Loop-Quantum-Gravity Vertex Amplitude

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Spin foam models are hoped to provide the dynamics of loop-quantum gravity. However, the most popular of these, the Barrett-Crane model, does not have the good boundary state space and there are indications that it fails to yield good low-energy n-point functions. We present an alternative dynamics that can be derived as a quantization of a Regge discretization of Euclidean general relativity, where second class constraints are imposed *weakly*. Its state space matches the $SO(3)$ loop gravity one and it yields an $SO(4)$-covariant vertex amplitude for Euclidean loop gravity.

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The *kinematics* of loop-quantum gravity (LQG) provides a well-understood background-independent language for a quantum theory of physical space [1–3]. Its *dynamics* is studied along two lines: Hamiltonian (as in the Schrödinger equation) [4] or covariant (as in Feynman's covariant quantum field theory). The key object that defines the dynamics in the second of these is the vertex amplitude, like the vertex $e\gamma^\mu$ ~~< that defines the dynamics of perturbative QED. What is the vertex of LQG?

The spin foam formalism [5] is viewed as a possible tool for answering this question. It can be derived in a remarkable number of distinct ways, which converge to the definition of transition amplitudes as a Feynman sum over spin foams. A spin foam is a two-complex (union of faces, edges, and vertices) colored with quantum numbers (spins associated to faces, intertwiners to edges); it can be loosely interpreted as a spin-network (colored graph) history. Its amplitude contains the product of vertex amplitudes; so vertices play a role similar to Feynman's covariant quantum field theory vertices [6,7]. This picture is nicely implemented in three dimensions by the Ponzano-Regge model [8], where the spin networks are precisely the LQG ones [9], and the vertex amplitude is given by the $6j$ Wigner symbol, which can be obtained as a matrix element of the Hamiltonian of 3D gravity [10].

Compelling and popular as it is, however, this picture has never been fully implemented in four dimensions. The best studied model in the 4D context is the Barrett-Crane (BC) model [11]. This is simple and elegant, has remarkable finiteness properties [12], and can be considered a modification of a topological BF theory, by means of constraints—called simplicity constraints—whose classical limit yields precisely the constraints that change BF theory into general relativity (GR). Furthermore, in the low-energy limit some of its n-point functions appear to agree with those computed from perturbative quantum GR [13]. However, the suspicion that something is wrong with the BC model has long been agitated [14]. Its boundary state space is similar, but does not exactly match, that of LQG; the volume operator is ill defined. Worse, recent calculations indicate that some n-point functions fail to yield the correct low-energy limit [15]. All these problems are related to the way the *intertwiner* quantum numbers (associated to the angles between the faces bounding the elementary quanta of space) are treated: these are fully constrained in the BC model by imposing the simplicity constraints as strong operator equations ($C_n\psi = 0$). But they are second class and imposing such constraints strongly may lead to the incorrect elimination of physical degrees of freedom [16].

It is therefore natural to try to implement in four dimensions the general picture discussed above by correcting the BC model [7,17]. In this Letter we show that this is possible, by properly imposing some of the constraints weakly ($\langle\phi C_n\psi\rangle = 0$), and that the resulting theory has remarkable features. First, its boundary quantum state space matches *exactly* that of $SO(3)$ LQG: no degrees of freedom are lost. Second, as the degrees of freedom missing in BC are recovered, the vertex may yield the correct low-energy n-point functions. Third, the vertex can be seen as defined on $SO(3)$ spin networks or $SO(4)$ ones, and is both $SO(3)$ and $SO(4)$ covariant. Finally, the theory can be obtained as a bona fide quantization of a discretization of Euclidean GR on a Regge triangulation. Here we give the definition of the theory, illustrate its main aspects and give only a rapid sketch of its derivation from Regge GR. Details are given elsewhere. We stress that we work in the Euclidean context: real Lorentzian gravity, which could be quite different, will be discussed elsewhere.

