Baryon-Number-Induced Chern-Simons Couplings of Vector and Axial-Vector Mesons in Holographic QCD

Sophia K. Domokos and Jeffrey A. Harvey

Enrico Fermi Institute and Department of Physics, University of Chicago, Chicago, Illinois 60637, USA

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We show that holographic models of QCD predict the presence of a Chern-Simons coupling between vector and axial-vector mesons at finite baryon density. In the Anti de Sitter/Conformal Field Theory dictionary, the coefficient of this coupling is proportional to the baryon number density and is fixed uniquely in the five-dimensional holographic dual by anomalies in the flavor currents. For the lightest mesons, the coupling mixes transverse ρ and a_1 polarization states. At sufficiently large baryon number densities, it produces an instability, which causes the ρ and a_1 mesons to condense in a state breaking both rotational and translational invariance.

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Introduction.—Models which use the gravity-gauge correspondence to treat strongly coupled QCD as a five-dimensional theory of gravity have progressed dramatically in recent years [1–3]. Particularly at high energies, these theories differ significantly from QCD—yet those models which incorporate light quarks [4] and chiral symmetry breaking of the form observed in QCD [5] do capture much of the important low-energy structure of the theory and give rise to a spectrum of mesons whose masses, decay constants, and couplings match those of QCD to within 20%.

The gravity-gauge approach includes both top-down models of QCD arising from D-brane constructions in string theory [5], and bottom-up phenomenological models, which attempt to capture the essential dynamics using a simple choice of five-dimensional metric (AdS_5) and a minimal field content consisting of a scalar X and gauge fields $A^a_{L\mu}$ and $A^a_{R\mu}$ [6,7]. These fields are holographically dual to the quark bilinear $\bar{q}^\alpha q^\beta$ and to the $SU(N_f)_L \times SU(N_f)_R$ flavor currents $\bar{q}_L\gamma^\mu t^a q_L$ and $\bar{q}_R\gamma^\mu t^a q_R$ of QCD, respectively.

These holographic models can be used to study QCD at finite baryon density [8,9]. In this Letter we focus on a novel effect, in which a Chern-Simons term leads to mixing between vector and axial-vector mesons. We will use the model introduced in [6,7] and for the most part follow the conventions and notation of [6].

The model.—We work in a slice of AdS_5 with metric

$$ds^2 = \frac{1}{z^2}(-dz^2 + dx^\mu dx_\mu), \qquad 0 < z \le z_m. \quad (1)$$

The fifth coordinate, z, is dual to the energy scale of QCD. We generate confinement by imposing an ir cutoff z_m and specifying the ir boundary conditions on the fields. The UV behavior, meanwhile, is governed by $z \to 0$.

In Anti de Sitter/Conformal Field Theory (AdS/CFT) calculations, boundary contributions to the action must be treated with care. In the full AdS space, the only boundary is in the UV (at $z = 0$). UV-divergent contributions to the action and to other quantities are canceled by counterterms. For details, see [10,11]. In the model at hand, the ir boundary at $z = z_m$ may contribute to the action. We follow the approach of [6,7] by (1) dropping ir boundary terms and (2) taking parameters normally fixed by ir boundary conditions on the classical solution as input parameters of the model.

We generalize the gauge symmetry to $U(N_f)_L \times U(N_f)_R$ and add a Chern-Simons term which gives the correct holographic description of the QCD flavor anomalies [3]. The Chern-Simons term does not depend on the metric and on general grounds will be present in any holographic dual description of QCD. The $U(1)$ axial symmetry in QCD is anomalous, but in the spirit of the large N_c approximation we treat it as an exact symmetry of QCD with massless quarks. Including the anomaly would not affect our conclusions.

