Dynamical Tunneling in Macroscopic Systems

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We investigate macroscopic dynamical quantum tunneling (MDQT) in the driven Duffing oscillator, characteristic for Josephson junction physics and nanomechanics. Under resonant conditions between stable coexisting states of such systems we calculate the tunneling rate. In macroscopic systems coupled to a heat bath, MDQT can be masked by driving-induced activation. We compare both processes, identify conditions under which tunneling can be detected with present day experimental means and suggest a protocol for its observation.

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The phase space of a classical system can have forbidden areas even in the absence of potential barriers, e.g., in the presence of external driving. Quantum mechanically, these areas can be crossed in a process called dynamical tunneling [1,2]. So far, dynamical tunneling has been observed experimentally in microscopic systems, i.e., cold atoms [3] with very low damping. Recent experimental progress has demonstrated many basic quantum features in macroscopic systems such as Josephson junctions or nanomechanical oscillators, overcoming the limitations posed by their coupling to the environment. Important for this success was the ability to reduce noise and cool to very low temperatures.

In this Letter we discuss the possibility of macroscopic dynamical tunneling (MDQT), i.e., involving a macroscopic degree of freedom, like the phase difference across a driven Josephson junction. Classically, for certain parameters, this system has two stable coexisting oscillations with different amplitudes. This driven system will feel the influence of its dissipative environment strongly even at temperature $T = 0$. We demonstrate that under experimentally accessible conditions the tunneling between the two classical states can indeed occur and be singled out from the background of thermal activation events. We suggest an experiment where MDQT can be directly observed. Our result can be applied to verify quantum physics in systems with weak nonlinearity such as nanomechanical oscillators. Quantum tunneling it is also a potential dark count error process in the Josephson bifurcation amplifier. Here the classical switching between the two driving-induced, coexisting states in a Josephson junction was used for high resolution dispersive qubit state detection [4–7].

Dynamical tunneling (in the absence of an environment) has been studied using the WKB approximation in the parametric driven oscillator [8]. Activation rates in the presence of an environment have been studied in bistable systems [9,10]. Dynamical tunneling with dissipation has been described numerically [11] and multiphoton resonances have been studied perturbatively [12].

We study a harmonically driven Duffing oscillator, as an approximate description of a wide range of macroscopic physical systems ranging Josephson junctions [4,13] and nanomechanical oscillators [14,15]. The driven Duffing oscillator is described by the Hamiltonian

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{m\Omega^2}{2}\hat{x}^2 - \gamma\hat{x}^4 + F(t)\hat{x}, \qquad (1)$$

where $F(t) = F_0(e^{i\nu t} + e^{-i\nu t})$ is the driving field with frequency ν. For subresonant driving, $\nu < \Omega$, and below a critical driving strength $F_0 < F_c$ two classical oscillatory states with different response amplitudes coexist. Considering a Josephson junction with capacitance C, critical current I_c, and driving current amplitude I we can identify x as the phase difference across the junction, $m = (\hbar/2e)^2C$, $\Omega = \sqrt{2eI_c/(\hbar C)}$, $F_0 = \hbar I/(2e)$, and $\gamma = m\Omega^2/24$.

Following the Caldeira-Leggett approach, we assume an Ohmic environment and describe it as a bath of harmonic oscillators

$$\hat{H}_E = \sum_i\left(\frac{m_i\omega_i^2\hat{x}_i^2}{2} + \frac{\hat{p}_i^2}{2m_i}\right) - \hat{x}\sum_i\lambda_i\hat{x}_i + \hat{x}^2\sum_i\frac{\lambda_i^2}{2m_i\omega_i^2},$$

with spectral density $J(\omega) = \pi\sum_i\lambda_i^2\delta(\omega-\omega_i)/(2m_i\omega_i) = m\kappa\omega\exp(-\omega/\omega_c)$ and ω_c a high frequency cutoff.

