

## Strong Monogamy of Bipartite and Genuine Multipartite Entanglement: The Gaussian Case

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We demonstrate the existence of general constraints on distributed quantum correlations, which impose a trade-off on bipartite and multipartite entanglement at once. For all  $N$ -mode Gaussian states under permutation invariance, we establish exactly a monogamy inequality, stronger than the traditional one, that by recursion defines a proper measure of genuine  $N$ -partite entanglement. Strong monogamy holds as well for subsystems of arbitrary size, and the emerging multipartite entanglement measure is found to be scale invariant. We unveil its operational connection with the optimal fidelity of continuous variable teleportation networks.

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Understanding the structure of entanglement distributed among many parties is central to diverse aspects of quantum information theory [1] and its manifold applications in condensed matter physics [2]. A direct consequence of the no-cloning theorem [3] is what one might call the no-sharing theorem: maximal entanglement cannot be freely shared. Suppose Alice is maximally entangled to both Bob and Charlie. Then she could exploit both channels to teleport two perfect clones of an unknown state, violating the linearity of quantum mechanics. Nonmaximal entanglement, however, can be shared; but this distribution is constrained to monogamy inequalities [4], as originally discovered by Coffman, Kundu, and Wootters (CKW) [5]. In the most general known form, monogamy imposes the following trade-off on bipartite entanglement distributed among  $N$  parties  $p_1 \dots p_N$ ,

$$E^{p_1|(p_2\dots p_N)} \geq \sum_{j \neq 1}^N E^{p_1|p_j}, \quad (1)$$

where  $E$  is a proper measure of bipartite entanglement. The left-hand side of inequality (1) is the bipartite entanglement between a probe subsystem  $p_1$  and the remaining subsystems taken as a whole. The right-hand side is the total bipartite entanglement between  $p_1$  and each of the other subsystems  $p_{j \neq 1}$  in the respective reduced states. Their difference represents the residual multipartite entanglement, not encoded in pairwise form. For  $N = 3$ , the residual entanglement quantifies the genuine tripartite entanglement shared by the three subsystems [5–7]. In Eq. (1), is known to hold for spin chains ( $N$ -qubit systems) [5,8] and harmonic lattices ( $N$ -mode Gaussian states) [6,9], with important consequences for the structure of correlations of those many-body systems [2,10,11].

*Is multipartite entanglement monogamous?*—In the present Letter, we wish to investigate if and to what extent sharing constraints can be established not only for bipartite but also for multipartite entanglement. In other words, is

there a suitable generalization of the tripartite analysis to arbitrary  $N$ , such that a genuine  $N$ -partite entanglement quantifier is naturally derived from a stronger monogamy inequality? This question is motivated by the fact that the residual multipartite entanglement emerging from the “weak” inequality (1) includes all manifestations of  $K$ -partite entanglement, involving  $K$  subsystems at a time, with  $2 < K \leq N$ . Hence, it severely overestimates the genuine  $N$ -partite entanglement for  $N > 3$ . It then seems compelling to further decompose the residual entanglement. How can one subsystem be entangled with the group of the remaining  $N - 1$  subsystems? Quite naturally, it can share individual pairwise entanglement with each of them, and/or genuine three-partite entanglement involving any two of them (and so on), and/or it can be genuinely  $N$ -party entangled with all of them. We then advance the hypothesis that these contributions are well-defined and mutually independent and check *a posteriori* that this is indeed true. Namely, we wish to verify whether in multipartite states, entanglement is strongly monogamous in the sense that the following equality holds:

$$E^{p_1|(p_2\dots p_N)} = \sum_{j=2}^N E^{p_1|p_j} + \sum_{k>j=2}^N E^{p_1|p_j|p_k} + \dots + E^{p_1|p_2|\dots|p_N}, \quad (2)$$

where  $E^{p_1|p_j}$  is the bipartite entanglement between parties 1 and  $j$ , while all the other terms are multipartite entanglements involving three or more parties. The last contribution in Eq. (2) is defined implicitly by difference and represents the residual  $N$ -partite entanglement. It depends in general on the probe system  $p_1$  with respect to which entanglement is decomposed. One then needs to define the genuine  $N$ -partite entanglement as the minimum over all the permutations of the subsystem indexes,  $E^{p_1|p_2|\dots|p_N} \equiv \min_{\{i_1, \dots, i_N\}} E^{p_1|i_2|\dots|i_N}$ . All the multipartite entanglement contributions appearing in Eq. (2) (except the last one)