The Lagrangian is thus

$$S = \int d^4x dz \sqrt{g} \mathrm{Tr}\Big[|DX|^2 + 3|X|^2 - \frac{1}{4g_5^2}(F_L^2 + F_R^2)\Big] + S_{\mathrm{CS}}. \quad (2)$$

The Chern-Simons term is given by

$$S_{\mathrm{CS}} = \frac{N_c}{24\pi^2}\int[\omega_5(A_L) - \omega_5(A_R)], \quad (3)$$

where $d\omega_5 = \mathrm{Tr}F^3$, N_c is the number of colors, and $A_{L,R} = \hat{A}_{L,R}\hat{t} + A^a_{L,R}t^a$ where t^a are the generators of $SU(N_f)_{L,R}$ normalized so that $\mathrm{Tr}t^a t^b = \delta^{ab}/2$ and $\hat{t} = \mathbf{1}/\sqrt{2N_f}$ is the generator of the $U(1)$ subalgebra of $U(N_f)$. In what follows, we take $N_f = 2$ so that $a = 1$, 2, 3. We will often work with the vector and axial-vector fields $V = (A_L + A_R)/2$ and $A = (A_L - A_R)/2$.

Classical background.—We expand around a nontrivial solution to the classical equations of motion for the scalar X. Following [6,7], we find the scalar background

$$X_0(z) = \left(\frac{1}{2}Mz + \frac{1}{2}\Sigma z^3\right) \equiv \frac{v(z)}{2}\mathbf{1}, \tag{4}$$

where the coefficient M of the non-normalizable term is proportional to the quark mass matrix, and Σ is the $\bar{q}q$ expectation value. We take both M and Σ to be diagonal: $M \equiv m_q\mathbf{1}$ and $\Sigma \equiv \sigma\mathbf{1}$. As shown in [6,7], we can fix the five-dimensional coupling g_5 by comparison with the vector current two-point function in QCD at large Euclidean momentum. This leads to the identification

$$g_5^2 = \frac{12\pi^2}{N_c}. \tag{5}$$

The model is thus defined by three parameters: z_m, m_q, and σ. Note that including the $U(1)$ gauge fields and Chern-Simons coupling does not mandate the addition of any new parameters. We use $z_m = 1/(346\text{ MeV})$, $m_q = 2.3$ MeV, and $\sigma = (308\text{ MeV})^3$, which correspond to values found through a global fit to seven observables (Model B) in [6].

A background with a static constant quark density is described by a classical solution to the equation of motion for the time component of the $U(1)$ vector gauge field $\hat{V}_\mu$, which is dual to the quark number current. Solving the $\hat{V}_0$ equation of motion at zero four-momentum yields

$$\hat{V}_0(z) = A + \tfrac{1}{2}Bz^2. \tag{6}$$

By the general philosophy of AdS/CFT, the coefficient of the non-normalizable term, A, is proportional to the coefficient with which the operator dual to $\hat{V}_0$ enters the gauge theory Lagrangian. Since $\hat{V}_\mu$ is dual to the quark number current, A must be proportional to the quark chemical potential. Meanwhile, the coefficient of the normalizable term, B, is proportional to the expectation value of the operator dual to $\hat{V}_0$: the quark number density. We now obtain the normalizations of A and B. The action evaluated for the background Eq. (6) is given by a boundary term:

$$S = \frac{1}{2g_5^2}\int d^4x \frac{1}{z}\hat{V}_0\partial_z\hat{V}_0|_{z=0} = \frac{1}{2g_5^2}AB\int d^4x. \tag{7}$$

At finite temperature and baryon number, the Euclidean action is equal to the grand canonical potential. Using Eq. (5), this implies that

$$AB = \frac{24\pi^2}{N_c}n_q\mu_q \tag{8}$$

with n_q the quark number density and μ_q the quark chemical potential. To fix A we separate $U(N_f)_{L,R}$ into $U(1)_{L,R}$ and $SU(N_f)_{L,R}$ components and note that the Chern-Simons term contains the coupling

$$\frac{N_c}{24\pi^2}\frac{3}{8}\int d^4xdz\epsilon^{MNPQ}(\hat{A}_0^L\text{Tr}F_{MN}^LF_{PQ}^L - \hat{A}_0^R\text{Tr}F_{MN}^RF_{PQ}^R), \tag{9}$$

