Non-Markovian Effects on the Dynamics of Entanglement

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A procedure that allows us to obtain the dynamics of N independent bodies each locally interacting with its own reservoir is presented. It relies on the knowledge of single-body dynamics and it is valid for any form of environment noise. It is then applied to the study of non-Markovian dynamics of two independent qubits, each locally interacting with a zero-temperature reservoir. It is shown that, although no interaction is present or mediated between the qubits, there is a revival of their entanglement, after a finite period of time of its complete disappearance.

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Entanglement is relevant to different fundamental [1,2] aspects of quantum theory and practical aspects of quantum-information processing [3]. Recently much interest has arisen in the evolution of the joint entanglement of a pair of qubits exposed to local noisy environments. The reason is related to the discovery by Yu and Eberly [4] that for this system, rather surprisingly, the Markovian dynamics of the joint qubits entanglement and single qubit decoherence may be rather different. The aspect that has mostly drawn attention is the possibility of a complete disappearance of entanglement at finite times. The occurrence of this phenomenon, termed "entanglement sudden death" (ESD), has been shown in a quantum optics experiment [5]. The intrinsic interest and potential importance of ESD, for example, in the application range of quantum error correction methods, has led to a flow of analysis that studies its appearance under different circumstances [6–13].

Disentanglement is related to the birth of body-environment correlations. It is therefore of interest to investigate the role played on its evolution by non-Markovian effects. In fact, although Markovian dynamics includes a level of backreaction, it neglects the entanglement that arises between bodies and bath modes during the evolution. Although some work has treated of this aspect [14–18], it should be considered an attractive theoretical challenge to extend the results obtained under various conditions in the Markovian regime to the non-Markovian case [19]. The aim of this Letter is to address this point first by adopting a procedure to obtain the dynamics of N independent bodies, locally interacting with reservoirs and without restriction on the nature of environmental noise, if the single-body dynamics is known. We shall then use this approach to explicitly investigate the entanglement dynamics of two qubits locally interacting with a zero-temperature non-Markovian environment.

To describe the method, we consider a system formed by two noninteracting parts $\tilde{A}$, $\tilde{B}$, each part consisting of a qubit $S = A$, B locally interacting, respectively, with a reservoir $R_S = R_A$, R_B. Each qubit and the corresponding reservoir are initially considered independent. For each part, the reduced density matrix evolution for the single qubit $S = A$, B is given by

$$\hat{\rho}^{S}(t) = \mathrm{Tr}_{R_S}\{\hat{U}^{\tilde{S}}(t)\hat{\rho}^{S}(0) \otimes \hat{\rho}^{R_S}(0)\hat{U}^{\tilde{S}\dagger}(t)\}, \tag{1}$$

where the trace is over the reservoir R_S degrees of freedom and $\hat{U}^{\tilde{S}}(t)$ is the time evolution operator for the part $\tilde{S}$. In terms of the Kraus operators $W^{S}_{\alpha,\beta}(t)$, the former equation becomes [20]

$$\hat{\rho}^{S}(t) = \sum_{\alpha\beta} W^{S}_{\alpha\beta}(t)\hat{\rho}^{S}(0)W^{\dagger S}_{\alpha\beta}(t). \tag{2}$$

The assumption of independent parts implies that the time evolution operator $\hat{U}^{\tilde{T}}(t)$ of the complete system $\tilde{T} = \tilde{A} + \tilde{B}$ factorizes as $\hat{U}^{\tilde{T}}(t) = \hat{U}^{\tilde{A}}(t) \otimes \hat{U}^{\tilde{B}}(t)$. It follows that the Kraus representation of the reduced density matrix for the two-qubit system $T = A + B$ reads like

$$\hat{\rho}^{T}(t) = \sum_{\alpha\beta}\sum_{\gamma\delta} W^{A}_{\alpha\beta}(t)W^{B}_{\gamma\delta}(t)\hat{\rho}^{T}(0)W^{\dagger A}_{\alpha\beta}(t)W^{\dagger B}_{\gamma\delta}(t). \tag{3}$$