The model we discuss is defined by a standard spin foam partition function

$$Z_{\mathrm{GR}} = \sum_{l_f, i_e} \prod_f \left(\dim \frac{l_f}{2}\right)^2 \prod_v A(l_f, i_e), \tag{1}$$

where the amplitude is given by

$$\begin{aligned} A(l_f, i_e) &= 15j_{SO(4)}\left(\frac{l_f}{2}, \frac{l_f}{2}; f(i_e)\right) \\ &= \sum_{i_e^+, i_e^-} 15j_{SO(4)}\left(\frac{l_f}{2}, \frac{l_f}{2}; i_e^+, i_e^-\right) \prod_{e\in v} f^{i_e}_{i_e^+ i_e^-}. \end{aligned} \tag{2}$$

Notation is as follows. The model is defined on a fixed cellular complex, dual to a 4D triangulation Δ. We do not discuss the recovery of triangulation independence [2,11,18]. The 2-skeleton of the complex defines the spin foam. This is made by faces f, edges e, and vertices v; dual, respectively, to triangles, tetrahedra, and 4-simplices of Δ. The sum in (1) is over an assignment of an *integer* spin l_f, or an irreducible representation (irrep) of $SO(3)$, to each face f, and an intertwiner i_e to each edge e. More precisely, the i_e sum is over a basis in the linear intertwiner space at each edge. Recall that an intertwiner is an element of the $SO(3)$ invariant subspace of the tensor product of the Hilbert spaces carrying the irreps associated to the four f's adjacent to an e. We use the usual basis given by the spin of the virtual link, under a fixed pairing of the four faces. $\dim j = 2j + 1$ is the dimension of the spin-j irrep, for $j \in \mathbb{Z}/2$. $15j_{SO(4)}$ is the Wigner $15j$ symbol of the group $SO(4)$. It is a function of 15 $SO(4)$ irreps. An $SO(4)$ irrep can be written as a pair of $SU(2)$ irreps, in the form (j^+, j^-), and the $SO(4)$ $15j$ is the product of two Wigner $SU(2)$ $15j$'s

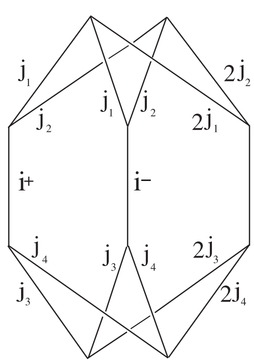
$$15j_{SO(4)}(j_f^+, j_f^-; i_e^+, i_e^-) = 15j(j_f^+, i_e^+)15j(j_f^-, i_e^-). \quad (3)$$

The last object to define, and our key ingredient, is the linear map f in the first line of (2). This is from the space of the $SO(3)$ intertwiners between the irreps $l_1, \ldots, l_4$, to the space of the $SO(4)$ intertwiners between the irreps $(j_1, j_1), \ldots, (j_4, j_4)$, where (and from now on)

$$j_f \equiv l_f/2. \quad (4)$$

The second line of (2) expresses this map in terms of its linear coefficients $f|i\rangle = \sum_{i^+, i^-} f^i_{i^+ i^-}|i^+, i^-\rangle$. These are defined as the evaluation of the spin network

$$f^i_{i^+ i^-} = \quad (5)$$



on the trivial connection. The amplitude can be equivalently written in the form

$$A(l_f, i_e) = \int_{SU(2)^5} dV_e \langle \bigotimes_{ee'} D^{l_f/2}(V_e) \otimes D^{l_f/2}(V_{e'}^{-1}), \bigotimes_e i_e \rangle. \quad (6)$$

Index contraction is dictated by the 4-simplex graph and the l_f intertwiner indices are contracted with the $\frac{l_f}{2} \otimes \frac{l_f}{2}$ representation indices of the matrices D. This concludes the definition of the model (for the general formalism, and details see [2]). We now comment on its features.

First, the boundary states of the theory are spanned by 4-valent graphs colored with $SO(3)$ spins and intertwiners. Second, the model is a simple modification of the BC model as follows. The BC model is given by

$$Z_{BC} = \sum_{j_f} \prod_f (\dim j_f)^2 \prod_v A_{BC}(j_f), \quad (7)$$

where here the sum is over half-integer spins and

$$A_{BC}(j_f) = 15j_{SO(4)}(j_f, j_f; i_{BC}). \quad (8)$$

The difference between the two theories is in the intertwiner state space. The common unconstrained intertwiner space is the $SO(4)$ intertwiner space between four simple representations $H_e = \text{Inv}(H_{(j_1,j_1)} \otimes \ldots \otimes H_{(j_4,j_4)})$. The BC theory constrains each intertwiner to be the unique "BC intertwiner" $|i_{BC}\rangle \equiv \sum_j (2j+1)|j, j\rangle$. The BC theory therefore constrains entirely the intertwiner degrees of freedom. In the model (1), instead, these remain free. More precisely, the states $f|i\rangle$ span a subspace K_e of H_e. The step from the single intertwiner i_{BC} to the space K_e is the essential modification made with respect to the BC model. Why this step?