where the indices M, N, P, Q run over 1, 2, 3, z and the trace is over $SU(N_f)$. Defining the $SU(N_f)_{L,R}$ instanton numbers by

$$n_{L,R} = \frac{1}{32\pi^2}\int d^3xdz\epsilon^{MNPQ}\text{Tr}F_{MN}^{L,R}F_{PQ}^{L,R} \tag{10}$$

and taking $\hat{A}_0^{L,R}$ constant, this reduces to the coupling

$$\frac{N_c}{2}\int dx^0(\hat{A}_0^Ln_L - \hat{A}_0^Rn_R). \tag{11}$$

Using the connection between instantons and Skyrmion configurations of the pion field carrying nonzero baryon number [12–16], we can interpret an instanton with $n_L = -n_R = N_b$ as a state with baryon number N_b. Equation (11) then fixes $A = \mu_b/N_c = \mu_q$ with μ_q the quark chemical potential; Eq. (8) fixes $B = 24\pi^2n_q/N_c$.

Quadratic action.—In vacuum, the spectrum of the theory consists of towers of scalar, vector, pseudoscalar, and axial-vector mesons given by mode expanding the five-dimensional fields along the holographic (z) direction and integrating over z. In this section, we identify the spectrum of excitations and their dispersion relations at nonzero baryon density by expanding the action to quadratic order around the background given by Eqs. (4) and (6).

We focus on the π mesons and the isospin triplet vector ρ and axial-vector a_1 mesons, ignoring contributions from heavier mesons and from the scalar σ which arises from fluctuations in the magnitude of X. Couplings similar to those for the $\rho - a_1$ mesons exist for the isoscalar ω and f_1 mesons. For simplicity, we omit these as well.

Pions arise as Nambu-Goldstone modes associated with the breaking of $U(N_f)_L \times U(N_f)_R$ to $U(N_f)_V$. We write $X(x, z) = X_0(z)\exp(i2\pi^at^a)$ and expand to quadratic order in π^a. The four-dimensional pion field is obtained by writing $\pi^a(x, z) = \pi^a(x)\psi_\pi(z)$. Similarly, the ρ^a and a_1 mesons appear by writing $V_\mu^a(x, z) = g_5\rho_\mu^a(x)\psi_\rho(z)$, $A_\mu^a(x, z) = g_5a_\mu^a(x)\psi_a(z)$. The wave functions $\psi_\pi(z)$, $\psi_\rho(z)$, and $\psi_a(z)$ are solutions of the quadratic equations of motion for fields with four-momentum $q^2 = m^2$ and with boundary conditions $\psi(0) = \partial_z\psi(z_m) = 0$. For details, see [6,7].

Making the above substitutions and expanding to quadratic order yields the four-dimensional action

$$\begin{aligned} S = \int d^4x\Big[&\tfrac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a - \tfrac{1}{2}m_\pi^2\pi^a\pi^a - \tfrac{1}{4}(\rho_{\mu\nu}^a)^2 \\ &- \tfrac{1}{4}(a_{\mu\nu}^a)^2 + \tfrac{1}{2}m_\rho^2\rho_\mu^a\rho^{a\mu} + \tfrac{1}{2}m_a^2a_\mu^aa^{a\mu} \\ &+ \mu\epsilon^{ijk}(\rho_i^a\partial_ja_k^a + a_i^a\partial_j\rho_k^a)\Big], \end{aligned} \tag{12}$$

with $\rho_{\mu\nu}$, $a_{\mu\nu}$ the field strengths for ρ_μ, a_μ. The Chern-Simons term with coefficient μ mixes the ρ and a_1 mesons. It arises from reduction of a term of the form $\int d\hat{V}\text{Tr}AdV$ in the expansion of Eq. (3).

As usual, to obtain Eq. (12) one must remove the mixing between a_μ^a and $\partial_\mu\pi^a$ by performing the transformation

$a_\mu^a \to a_\mu^a + \xi \partial_\mu \pi^a$ and then rescaling the pion field to obtain a canonical kinetic energy term [17]. This leads to a pion contribution to the Chern-Simons term. A total spatial derivative, it does not contribute to the equations of motion and may be dropped.