We transform this Hamiltonian using the unitary operator $\hat{U} = \exp(i\nu t(\hat{a}^\dagger\hat{a} + \sum_i\hat{b}_i^\dagger\hat{b}_i)$ similar to Ref. [9], where $\hat{a}$ and $\hat{b}_i$ are the annihilation operators for the system and bath oscillators. Dropping the fast rotating terms in the rotating wave approximation (RWA), we obtain

$$\hat{H}_{\rm tot} = \hat{H}_0^{(\delta)} - x\sum_i\lambda_i\hat{x}_i + \sum_i\frac{\tilde{m}\tilde{\omega}_i^2\hat{x}_i^2}{2} + \frac{\hat{p}_i^2}{2\tilde{m}_i}, \qquad (2)$$

where, up to a constant we have

$$\hat{H}_0^{(\delta)} = \frac{\tilde{m}\tilde{\Omega}^2}{2}\hat{x}^2 + \frac{\hat{p}^2}{2\tilde{m}} - \frac{6\gamma}{4\tilde{m}^2\tilde{\Omega}^4}\left(\frac{\tilde{m}\tilde{\Omega}^2}{2}\hat{x}^2 + \frac{\hat{p}^2}{2\tilde{m}}\right)^2 + F_0\hat{x}. \qquad (3)$$

We thus obtain a time independent Hamiltonian at the expense of a form that is not separable in $\hat{p}$ and $\hat{x}$. This transformation reduces the frequency $\tilde{\Omega} = \Omega\delta$ and increases the mass $\tilde{m} = m/\delta$ of the oscillators by $\delta_i = (\omega_i - \nu)/\omega_i$ in the case of the bath and $\delta = (\Omega - \nu)/\Omega + \kappa\omega_c/(\pi\Omega^2)$ for the main oscillator, where the second term describes a deterministic force induced by dragging the system through its environment.

We concentrate at first on quantum tunneling in the absence of bath fluctuations and study the system in phase space. The classical Hamilton function $H_0^{(\delta)}(x, p)$ is portrayed in Figs. 1(b) and 1(c) for a subcritical driving strength $F_0 < F_c = 2/9(2\tilde{m}^3\tilde{\Omega}^6/\gamma)^{1/2}$. It has three extremal points: saddle (s), minimum (m), and maximum (M) with phase space coordinates (x_e, p_e), where $e \in \{m, s, M\}$. The curves of constant quasienergy $H_0^{(\delta)}(x, p) = E$ represent classical trajectories. In the bistability region $E \in (E_m, E_s)$ where $E_e = H_0^{(\delta)}(x_e, p_e)$ there are always two periodic classical trajectories, around the two stable points (m) and (M), with a small and large amplitude, respectively.

Using this phase space, we outline an experiment to observe MDQT during the transient evolution of the system. Without driving, the system relaxes to its ground state centered around (m). Then, after turning on the driving field, one records the time needed for a transition to the large orbit as a function of a parameter of the drive, e.g., frequency ν. When two quantized levels pertaining to the two oscillatory states are close in quasienergy, tunneling can occur, and enhance the total switching rate.

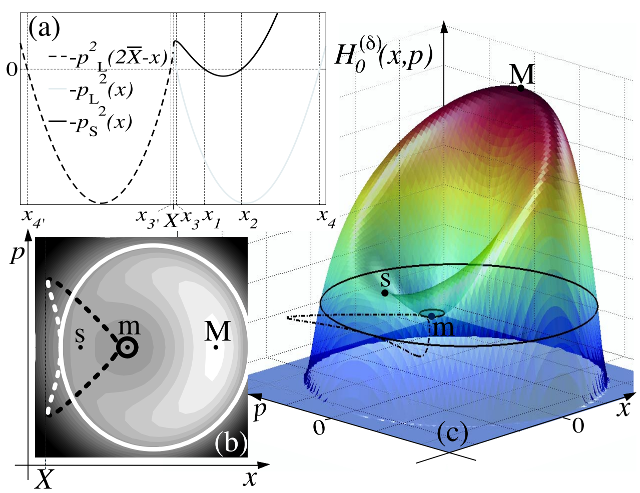

FIG. 1 (color online). Illustration of the Hamilton function and the potential landscape. (a) $-p_{S,L}^2(x, E)$: the potential changes with E; classical turning points are found at $p(x_i, E) = 0$. (b),(c) $H_0^{(\delta)}(x, p)$; in (b) white corresponds to high, black to low quasienergy; (b) white lines corresponds to the L, black ones to the S branch; continuous lines correspond to real and dashed ones to imaginary valued momentum.

