial S_{\rm HMC}}{\partial\tilde{\phi}_n}=-\alpha_n\frac{\partial S_{\rm eff}}{\partial\tilde{\phi}_n} \quad (7)$$

along the fictitious time τ for a fixed interval τ_f. The real coefficients α_n should be optimized based on the idea of the Fourier acceleration [13]. The τ evolution, if treated exactly, conserves the action $S_{\rm HMC}$. In practice, we discretize it in such a way that the reversibility is maintained (the leap-frog discretization). Because of the discretization, the action $S_{\rm HMC}$ changes by a small amount, say $\Delta S_{\rm HMC}$. In order to satisfy the detailed balance, we accept the new configuration with the probability $\min(1, e^{-\Delta S_{\rm HMC}})$, which is the usual Metropolis procedure. Before we start a new τ evolution, we refresh the $\tilde{\Pi}_n$ variables by drawing Gaussian random numbers which follow from the action $S_{\rm HMC}$. This procedure is necessary for avoiding the ergodicity problem. (The step size $\Delta\tau$ should be optimized for fixed τ_f by maximizing the acceptance rate times $\Delta\tau$. Then τ_f should be optimized by minimizing the autocorrelation time in units of step in the τ evolution.)

The main part of the computation is the evaluation of the term in Eq. (7) given by

$$\frac{\partial S_{\rm eff}}{\partial \tilde{\phi}_n} = \beta[\{(n\omega)^2 + m^2\}\tilde{\phi}_{-n} + 4mg(\tilde{\phi}^3)_{-n} + 3g^2(\tilde{\phi}^5)_{-n}] - \mathrm{tr}\left(\frac{\partial \mathcal{M}}{\partial \tilde{\phi}_n}\mathcal{M}^{-1}\right). \quad (8)$$

The convolution requires $O(\Lambda^2)$ calculations, while the inverse $\mathcal{M}^{-1}$ requires $O(\Lambda^3)$ calculations.

As usual, we extract masses from the exponential decay of the two-point functions

$$G_B(t) \equiv \langle \phi(0)\phi(t)\rangle = b_0 + 2\sum_{n=1}^{\Lambda} b_n \cos(\omega n t), \quad (9)$$

$$G_F(t) \equiv \langle \psi(0)\bar{\psi}(t)\rangle = \sum_{n=-\Lambda}^{\Lambda} c_n e^{-i\omega n t}, \quad (10)$$

where we have defined $b_n \equiv \langle |\tilde{\phi}_n|^2\rangle$ and $c_n \equiv \langle (\mathcal{M}^{-1})_{nn}\rangle$. For the fermion, it proved convenient to consider, instead of (10), a symmetrized correlator

$$G_F^{(\rm sym)}(t) \equiv \frac{1}{2}\{G_F(t) + G_F(-t)\} = c_0 + 2\sum_{n=1}^{\Lambda} \mathrm{Re}(c_n)\cos(\omega n t), \quad (11)$$

where we have used the fact $(\mathcal{M}_{nk})^* = \mathcal{M}_{-n,-k}$. In fact the functions (9) and (11) with respect to t oscillate with the frequency of the order of cutoff. This is nothing but the Gibbs phenomenon due to the sharp cutoff in the sum over Fourier modes. To overcome this problem, we note that the coefficients b_n behave as $b_n \sim \frac{d_1}{n^2} + \frac{d_2}{n^4}$ at large n as can be shown from general arguments. We obtain the coefficients d_1 and d_2 from the results at $n = \Lambda - 1, \Lambda$, and extend the sum in (9) over n up to 1000 assuming the above asymptotic form. We make an analogous analysis for $\mathrm{Re}(c_n)$ in (11). In this way we are able to see clear exponential behaviors, and extract the corresponding masses. The results for $\beta = 1$, $m = 10$, $g = 100$ are plotted against $1/\Lambda$ in Fig. 1. (Note that the effective coupling constant is $g/m^2 = 1$.) We find that the finite Λ effects are $O(1/\Lambda)$, and that the data points for the boson and the fermion lie on top of each other. Thus in our formalism, the effect of supersymmetry breaking by the cutoff disappears much faster than $1/\Lambda$. In the same figure we also plot the results obtained from lattice formulations for comparison. (Matching the number of degrees of freedom, we make an identification $\Lambda = \frac{L}{2}$, where L is the number of sites.)