Given the basis $\{|0\rangle, |1\rangle\}$ for each qubit, inserting unity operators $I = |0\rangle\langle 0| + |1\rangle\langle 1|$ between Kraus operators and density matrices in Eq. (2), it follows that the dynamics of each qubit has the form

$$\begin{aligned} \rho^{A}_{ii'}(t) &= \sum_{ll'} A^{ll'}_{ii'}(t)\rho^{A}_{ll'}(0), \\ \rho^{B}_{jj'}(t) &= \sum_{mm'} B^{mm'}_{jj'}(t)\rho^{B}_{mm'}(0). \end{aligned} \tag{4}$$

Adopting the same procedure for $\hat{\rho}^{T}$ in the form of Eq. (3), the dynamics of the two-qubit system is given by

$$\rho^{T}_{ii',jj'}(t) = \sum_{ll',mm'} A^{ll'}_{ii'}(t)B^{mm'}_{jj'}(t)\rho^{T}_{ll',mm'}(0), \tag{5}$$

where the indexes i, j, l, $m = 0, 1$. Equations (4) and (5) clearly show that the dynamics of two-qubit density matrix elements can be obtained by knowing that of the single qubit. The validity of the above procedure can be straightforwardly extended to any multipartite and multilevel sys-

tem (qudit), provided that the different parts, "qudit + reservoir," are independent.

We now apply the results obtained above to study non-Markovian effects on the entanglement dynamics of two qubits, each interacting only and independently with its local environment. To this aim, we shall consider the single "qubit + reservoir" Hamiltonian given by

$$H = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k b_k^\dagger b_k + (\sigma_+ B + \sigma_- B^\dagger), \quad (6)$$

with $B = \sum_k g_k b_k$, where ω_0 is the transition frequency of the two-level system (qubit) and $\sigma_\pm$ are the system raising and lowering operators while the index k labels the field modes of the reservoir with frequencies ω_k, $b_k^\dagger$ and b_k are the modes' creation and annihilation operators, and g_k are the coupling constants. The Hamiltonian of Eq. (6) may describe various systems as, for example, a qubit formed by an exciton in a potential well environment. However, to fix our ideas we shall take it to represent a qubit formed by the excited and ground electronic state of a two-level atom interacting with the reservoir formed by the quantized modes of a high-Q cavity. At zero-temperature, this Hamiltonian represents one of the few open quantum systems amenable to an exact solution [21]. The dynamics of qubit S is known to be described by the reduced density matrix [22,23]

$$\hat{\rho}^S(t) = \begin{pmatrix} \rho_{11}^S(0) P_t & \rho_{10}^S(0)\sqrt{P_t} \\ \rho_{01}^S(0)\sqrt{P_t} & \rho_{00}^S(0) + \rho_{11}^S(0)(1 - P_t) \end{pmatrix}, \quad (7)$$

where the function P_t obeys the differential equation

$$\dot{P}_t = -\int_0^t dt_1 f(t - t_1) P_{t_1}, \quad (8)$$

and the correlation function $f(t - t_1)$ is related to the spectral density $J(\omega)$ of the reservoir by

$$f(t - t_1) = \int d\omega J(\omega) \exp[i(\omega_0 - \omega)(t - t_1)]. \quad (9)$$

The exact form of P_t thus depends on the particular choice for the spectral density of the reservoir. Because the Hamiltonian of Eq. (6) represents a model for the damping of an atom in a cavity, we consider then the case of a single excitation in the atom-cavity system. For the effective spectral density $J(\omega)$, we take the spectral distribution of an electromagnetic field inside an imperfect cavity supporting the mode ω_0, resulting from the combination of the reservoir spectrum and the system-reservoir coupling with γ_0 related to the microscopic system-reservoir coupling constant, of the form [22]

$$J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{(\omega_0 - \omega)^2 + \lambda^2}. \quad (10)$$