involve  $K$  parties, with  $K < N$ , and are of the form  $E^{p_1|p_2|\dots|p_K}$ . Each of these terms is defined by Eq. (2) when the left-hand side is the  $1|(K-1)$  bipartite entanglement  $E^{p_1|(p_2\dots p_K)}$ . The  $N$ -partite entanglement is thus, at least in principle, computable in terms of the known  $K$ -partite contributions, once Eq. (2) is applied recursively for all  $K = 2, \dots, N-1$ . To assess  $E^{p_1|p_2|\dots|p_N}$  as a proper quantifier of  $N$ -partite entanglement, one needs first to show its nonnegativity on all quantum states. This property in turn implies that Eq. (2) can be recast as a sharper monogamy inequality, constraining both bipartite and genuine  $K$ -partite ( $K \leq N$ ) entanglements in  $N$ -partite systems. Such a constraint on distributed entanglement is then a strong generalization of the original CKW inequality [5], implying it, and reducing to it in the special case  $N = 3$ .

*Entanglement distribution under permutation invariance.*—A prominent role in multiparty quantum information science is played by permutation-invariant (“fully symmetric”) quantum states. In practical applications, symmetric states are the privileged resources for most communication protocols [12], while from a theoretical perspective, they are basic test grounds for investigating structural aspects of multipartite entanglement both for continuous [13] and discrete [14] variable systems. For our purposes, specializing to such symmetric states yields a significant simplification in Eq. (2), as the multipartite entanglements will only depend on the total number of parties involved in each contribution. Equation (2) thus reduces to  $E^{p_1|(p_2\dots p_N)} = \sum_{K=1}^{N-1} \binom{N-1}{K} E^{p_1|\dots|p_{K+1}}$ , as pictorially depicted in Fig. 1. In this particular instance, by reexpressing the  $K$ -partite contributions in terms of  $K'$ -partite ones ( $\forall K' < K$ ), the recursion can be completely resolved, and the proposed measure of genuine multipartite entanglement in  $N$ -party permutation-invariant quantum states takes the following closed form in terms of a finite sum, with alternating signs, of bipartite entanglements:

$$E^{p_1|p_2|\dots|p_N} = \sum_{K=1}^{N-1} \binom{N-1}{K} (-1)^{K+N+1} E^{p_1|(p_2\dots p_{K+1})}. \quad (3)$$

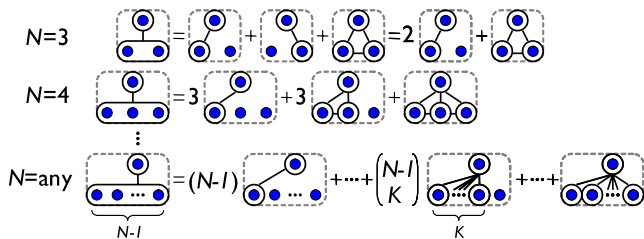


FIG. 1 (color online). The structure of multipartite entanglement in a permutation-invariant state of  $N$  parties. The bipartite  $1 \times (N-1)$  entanglement is decomposed into all the multipartite entanglements shared by the single parties. The rightmost graph in each row depicts the genuine  $N$ -partite quantum correlations.

All the considerations so far are not relying on any specific Hilbert space dimensionality.

*Strong monogamy of Gaussian entanglement.*—In the following, we demonstrate that entanglement indeed distributes according to the strong monogamy construction Eq. (2) in arbitrary (pure and mixed)  $N$ -mode Gaussian states on harmonic lattices, endowed with permutation invariance. We extract a proper, computable measure of genuine multipartite entanglement by explicitly evaluating Eq. (3), with  $E$  denoting in this case the Gaussian contangle [6]. Such  $N$ -partite entanglement turns out to be monotone in the optimal fidelity of  $N$ -party teleportation networks with symmetric Gaussian resources [15,16], thus acquiring an operational interpretation and a direct experimental accessibility.