Supersymmetric matrix quantum mechanics.—Here we consider a model with four supercharges defined by

$$S = \frac{1}{g^2}\int_0^\beta dt \mathrm{tr}\left\{\frac{1}{2}(D_t X_i)^2 - \frac{1}{4}[X_i, X_j]^2 + \bar{\psi} D_t \psi - \bar{\psi}\sigma_i[X_i, \psi]\right\}, \quad (12)$$

where $D_t = \partial_t - i[A(t), \cdot]$ represents the covariant derivative with the gauge field $A(t)$ being an $N \times N$ Hermitian matrix. The bosonic matrices $X_i(t)$ $(i = 1, 2, 3)$ are $N \times N$ Hermitian, and the fermionic matrices $\psi_\alpha(t)$ and $\bar{\psi}_\alpha(t)$ $(\alpha = 1, 2)$ are $N \times N$ matrices with complex Grassmann entries. The 2×2 matrices σ_i are the Pauli matrices. The model can be obtained formally by dimensionally reducing 4D $\mathcal{N} = 1$ U(N) super Yang-Mills theory to 1D, and it can be viewed as a one-dimensional U(N) gauge theory. (The totally reduced model has been studied by Monte Carlo simulation in Refs. [14].) Let us assume the boundary conditions to be periodic for bosons and antiperiodic for fermions. The extent β in the Euclidean time direction then corresponds to the inverse temperature $\beta \equiv 1/T$. The parameter g in (12) can always be absorbed by an appropriate rescaling of the matrices and the time coordinate t. Hence we set $g = \frac{1}{\sqrt{N}}$ without loss of generality.

Let us take the static diagonal gauge $A(t) = \frac{1}{\beta}\mathrm{diag}(\alpha_1, \ldots, \alpha_N)$, where α_a $(a = 1, \ldots, N)$ can be chosen to lie within the interval $(-\pi, \pi]$ by making a gauge transformation with a nonzero winding number [15]. We have to add to the action a term $S_{\rm FP} = -\sum_{a<b} 2\ln|\sin\frac{\alpha_a - \alpha_b}{2}|$, which appears from the Faddeev-Popov procedure, and the integration measure for α_a is taken to be uniform.

We expand $X_i^{ab}(t) = \sum_{n=-\Lambda}^{\Lambda} \tilde{X}_{in}^{ab} e^{i\omega n t}$ and $\psi_\alpha^{ab}(t) = \sum_{r=-\lambda}^{\lambda} \tilde{\psi}_{\alpha r}^{ab} e^{i\omega r t}$ into Fourier modes, and similarly for $\bar{\psi}$, where r takes half-integer values due to the antiperiodic boundary conditions, and $\lambda \equiv \Lambda - 1/2$. Equation (12) can then be written as

$$\begin{aligned} S = N\beta\Bigg[&\frac{1}{2}\sum_{n=-\Lambda}^{\Lambda}\left(n\omega - \frac{\alpha_a - \alpha_b}{\beta}\right)^2 \tilde{X}_{i,-n}^{ba}\tilde{X}_{in}^{ab} - \frac{1}{4}\mathrm{tr}([\tilde{X}_i, \tilde{X}_j]^2)_0\Bigg] \\ &+ N\beta\sum_{r=-\lambda}^{\lambda}\Bigg[i\left(r\omega - \frac{\alpha_a - \alpha_b}{\beta}\right)\tilde{\bar{\psi}}_{\alpha r}^{ba}\tilde{\psi}_{\alpha r}^{ab} - (\sigma_i)_{\alpha\beta}\mathrm{tr}\{\tilde{\bar{\psi}}_{\alpha r}([\tilde{X}_i, \tilde{\psi}_\beta])_r\}\Bigg]. \end{aligned} \quad (13)$$