Since the ρ has $J^{PC} = 1^{--}$ and the a_1 has $J^{PC} = 1^{++}$, the Chern-Simons coupling is even under P and odd under C. This is indeed consistent with a background having nonzero baryon number, which preserves P and violates C: the coupling is rotationally invariant but not Lorentz invariant due to the preferred rest frame of the baryons.

We can deduce the existence of the Chern-Simons coupling in four-dimensional terms as follows. The reduction of the five-dimensional Chern-Simons term [5,18] gives rise to the usual gauged Wess-Zumino-Witten action [19–21], as well as a set of couplings which arise from inexact bulk terms. These include a $\rho - a_1 - \omega$ coupling which, in the presence of a coherent ω field in nuclear matter, gives rise to a coupling of the form given in Eq. (12). The $\rho - a_1 - \omega$ coupling has been considered previously in a general discussion of chiral effective Lagrangians [22] and is implicit in the formulas of [23]. Related terms appear in [18,24]. In AdS/QCD, different forms of the gauged Wess-Zumino-Witten action can be obtained by the addition of UV counterterms [25], but these will not cancel the Chern-Simons coupling and lead to explicit breaking of chiral symmetry beyond that given by the quark mass term in Eq. (4).

The mass of the ρ meson is given by $m_\rho = 2.405/z_m$, while m_{a_1} must be determined from a numerical solution of the equation of motion. Model B of [6] finds $m_\rho = 832$ MeV, $m_{a_1} = 1200$ MeV which should be compared to the experimental values $m_\rho = 775.8 \pm 0.5$ MeV and $m_{a_1} = 1230 \pm 40$ MeV [26]. The parameter μ in the Chern-Simons coupling is given by

$$\mu = 18\pi^2 n_b z_m^2 I, \tag{13}$$

where I is the dimensionless overlap integral

$$I = \frac{1}{z_m^2}\int_0^{z_m} dz z \psi_\rho(z)\psi_{a_1}(z). \tag{14}$$

Numerical evaluation of the integral gives $I = 0.54$. A typical baryon density in nuclear matter, $n_b^0 \simeq 0.16/\mathrm{F}^3$, gives

$$\mu \simeq 1.05\ \mathrm{GeV}\left(\frac{n_b}{n_b^0}\right). \tag{15}$$

Phenomenological applications.—We now outline two potentially observable consequences of the Chern-Simons coupling between the ρ and a_1. Details will appear elsewhere.

Mixing of transverse ρ and a_1 states.—We consider plane-wave solutions to the equations of motion resulting from Eq. (12), dropping the pion fields and focusing on the ρ and a_1 dispersion relation and polarization vectors. Without loss of generality, we consider propagation along x^3:

$$\rho_\mu(x) = \epsilon_\mu^\rho(q) e^{-iq\cdot x}, \qquad a_\mu(x) = \epsilon_\mu^a(q) e^{-iq\cdot x} \tag{16}$$

with $q = (q_0, 0, 0, q_3)$. For convenience, we suppress the $SU(2)$ indices in the following. The components ρ_0, ρ_3, a_0, and a_3 have standard dispersion relations, unaffected by the Chern-Simons coupling. The transverse components ρ_1, ρ_2, a_1, and a_2 mix through a derivative coupling. The equations of motion yield the dispersion relation for the transverse polarizations

$$q_0^2 - q_3^2 = \tfrac{1}{2}(m_\rho^2 + m_{a_1}^2) \pm \tfrac{1}{2}\sqrt{(m_{a_1}^2 - m_\rho^2)^2 + 16\mu^2 q_3^2}. \tag{17}$$