The reduction of the intertwiner space to the sole i_{BC} vector is commonly motivated by the imposition of the off-diagonal simplicity constraints. For each couple of faces f, f' adjacent to e, consider the pseudoscalar $SO(4)$ Casimir operator

$$\hat{C}_{ff'} = \epsilon_{IJKL} \hat{J}_f^{IJ} \hat{J}_{f'}^{KL} \quad (9)$$

on the representation $(H_{(j_f,j_f)} \otimes H_{(j_{f'},j_{f'})})$. ($\epsilon_{IJKL}$ is the fully antisymmetric object and summation over repeated indices is understood.) Here $f \neq f'$ and $\hat{J}_f^{IJ}$ with $I, J = 1, \ldots, 4$ are the generators of $SO(4)$ in $H_{(j_f,j_f)}$. These are the quantum operators corresponding to the classical bivector J_f^{IJ} associated to the face f. $C_{ff'} = \epsilon_{IJKL} J_f^{IJ} J_{f'}^{KL}$ vanishes in the classical theory because the bivectors of the faces in a single tetrahedron span a 3D space and therefore their external products are zero. These are the off-diagonal simplicity constraints. (The diagonal ones $\hat{C}_{ff} = 0$ constrains the irrep of each f to be simple.) In BC theory, the constraints $\hat{C}_{ff'} = 0$ are imposed *strongly* on H_e, and the only solution is i_{BC} [19]. But these constraints do not commute with one another, and are therefore second class. Imposing such constraints strongly is a well-known way of erroneously killing physical degrees of freedom.

An alternative way to write the classical off-diagonal simplicity constraints is the following. As noted, these constraints impose the faces of the tetrahedron to lie on a common 3D subspace of 4D spacetime. If and only if they are satisfied, there is a direction n^I orthogonal to all the faces: the direction normal to the tetrahedron. The J_f have vanishing components in this direction. Choose coordinates in which $n^I = (0, 0, 0, 1)$ and let i, j be indices that run over the first 3 coordinates only. Then we have $2C_4 \equiv \frac{1}{2}J_f^{IJ}J_f^{IJ} = \frac{1}{2}J_f^{ij}J_f^{ij} \equiv C_3$. The off-diagonal simplicity constraints can therefore be written as the requirement that there is a common direction n such that

$$2C_4 - C_3 = 0 \tag{10}$$

for all the faces of the tetrahedron. Can this constraint be imposed quantum mechanically on H_e?

In the quantum context, $\hat{C}_4 \equiv \frac{1}{4}\hat{J}_f^{IJ}\hat{J}_f^{IJ}$ is the quadratic Casimir operator of $SO(4)$, with eigenvalues $j^+(j^+ + 1)\hbar^2 + j^-(j^- + 1)\hbar^2$, while $\hat{C}_3 \equiv \frac{1}{2}\hat{J}_f^{ij}\hat{J}_f^{ij}$ is the quadratic Casimir operator of the $SO(3)$ subgroup of $SO(4)$ that leaves n^I invariant, with eigenvalues $j(j+1)\hbar^2$, where we have momentarily restored $\hbar \neq 1$ units for clarity. A simple $SO(4)$ irrep (j, j) transforms under the $SO(3)$ subgroup in the representation $j \otimes j = 0 \oplus \ldots \oplus 2j$. The $2j$ component, namely, the highest $SO(3)$ irreducible, is characterized by the relation

$$\sqrt{2\hat{C}_4 + \hbar^2} - \sqrt{\hat{C}_3 + \hbar^2/4} - \hbar/2 = 0. \tag{11}$$

But this relation reduces precisely to (10) when $\hbar \to 0$, and therefore can be considered as a possible quantum version of the classical constraint (10). Imposing the constraints on each face thus selects from $(H_{(j_{f_1}, j_{f_1})} \otimes \ldots \otimes H_{(j_{f_4}, j_{f_4})})$ the space formed by the tensor product of the highest $SO(3)$ irreducibles. This depends on *which* $SO(3)$ subgroup is chosen, but if we project to the $SO(4)$-invariant space, the dependence drops out because all $SO(3)$ subgroups in $SO(4)$ are conjugate. A direct calculation shows that what we obtain is precisely the intertwiner space K_e defined above. Finally, it is easy to check that the off-diagonal simplicity constraints are all weakly zero in this space: they are antisymmetric in the i^+, i^- indices, while the states $f|i\rangle$ are symmetric.