The correlation function (9) corresponding to the spectral density of Eq. (10) has an exponential form with λ as the decay rate. The parameter λ, defining the spectral width of the coupling, is then connected to the reservoir correlation time τ_B by the relation $\tau_B \approx \lambda^{-1}$. On the other hand, the parameter γ_0 can be shown to be related to the decay of the excited state of the atom in the Markovian limit of flat spectrum. The relaxation time scale τ_R over which the state of the system changes is then related to γ_0 by $\tau_R \approx \gamma_0^{-1}$.

Using the spectral density of Eq. (10) in the correlation function of Eq. (9), in the subsequent analysis of the function P_t of Eq. (8), typically a weak and a strong coupling regime can be distinguished. For a weak regime we mean the case $\gamma_0 < \lambda/2$, that is, $\tau_R > 2\tau_B$. In this regime the relaxation time is greater than the reservoir correlation time and the behavior of P_t is essentially a Markovian exponential decay controlled by γ_0. In the strong coupling regime, that is, for $\gamma_0 > \lambda/2$, or $\tau_R < 2\tau_B$, the reservoir correlation time is greater than or of the same order as the relaxation time and non-Markovian effects become relevant. For this reason we are interested in this regime and we shall mainly limit our considerations to this case. Within this regime, the function P_t assumes the form [22,23]

$$P_t = e^{-\lambda t}\left[\cos\left(\frac{dt}{2}\right) + \frac{\lambda}{d}\sin\left(\frac{dt}{2}\right)\right]^2, \quad (11)$$

where $d = \sqrt{2\gamma_0\lambda - \lambda^2}$. P_t presents oscillations describing the fact that the decay of the atom excited state is induced by the coherent processes between the system and the reservoir. In particular, the function P_t has discrete zeros at $t_n = 2[n\pi - \arctan(d/\lambda)]/d$, with n integer. We note that the solution in the weak coupling regime can be obtained by the former one simply substituting the harmonic functions with the corresponding hyperbolic ones and d with id.

Now we are ready to use, following the procedure described before, the evolution of the reduced density matrix elements for the single qubit to construct the reduced density matrix $\hat{\rho}^T$ for the two-qubit system. In the standard product basis $\mathcal{B} = \{|1\rangle \equiv |11\rangle, |2\rangle \equiv |10\rangle, |3\rangle \equiv |01\rangle, |4\rangle \equiv |00\rangle\}$, using Eqs. (4), (5), and (7), we obtain the diagonal elements

$$\begin{aligned} \rho_{11}^T(t) &= \rho_{11}^T(0) P_t^2, \\ \rho_{22}^T(t) &= \rho_{22}^T(0) P_t + \rho_{11}^T(0) P_t(1 - P_t), \\ \rho_{33}^T(t) &= \rho_{33}^T(0) P_t + \rho_{11}^T(0) P_t(1 - P_t), \\ \rho_{44}^T(t) &= 1 - [\rho_{11}^T(t) + \rho_{22}^T(t) + \rho_{33}^T(t)], \end{aligned} \quad (12)$$

and the nondiagonal elements

$$\begin{aligned} \rho_{12}^T(t) &= \rho_{12}^T(0) P_t^{3/2}, \qquad \rho_{13}^T(t) = \rho_{13}^T(0) P_t^{3/2}, \\ \rho_{14}^T(t) &= \rho_{14}^T(0) P_t, \qquad \rho_{23}^T(t) = \rho_{23}^T(0) P_t, \\ \rho_{24}^T(t) &= \sqrt{P_t}[\rho_{24}^T(0) + \rho_{13}^T(0)(1 - P_t)], \\ \rho_{34}^T(t) &= \sqrt{P_t}[\rho_{34}^T(0) + \rho_{12}^T(0)(1 - P_t)], \end{aligned} \quad (13)$$