The lower sign in Eq. (17) gives a state which is pure ρ as $q_3 \to 0$. At nonzero q_3, it is a mixture of transverse ρ and a_1 states with orthogonal polarization vectors:

$$\epsilon_1^a = \frac{i\mathcal{M}^2(q_3)}{2\mu q_3}\epsilon_2^\rho, \qquad \epsilon_2^a = -\frac{i\mathcal{M}^2(q_3)}{2\mu q_3}\epsilon_1^\rho, \tag{18}$$

where we have defined $\Delta^2 = m_{a_1}^2 - m_\rho^2$ and $\mathcal{M}^2(q_3) = (\sqrt{\Delta^4 + 16\mu^2 q_3^2} - \Delta^2)/2$. The upper sign in Eq. (17) gives a pure a_1 state for $q_3 = 0$, while for nonzero q_3,

$$\epsilon_1^\rho = -\frac{i\mathcal{M}^2(q_3)}{2\mu q_3}\epsilon_2^a, \qquad \epsilon_2^\rho = \frac{i\mathcal{M}^2(q_3)}{2\mu q_3}\epsilon_1^a. \tag{19}$$

For μ greater than some momentum-dependent critical value, the dispersion relation Eq. (17) leads to tachyonic modes (modes having $dq_0/dq_3 > 1$). For very large momenta, this critical value becomes

$$\mu_{\mathrm{crit}} = \sqrt{(m_\rho^2 + m_{a_1}^2)/2} \simeq 1.09\ \mathrm{GeV}. \tag{20}$$

For a range of μ below μ_{crit} the dispersion relation with the lower sign in Eq. (17) exhibits interesting anomalous behavior, the analysis of which is beyond the scope of this Letter.

It would be interesting to explore signatures of these mixed polarization states in the quark-gluon plasma and in nuclear matter.

Vector meson condensation.—To identify the tachyonic instability which occurs for $\mu > \mu_{\mathrm{crit}}$ we start with the energy density corresponding to Eq. (12) for the diagonal component of the ρ and a fields, $a^a = a\delta^{a3}$, $\rho^a = \rho\delta^{a3}$. Completing the square and dropping the terms involving the electric components of the field strengths, which play no role in the instability, we find

$$\begin{aligned}\mathcal{H} &= \tfrac{1}{2}(m_a^2 - \mu^2)\vec{a}\cdot\vec{a} + \tfrac{1}{2}(m_\rho^2 - \mu^2)\vec{\rho}\cdot\vec{\rho} \\ &\quad + \tfrac{1}{2}(\vec{B}_a - \mu\vec{\rho})^2 + \tfrac{1}{2}(\vec{B}_\rho - \mu\vec{a})^2,\end{aligned} \tag{21}$$

where $\vec{B}_\rho = \vec{\nabla}\times\vec{\rho}$, $\vec{B}_a = \vec{\nabla}\times\vec{a}$.

Applying the ansatz

$$\vec{a} = v\cos(\mu x_3)\hat{x}_2, \qquad \vec{\rho} = v\sin(\mu x_3)\hat{x}_1, \tag{22}$$

the last two terms in Eq. (21) vanish, while the average of the first two terms over x_3 is negative for $\mu^2 > \mu^2_{\rm crit}$, leading to an instability to $v \neq 0$. Understanding the stabilization of the configuration Eq. (22) requires generalizing $\mathcal{H}$ to include higher order terms. Note that Eq. (22) breaks both rotational and translational symmetry, exhibiting a structure similar to the smectic phase of liquid crystals which includes an interesting set of topological defects.

The critical value Eq. (20) is remarkably close to the estimate Eq. (15) for μ at ordinary nuclear densities. If this model is accurate then there should be a condensate of vector and axial-vector mesons in nuclear matter with baryon densities at or slightly above n_b^0. In ordinary nuclei, there are finite size effects as well as other corrections to the ρ and a_1 interactions which will have to be included to determine whether this condensate occurs. Neutron stars are more likely to produce such a condensate, as they are thought to contain matter at a density somewhat greater than n_b^0. The interplay between this condensate and other conjectured effects in nuclear matter, such as color superconductivity, pion condensation, and kaon condensation [27] deserves further study.

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