We describe tunneling using the semiclassical WKB approximation which is an expansion in $\hbar$ close to the least-action path. To find that path we solve the equation $H_0^{(\delta)}(x, p) = E$ and obtain four coexisting momentum branches $\pm p_{L,S}(x, E)$ where

$$p_{S,L}(x, E) = \tilde{m}\,\tilde{\Omega}\sqrt{\frac{2\tilde{m}\tilde{\Omega}^2}{3\gamma} - x^2 \mp \sqrt{\frac{8F_0}{3\gamma}}\sqrt{x - X}}, \quad (4)$$

with $X = E/F_0 - (\tilde{m}\tilde{\Omega}^2)^2/(6F_0\gamma)$. This configuration is reminiscent of Born-Oppenheimer surfaces in molecular physics where dynamical tunneling has also been studied [1]. A real-valued $p_{S,L}$ corresponds to a classically allowed area with an oscillating WKB wave function, a complex-valued one to a classically forbidden area with a decaying wave function. At $x = X$, both trajectories have the same momentum and position and connect. Here $\dot{x} = \partial_p H_0^{(\delta)}(x, p) = 0$ but $p \neq 0$ such that the motion changes direction and continues on a different momentum branch. For all $x < X$ both $p_{S,L}(x, E)$ are complex. The tunneling least-action trajectory which connects the two allowed regions only passes through the region $x > X$. Here the $p_{S,L}$ are either real or purely imaginary, i.e., $-p_{S,L}^2 \in \mathbb{R}$. Thus the forbidden area with $x < X$ does not influence the quantization rules within the WKB approximation. To study the region where $x > X$, we mirror the solution $p_L(x, E)$ around the X point as shown in Fig. 1(a) and obtain a double well "potential." The small and large amplitude oscillation states are localized in the right- and left-hand wells, respectively, and are separated by a "potential barrier" where the momentum is purely imaginary. We apply the WKB theory in this "potential" in order to determine the tunnel splitting in the limit of a low transmission through the forbidden region. The classical turning points x_i are given by $p_{S,L}(x_i, E) = 0$; see Fig. 1(a). The bound state energies at zero transmission are given by the Sommerfeld energy quantization rules

$$S_{12}(E) = \pi n + \pi/2, \quad S_{4'3'}(E) = \pi m + \pi/2, \quad n, m \in \mathbb{Z}, \quad (5)$$

where $S_{ij}(E) = \int_{x_i}^{x_j} \mathrm{sgn}(x - X)|p(x, E)|dx/\hbar$ and the negative sign on the left-hand side of X is due to mirroring. Whenever a pair of energies from either well is degenerate, resonant tunneling through the barrier can occur. This induces coupling between the two wells and lifts the degeneracy. The level crossings become avoided crossings at finite transmission and the full WKB condition reads

$$\cot S_{12}(E) \cot S_{4'3'}(E) = \exp(-2S_{3'1}(E))/4. \quad (6)$$

We expand the quasienergy E and the actions S_{ij} in a series of $\xi = 1/4\exp(-2S_{3'1})$ around the level crossings with quasienergy E_0 where Eqs. (5) are simultaneously satisfied. The first energy correction $E_1\xi$ is obtained straightforwardly from $\partial_E S_{12}|_{E_0}\partial_E S_{4'3'}|_{E_0}(E_1\xi)^2 = \xi$, and the tunneling rate is obtained directly from the energy splitting at the avoided level crossings

$$\Gamma_t = \frac{2E_1\xi}{\hbar\pi} = \frac{\exp(-S_{3'1})}{\hbar\pi\sqrt{\partial_E S_{12}\partial_E S_{4'3'}}}\bigg|_{E_0}. \quad (7)$$

This can be evaluated in closed form involving elliptic integrals for S_{ij} and we obtain the exact expressions

$$\partial_E S_{12}|_m = \partial_E S_{4'3'}|_m = \pi/(\hbar\Omega_m),$$
$$\partial_E S_{12}|_s = \partial_E S_{3'1}|_m = \infty, \qquad \partial_E S_{3'1}|_s = \pi/(\hbar|\Omega_s|),$$

where $\Omega_e = \sqrt{\partial^2_{xx}H_0^{(\delta)}\partial^2_{pp}H_0^{(\delta)}}|_e$ and $e \in \{m, s, M\}$. Thus, for S_{12} at (m) and $S_{3'1}$ at (s) we reproduce the harmonic oscillator result. The saddle point "frequency" Ω_s is imaginary as expected.