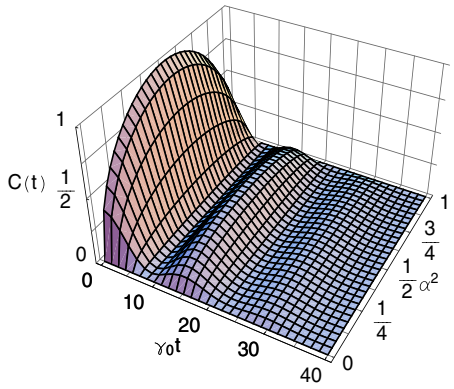

FIG. 1 (color online). Concurrence for the initial state $\alpha|01\rangle + \beta|10\rangle$ as a function of the dimensionless quantity $\gamma_0 t$ and α^2, in a realistic CQED condition ($\lambda = 0.1\gamma_0$).

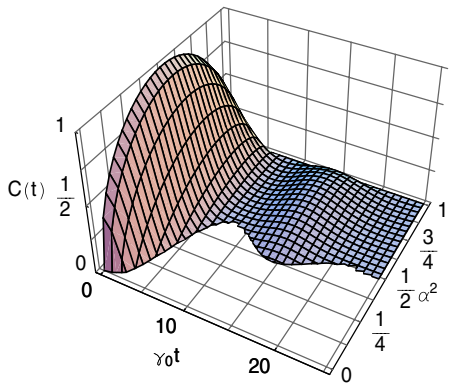

FIG. 2 (color online). Concurrence for the initial state $\alpha|00\rangle + \beta|11\rangle$ as a function of the dimensionless quantity $\gamma_0 t$ and α^2, in a realistic CQED condition ($\lambda = 0.1\gamma_0$).

and $\rho^T_{ij}(t) = \rho^{T*}_{ji}(t)$, where $\rho^T(t)$ is a Hermitian matrix. In order to follow the entanglement dynamics of the bipartite system, we use Wootters concurrence [24]. This is obtained from the density matrix $\hat{\rho}^T$ for qubits A and B as $C_{\hat{\rho}^T}(t) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$, where the quantities λ_i are the eigenvalues of the matrix ζ,

$$\zeta = \hat{\rho}^T(\sigma^A_y \otimes \sigma^B_y)\hat{\rho}^{T*}(\sigma^A_y \otimes \sigma^B_y), \tag{14}$$

arranged in decreasing order. Here $\hat{\rho}^{T*}$ denotes the complex conjugation of $\hat{\rho}^T$ in the standard basis, and σ_y is the well-known Pauli matrix expressed in the same basis. The concurrence varies from $C = 0$ for a disentangled state to $C = 1$ for a maximally entangled state.

The form of Eqs. (12) and (13) is such that one can study the entanglement evolution for any initial state. We shall, however, restrict our analysis to the initial entangled states

$$|\Phi\rangle = \alpha|01\rangle + \beta|10\rangle, \qquad |\Psi\rangle = \alpha|00\rangle + \beta|11\rangle, \tag{15}$$

where α is real, $\beta = |\beta|e^{i\delta}$, and $\alpha^2 + |\beta|^2 = 1$. For these two entangled states, the initial total density matrix has an X structure [19], which is maintained during the evolution, as is easily seen from Eqs. (12) and (13). In particular, the concurrence, for the initial states of Eq. (15), is given by

$$\begin{aligned} C_\Phi(t) &= \max\{0, 2|\rho^T_{23}(t)| - 2\sqrt{\rho^T_{11}(t)\rho^T_{44}(t)}\}, \\ C_\Psi(t) &= \max\{0, 2|\rho^T_{14}(t)| - 2\sqrt{\rho^T_{22}(t)\rho^T_{33}(t)}\}, \end{aligned} \tag{16}$$

and using Eqs. (12) and (13) we obtain

$$\begin{aligned} C_\Phi(t) &= \max\{0, 2\sqrt{1-\alpha^2}P_t\alpha\}, \\ C_\Psi(t) &= \max\{0, 2\sqrt{1-\alpha^2}P_t[\alpha - \sqrt{1-\alpha^2}(1-P_t)]\}. \end{aligned} \tag{17}$$