We close by sketching the derivation of the model as a quantization of a discretization of GR (see [20]). Fix an oriented triangulation Δ and restrict the metric to be a Regge one on Δ: flat within each 4-simplex, with curvature on the triangles. To describe it, choose a cotetrad one-form $e^I(t)$ for each tetrahedron t (notice the change of notation $e \to t$), and also one $e^I(v)$ for each simplex. The two are related by an $SO(4)$ element $V_{vt} \equiv V_{tv}^{-1}$. For each face in each tetrahedron, define $B_f(t) = \int_f \star[e(t) \wedge e(t)]$, where the star is Hodge duality in R^4. $B_f(t)$ and $B_f(t')$ are related by $B_f(t)U_{tt'} = U_{tt'}B_f(t')$, where $U_{tt'} = V_{tv}V_{vt''}\ldots V_{v^n t'}$ is the product of the group elements around the oriented perimeter of f, from t to t'. The bulk action can be written as

$$S_{\text{bulk}}[e] = \frac{1}{2}\sum_f \text{Tr}[B_f(t)U_f(t)], \tag{12}$$

where $U_f(t)$ is the product of the group elements $V_{tv}V_{vt'}$ around the full perimeter of f. The boundary terms of the action can be written as

$$S_{\text{boundary}}[e] = \frac{1}{2}\sum_f \text{Tr}[B_f(t)U_{tt'}], \tag{13}$$

where $U_{tt'}$ is the product of the group elements of the part of the perimeter which is not in the boundary. We take $B_f(t)$ and V_{tv} as basic variables, and take into account the constraints on B_f. These are the closure constraint

$$\sum_{f\in t} B_f(t) = 0 \tag{14}$$

and the simplicity constraints (9), for all f, f' (possibly equal) in t. (The constraints relating triangles that meet only at one point, which appear in other formulations, are automatically solved by the choice of variables.)

On the boundary, the boundary coordinates are the $B_f(t)$ for the boundary triangles f. Each f has only two adjacent tetrahedra t, t' on the boundary. The conjugate momentum [as can be seen from (13)] is a group element for each f. Therefore the canonical boundary variables are the same as those of $SO(4)$ lattice gauge theory. We can thus choose the Hilbert space of $SO(4)$ lattice gauge theory as our unconstrained Hilbert space. This can be represented as the L^2 space on the product of one $SO(4)$ per triangle. However, in order to match with the solution of the constraints considered, we interpret the $SO(4)$ generators J_f (the right invariant vector fields) that are defined on this space as the quantum operators corresponding to the *dual* $\star B_f$ of the variable B_f, namely, to the geometrical bivectors $\int_f e \wedge e$ associated to the triangles [21]. The constraint (14) gives gauge-invariance at each tetrahedron, and reduces the space of states to the space of the $SO(4)$ spin networks on the graph dual to the boundary triangulation. The simplicity constraints (9), as above, reduce each $SO(4)$ link representation to a simple one, and the intertwiner spaces to K_e. The resulting space of states is not only mathematically isomorphic to the corresponding one of $SO(3)$ LQG, (the two spaces are spanned by spin-networks labeled by the same spins and intertwiners) but it can also be physically identified with it, because we have an explicit identification of eigenstates of quantum operators with the same classical analogs (such as the areas of faces).

Finally, coming to the dynamics, we can evaluate the amplitude of a single 4-simplex v, in order to derive the form of the vertex amplitude. Fixing the ten $B_{tt'} \equiv B_f(t)$ variables on the boundary, this can be formally written

$$A[B_{tt'}] = \int dV_{vt} e^{i\sum \text{Tr}[B_{tt'}V_{tv}V_{vt'}]}. \tag{15}$$

Transforming to the conjugate variables gives

$$A[U_{tt'}] = \int dB_{tt'} e^{-i\sum \mathrm{Tr}[B_{tt'}U_{tt'}]} A[B_{tt'}] = \int dV_{vt} \prod_{tt'} \delta(U_{tt'} V_{t'v} V_{vt}). \tag{16}$$

This is the amplitude. We transform back to the spin-network basis, using the $SO(4)$ spin-network functions $\Psi_{j^{\pm}_{tt'}, i^{\pm}_t}(U_{tt'})$

$$A[j^{\pm}_{tt'}, i^{\pm}_t] = \int dU_{tt'} \Psi_{j^{\pm}_{tt'}, i^{\pm}_t}(U_{tt'}) A[U_{tt'}] = \int dV_{vt} \Psi_{j^{\pm}_{tt'}, i^{\pm}_t}(V_{tv} V_{vt'}) = 15 j_{SO(4)}(j^{+}_{tt'}, j^{-}_{tt'}; i^{+}_t, i^{-}_t). \tag{17}$$

Combining this $15 j_{SO(4)}$ amplitude with the constraints discussed, gives the model (1) and (2).

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