Some preliminaries are in order. We consider a continuous variable (CV) system consisting of  $N$  canonical bosonic modes. Pure, fully symmetric (permutation-invariant),  $N$ -mode Gaussian states provide key resources for essentially all the so-far implemented multiparty CV quantum information protocols [17]. They can be experimentally prepared by sending a single-mode squeezed state with squeezing  $r_m$  in momentum and  $N-1$  single-mode squeezed states with squeezing  $r_p$  in position, through a network of  $N-1$  beam splitters with tuned transmittivities, as detailed in [15,16]. Up to local unitaries, such states are completely specified by the  $2N \times 2N$  covariance matrix (CM)  $\sigma^{(N)}$  of the second canonical moments, explicitly given in Ref. [13], which is parametrized by the average squeezing  $\bar{r} \equiv (r_m + r_p)/2$ . In general, the determinant of the reduced  $K$ -mode CM  $\sigma_K^{(N)}$  of a fully symmetric  $N$ -mode pure Gaussian state is given by  $\det \sigma_K^{(N)} = [2K^2 - 2NK + 2(N-K) \cosh(4\bar{r})K + N^2]/N^2$ . The determinant of the CM  $\sigma$  is related to the purity  $\mu$  of a Gaussian state by  $\mu = (\det \sigma)^{-1/2}$ . In order to evaluate Eq. (3), we have to compute all the bipartite  $1 \times K$  entanglements. In general, the bipartite entanglement between  $L$  modes and  $K$  modes of a fully symmetric Gaussian state can be concentrated by local unitary operations onto two modes only, a process known as unitary localizability [13]. This effective two-mode Gaussian state is a minimum-uncertainty mixed state, completely specified by its global (two-mode) and its two local (single-mode) purities, given by  $[\det \sigma_{L+K}^{(N)}]^{-1/2}$ ,  $[\det \sigma_L^{(N)}]^{-1/2}$ , and  $[\det \sigma_K^{(N)}]^{-1/2}$ , respectively. The entire family of Gaussian entanglement measures, including the Gaussian contangle  $G_\tau$  [6], can be computed in closed form for minimum-uncertainty mixed states [18]. Equipped with these results, we can analyze the general instance of a mixed,  $N$ -mode fully symmetric Gaussian state with CM  $\sigma_N^{(N+M)}$ , obtained from a pure  $(N+M)$ -mode one by tracing out  $M$  modes. The genuine  $N$ -partite entanglement shared by the  $N$  individual modes, according to Eq. (3), acquires the following explicit expression:

$$G_{\tau}^{\text{res}}(\sigma_N^{(N+M)}) = \sum_{j=0}^{N-2} \binom{N-1}{j} (-1)^j f_j^{\tau},$$

$$f_j^{\tau} \equiv \text{arcsinh}^2 \left[ \frac{2\sqrt{N-1-j} \sinh(2\bar{r})}{\sqrt{M+N}\sqrt{e^{4\bar{r}}(j+M)} + N-j} \right],$$
(4)

which depends only on the average squeezing  $\bar{r}$  and the number of modes  $N$  and  $M$ . Equation (4) provides a closed, analytical formula for the genuine  $N$ -partite Gaussian contangle  $G_{\tau}^{\text{res}}$  of permutation-invariant Gaussian states, as emerging from the assumed strong monogamy constraint. In Fig. 2, we plot Eq. (4) for pure states ( $M = 0$ ) of up to  $N = 10^3$  modes. It is apparent that the following holds:

*Theorem*  $G_{\tau}^{\text{res}}(\sigma_N^{(N+M)}) \geq 0$ .

*Sketch of the proof.*—We consider general sums of the form  $F = \sum_{j=0}^{N-2} \binom{N-1}{j} (-1)^j f_j$  where  $f_j$  is a decreasing sequence of the integer  $j$  with  $f_{N-1} = 0$ . Positivity (for any  $N$ ) of such sums depends on the decay rate of  $f_j$  with  $j$ . We choose for comparison a sequence of the form  $\tilde{f}_j(a, b, c) = (N-1-j)/[(N-1)(c + bj^a)]$ . The corresponding alternating sum  $\tilde{F}(a, b, c)$  is positive for  $a \leq 1$ , and specifically  $\tilde{F}(1, b, c) = \Gamma(c/b)\Gamma(N-1)/[b\Gamma(N-1+c/b)]$  where  $\Gamma$  is the Euler function. Now,  $f_j^{\tau}$  in Eq. (4) is bounded both from above and from below by functions of the form  $\tilde{f}_j(1, b, c)$ , for any  $N, M, \bar{r}$  [19]. Hence,  $f_j^{\tau} = O(\tilde{f}_j|_{a=1})$ , which yields that the corresponding sum, Eq. (4), is nonnegative as well. ■