The time behavior of the concurrences C_Φ and C_Ψ as a function of α^2 and the dimensionless quantity $\gamma_0 t$ are plotted for P_t given by Eq. (11) in Figs. 1 and 2 (for $\lambda/\gamma_0 = 0.1$). This is a condition that can be well within the current experimental capabilities. In fact, cavity quantum electrodynamics (CQED) experimental configurations have been realized using Rydberg atoms with lifetimes $T_{\rm at} \approx 30$ ms, inside Fabry-Perot cavities with quality factors $Q \approx 4.2 \times 10^{10}$ corresponding to cavity lifetimes $T_{\rm cav} \approx 130$ ms [25]; these values correspond to $2\lambda/\gamma_0 \approx 0.2$.

Figure 1 shows that, in the non-Markovian regime, the concurrence C_Φ periodically vanishes according to the zeros of the function P_t, with a damping of its revival amplitude. This behavior is evidently different from Markovian, where in contrast C_Φ decays exponentially and vanishes only asymptotically [7]. The Markovian decay rate is, however, larger than the initial non-Markovian one, as shown in Fig. 3 for the maximally entangled case $\alpha^2 = 1/2$.

Figure 2 shows that the entanglement represented by C_Ψ has a similar behavior to C_Φ for $\alpha^2 \geq 1/2$. In contrast, for $\alpha^2 < 1/2$ two ranges of parameter may be distinguished. In one there is ESD because C_Ψ vanishes permanently after a finite time, similar to the Markovian case [4,7]. In the second, revival of entanglement appears after periods of times when disentanglement is complete. This behavior is

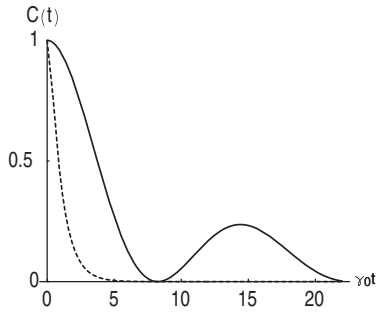

FIG. 3. Concurrence for the initial state $(|01\rangle + |10\rangle)/\sqrt{2}$ as a function of the dimensionless quantity $\gamma_0 t$ in non-Markovian regime (solid line; $\lambda = 0.1\gamma_0$) and Markovian regime (dashed line; $\lambda = 5\gamma_0$).

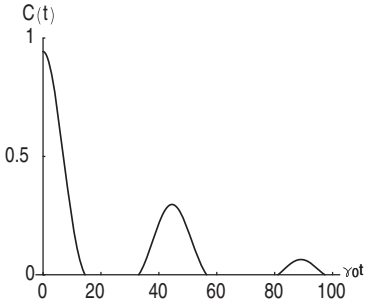

FIG. 4. Concurrence for the initial state $\alpha|00\rangle + \beta|11\rangle$ as a function of the dimensionless quantity $\gamma_0 t$ for $\alpha^2 = 1/3$, in strong coupling ($\lambda = 0.01\gamma_0$).

more evident in the plot of Fig. 4 obtained under stronger non-Markovian conditions. This revival phenomenon is induced by the memory effects of the reservoirs, which allows the two-qubit entanglement to reappear after a dark period of time, during which the concurrence is zero. This phenomenon of revival of entanglement after finite periods of ''entanglement death'' appears to be linked to the environment-single qubit non-Markovian dynamics. In this sense this result differs from the revival of entanglement previously obtained in the presence of interaction among qubits or because of their interaction with a common reservoir [10–12]. The physical conditions examined here are, moreover, more similar to those typically considered in quantum computation, where qubits are taken to be independent and where qubits interact with non-Markovian environments typical of solid state microdevices [26].