The algorithm for simulating (13) is analogous to the previous model. Here we introduce the auxiliary variables $\Pi_i(t)$ and p_a, which are $N \times N$ Hermitian matrices conjugate to $X_i(t)$ and N real variables conjugate to α_a, respectively. The fermion determinant is real positive, and the computational effort for one step in the τ evolution is proportional to $\Lambda^3 N^6$. Figures 2 and 3 show the results for the energy and the Polyakov line, respectively, for the bosonic and SUSY cases.

In the bosonic case we also plot the results from lattice simulation with the lattice spacing $a = 0.02$. (The number of lattice sites is given by $L = 1/(Ta)$, which is 50 for $T = 1$.) The results obtained by our new method approach the lattice result as Λ is increased.

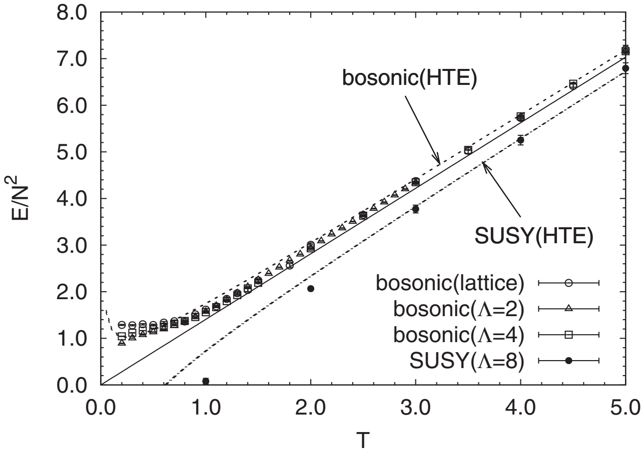

FIG. 2. The energy (normalized by N^2) is plotted against temperature for the matrix quantum mechanics with $N = 4$.

In the SUSY case, our preliminary results with $\Lambda = 8$ reproduce the asymptotic behavior at high T obtained by the high temperature expansion (HTE) up to the next-leading order [16]. (The solid lines represent the results at the leading order of HTE, which are the same for the bosonic and SUSY cases.) Note that our method is also applicable at low T, where the HTE is no more valid.

Summary and concluding remarks.—In this Letter we have proposed a new simulation method, which enables nonperturbative studies of supersymmetric gauge theories in one dimension. For practical implementation, the idea of the hybrid Monte Carlo algorithm seems to be quite useful. In particular, the Fourier acceleration requires no extra cost, since we deal with the Fourier modes directly. The continuum limit is achieved much faster than one would expect naively from the number of degrees of freedom. This is understandable since the Fourier modes omitted by the cutoff scheme are naturally suppressed by the kinetic term. The most interesting case with 16 supercharges is currently under investigation [17].

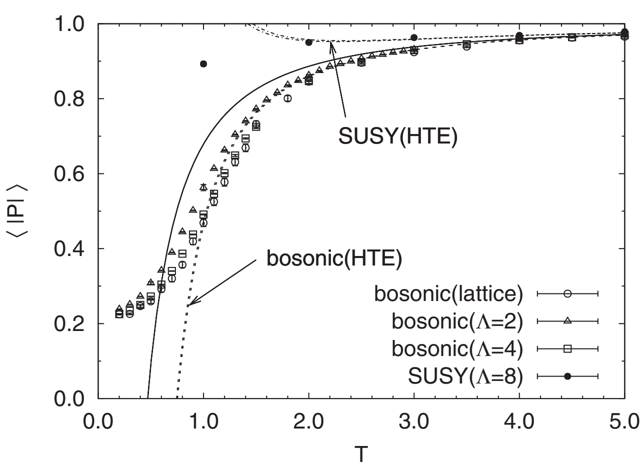

FIG. 3. The same as Fig. 2 but for the absolute value of the Polyakov line.

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