We have thus demonstrated that multipartite entanglement, once properly quantified, is strongly monogamous, in particular, in Gaussian states on permutation-invariant harmonic lattices. Similarly, one can show that  $G_{\tau}^{\text{res}}$  monotonically increases with the average squeezing  $\bar{r}$ , while it decreases with  $N$ , eventually becoming identically null in the field limit  $N \rightarrow \infty$ . For mixed states,  $G_{\tau}^{\text{res}}$  decreases with the number  $M$  of the traced-out modes, i.e., with the mixedness, as expected. The monotonically increasing dependence of the  $N$ -partite entanglement on the squeezing resource  $\bar{r}$ , directly yields it to be an entanglement monotone under Gaussian local operations and classical communications [6,18] which preserve the state symmetry

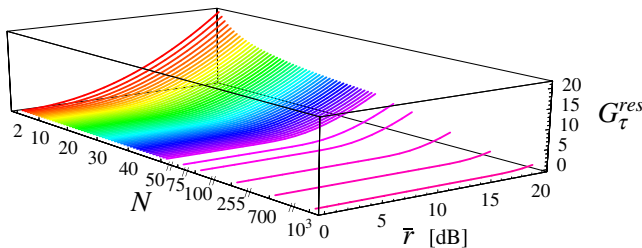


FIG. 2 (color online). Genuine multipartite entanglement  $G_{\tau}^{\text{res}}$  (in ebits) of pure permutation-invariant  $N$ -mode Gaussian states  $\sigma_N$ , plotted as a function of  $N$  and of the squeezing  $\bar{r}$  (in decibels).

(i.e., which produce the same local action on every single mode). The monotonically decreasing dependence on  $N$  can be understood as well, since, according to the strong monogamy decomposition, with increasing number of modes the residual nonpairwise entanglement can be encoded in so many different multipartite forms, that the genuine  $N$ -partite contribution is actually frustrated.

*Operational connection with teleportation networks.*—An interesting experimental setting can be considered, which provides an operational meaning to Eq. (4) as a *bona fide* measure of genuine  $N$ -partite entanglement. Permutation-invariant pure Gaussian states ( $M = 0$ ) can be successfully employed as shared resources to implement  $N$ -party teleportation networks, where two parties (Alice and Bob) perform CV teleportation of unknown coherent states, with the assistance of the other  $N-2$  cooperating parties [15]. The optimal fidelity  $\mathcal{F}_N^{\text{opt}}$  of the process, which quantifies operationally the shared  $N$ -partite entanglement, has been computed in Ref. [16]. In full qualitative and quantitative analogy with  $G_{\tau}^{\text{res}}$ ,  $\mathcal{F}_N^{\text{opt}}$  always lies above the classical threshold  $\mathcal{F}^{\text{cl}} \equiv 1/2$  (which quantifies the best possible transfer without using entanglement [15]), it is monotonically increasing with the squeezing  $\bar{r}$ , and monotonically decreasing with  $N$ . In fact, for any  $N$ , the genuine multipartite entanglement can be recast as a monotonic function of  $\mathcal{F}^{\text{opt}}$ , which is explicitly obtained by substituting  $\bar{r} = \frac{1}{4} \log\{1 + [N(2\mathcal{F}^{\text{opt}} - 1)]/[2(\mathcal{F}^{\text{opt}} - 1)^2]\}$  in Eq. (4) [16], with  $M = 0$ . Experimentally, this means that one does not need a full tomographic reconstruction of the  $N$ -mode states to measure  $N$ -partite entanglement: it can be indirectly quantified by the success of the teleportation protocol. Actually, it is enough to measure the quadrature squeezing in any single mode, and thus  $\bar{r}$ , to have a complete information on any form of multipartite entanglement of symmetric Gaussian states. From a broader perspective, the equivalence between operational entanglement quantifiers (optimal fidelity) and monogamy-based measures (residual contangle), entails that there is a unique form of genuine  $N$ -partite entanglement (for any  $N$ ) in symmetric Gaussian states, generalizing the results known for  $N = 2, 3$  [6,16,18].