The above analysis can easily be extended to study entanglement dynamics starting from different initial conditions and to take into account finite temperature effects. In particular, starting from a Werner state [27] one gets an entanglement behavior structurally similar to that obtained in this Letter for the states of Eq. (15) with ESD and entanglement revival periods. The details of the evolution for this case and finite temperature effects for non-Markovian dynamics will be considered elsewhere.

In conclusion, we have presented a procedure that allows one in principle to obtain the dynamics of a system of N independent bodies, each locally interacting with an environment, as long as the single system dynamics is known. This procedure is valid for any form of single-body-environment interaction. It has been applied to the case of two qubits interacting with the environment where non-Markovian effects are present. In particular, the model described has been identified with a system made by two atoms each in a high-Q cavity. For this case the Hamiltonian dynamics of the single qubit can be solved exactly and no problems about map positivity arise. We have found that non-Markovian effects influence the entanglement dynamics and may give rise to a revival of entanglement even after complete disentanglement has been present for finite time periods. This effect, arising for completely independent systems, is only a consequence of the non-Markovian behavior of the single qubit-reservoir dynamics.

These results show that entanglement dynamics may present facets that one simply may not expect from single qubit dynamics and may also lead to the possibility of recovering the entanglement initially present.

[1] J. S. Bell, Physics **1**, 195 (1964).
[2] C. H. Bennett *et al.*, Phys. Rev. Lett. **70**, 1895 (1993).
[3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[4] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).
[5] M. P. Almeida *et al.*, Science **316**, 579 (2007).
[6] T. Yu and J. H. Eberly, Phys. Rev. Lett. **97**, 140403 (2006).
[7] M. F. Santos, P. Milman, L. Davidovich, and N. Zagury, Phys. Rev. A **73**, 040305(R) (2006).
[8] L. Díosi, Lect. Notes Phys. **622**, 157 (2003).
[9] A. R. R. Carvalho, F. Mintert, S. Palzer, and A. Buchleitner, Eur. Phys. J. D **41**, 425 (2007).
[10] Z. Ficek and R. Tanás, Phys. Rev. A **74**, 024304 (2006).
[11] Z. Sun, X. Wang, and C. P. Sun, Phys. Rev. A **75**, 062312 (2007).
[12] Y. Hamdouni, M. Fannes, and F. Petruccione, Phys. Rev. B **73**, 245323 (2006).
[13] M. A. Cirone, G. Compagno, G. M. Palma, R. Passante, and F. Persico, Europhys. Lett. **78**, 30003 (2007).
[14] M. Ban, S. Kitajima, and F. Shibata, J. Phys. A **38**, 7161 (2005).
[15] M. Ban, J. Phys. A **39**, 1927 (2006).
[16] K.-L. Liu and H.-S. Goan, arXiv:0706.0996.
[17] M. Yonac, T. Yu, and J. H. Eberly, J. Phys. B **39**, S621 (2006).
[18] I. Glendinning, M. Jacob, and M. Nölle, arXiv:quant-ph/0604030.
[19] T. Yu and J. H. Eberly, Quantum Inf. Comput. **7**, 459 (2007).
[20] K. Kraus, *States, Effect, and Operations: Fundamental Notions in Quantum Theory* (Springer-Verlag, Berlin, 1983).
[21] B. M. Garraway, Phys. Rev. A **55**, 2290 (1997).
[22] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, New York, 2002).
[23] S. Maniscalco and F. Petruccione, Phys. Rev. A **73**, 012111 (2006).
[24] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[25] S. Kuhr *et al.*, Appl. Phys. Lett. **90**, 164101 (2007).
[26] I. de Vega, D. Alonso, and P. Gaspard, Phys. Rev. A **71**, 023812 (2005).
[27] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).