*Monogamy beyond single modes and promiscuity.*—The standard monogamy inequalities established so far for spins and harmonic lattices focus on multipartitions where each subsystem consists only of one elementary unit (qubit or mode) [5,6,8,9]. We will now generalize the strong monogamy constraint to an arbitrary number of modes per subsystem. As the unitary localizability of symmetric Gaussian entanglement applies to general  $L \times K$  bipartitions [13], Eq. (3) can be evaluated explicitly for subsystems of arbitrary dimension. An important instance is when the subsystem permutation invariance is preserved: namely, when we consider a  $(nN)$ -mode fully symmetric Gaussian state, multipartitioned in  $N$  subsystems, each being a “molecule” made of  $n > 1$  modes. In this case,



one immediately sees that  $\det \sigma_{nK}^{(nN)}$  does not depend on the integer scale factor  $n$ . Therefore, Eq. (4) describes in general the molecular  $N$ -partite entanglement in a permutation-invariant  $(nN)$ -mode harmonic lattice, which is independent of the size  $n$  of the molecule:  $N$ -partite entanglement is, in this sense, scale invariant. This is relevant in view of practical exploitation of Gaussian resources for communication tasks [17]: adding redundancy, e.g., by doubling the size of the individual subsystems, yields no advantage for the multiparty-entangled resource. Importantly, the positivity of Eq. (4) directly entails that strong monogamy holds as well as a constraint on entanglement distributed among subsystems formed of arbitrarily many modes, under permutation invariance. On the other hand, if we keep the number of modes  $N$  fixed, this argument together with the fact that Eq. (4) decreases with  $N$ , implies as a general rule that a smaller number of larger molecules shares strictly more entanglement than a larger number of smaller molecules.

All forms of  $K$ -partite entanglement ( $2 \leq K \leq N$ ) are indeed simultaneously coexisting in  $N$ -partite (pure or mixed) permutation-invariant Gaussian states and, being the general expression Eq. (4) an increasing function of  $\bar{r}$ , they are all increasing functions of each other and mutually enhanced. This structural property of distributed entanglement is known as *promiscuity* and is peculiar to high-dimensional (in the limit, infinite) spaces [20]. In fact, monogamy (already in its weak form) acts in low-dimensional spaces like those of qubits, such as to make bipartite and genuine multipartite entanglements mutually incompatible [7]. In Gaussian states of CV systems, full promiscuity actually occurs under permutation invariance and is perfectly compatible with strong monogamy of multipartite entanglement. This generalizes the known results originally obtained in permutation-invariant three-mode Gaussian states, which due to promiscuity have been dubbed the simultaneous analogues of Greenberger-Horne-Zeilinger and  $W$  states of three qubits [6]. Moreover, unlimited promiscuity occurs in a family of nonsymmetric four-mode Gaussian states, and strong monogamy holds as well in that case [20]. This fact suggests that the approach presented here may retain its validity beyond the fully symmetric scenario.

*Concluding remarks.*—In this Letter, we have addressed and analytically solved the problem of quantifying genuine multipartite entanglement among (groups of) modes in Gaussian states on permutation-invariant harmonic lattices. Such entanglement is experimentally accessible and operationally related to the fidelity of teleportation networks. The results obtained for the Gaussian scenario rest on a more general approach that postulates the existence of stronger monogamy constraints on distributed bipartite and multipartite entanglement. In this respect, permutation-invariant states lend themselves naturally to be investigated via our framework.

Our analysis bears a promising potential in the context of quantum cryptography: (weak) monogamy of entanglement is the only requirement that any physical theory must fulfill to make two-party quantum key distribution unconditionally secure [21]. Strong monogamy may likely play the same role as soon as multiparty secure communication schemes (such as Byzantine agreement, which in the CV case is solved with fully symmetric Gaussian states [22]) are concerned. Further investigation is needed on such an intriguing topic, as well as on the demonstration of the strong monogamy property in other systems, like  $N \geq 4$  qubits, possibly making use of the techniques introduced in Ref. [23].

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