We simplify Eq. (7) by locally approximating $H_0^{(\delta)}$ close to the extremal points by harmonic oscillators, i.e., assuming that S_{ij} are linear functions of E. This approximation holds for all S_{ij} simultaneously when E is far enough from both extremal points $E_{s,m}$, as it is the case for the ground state $E_m + \hbar\Omega_m/2$ of the small amplitude well. In this approximation $S_{3'1}(E) \approx \pi(E_s - E)/(\hbar|\Omega_s|)$ and thus we find a compact approximation

$$\Gamma_t \approx \frac{\Omega_m}{\pi^2}\exp\left(-\frac{\pi(E_s - E_m - \hbar\Omega_m/2)}{\hbar|\Omega_s|}\right). \quad (8)$$

Our calculations rely on a series of assumptions. To test them, we compare the results to a full numerical diagonalization of $\hat{H}_0^{(\delta)}$ taking a basis of the first $2N$ Fock states. At $F_0 = 0$, the number of levels that cover the bistability region is $N = \hbar\Omega(2m\Omega)^2/(6\gamma\hbar^2)$. As shown for a representative set of data in Fig. 2, we find good agreement between these numerically exact results and the predictions of Eqs. (5) and (7) and also (8).

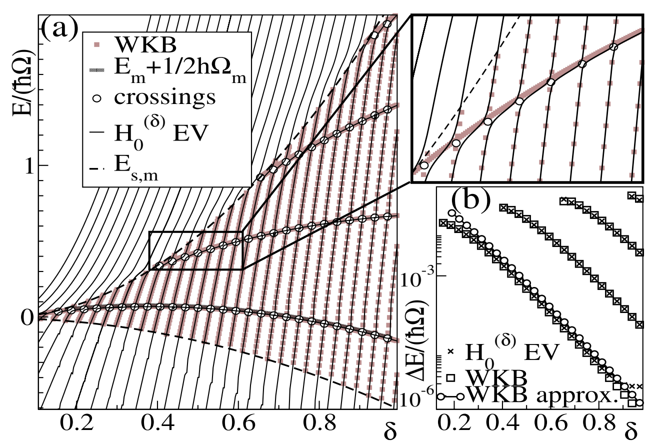

FIG. 2 (color online). (a) Quantized energies: eigenvalues (EV) of $\hat{H}_0^{(\delta)}$ versus WKB. $\hat{H}_0^{(\delta)}$ was represented in the number state basis considering $2N$ levels. (b) Tunneling-induced energy splittings at level crossings. Frequency sweep at $m\Omega/\hbar = 2$, $\gamma = m\Omega^2/24$, $\kappa\omega_c/\Omega^2 = 0.1$, and $F_0 = 0.5F_c(\nu)$.

Quantum tunneling is significant only close to level crossings. It always competes with the activation over the barrier, which occurs at all energies and is based on classical fluctuations due to coupling to a heat bath. A rather detailed treatment of a similar process has been given in Refs. [10]. We now estimate these effects and compare them to the quantum tunneling rate. When modeling activation, it is crucial to consider that we are working in a frame rotating relative to the heat bath, which is fixed in the laboratory.

We start from Eq. (2). As we will adopt the mean-first-passage time approach [16], it is sufficient to approximate the system Hamiltonian close to its minimum in phase space by $\hat{H}_0^{(\delta)} \approx \hat{p}^2/(2m_{\rm eff}) + V(\hat{x})$ where the effective mass is determined by the curvature of the Hamilton function $m_{\rm eff}^{-1} = \partial^2_{pp}H_0^{(\delta)}(x, p)|_m$ and the effective potential is $V(x) = H_0^{(\delta)}(x, p_m)$. In this approximation we obtain a quantum Langevin equation

$$m_{\rm eff}\ddot{x} + \partial_x V(x) - x\int_0^\infty d\omega\frac{2J(\omega)}{\pi\omega} + m_{\rm eff}\int_0^t \tilde{\kappa}(t-s)\dot{x}(s)ds = \xi(t),$$

where

$$\tilde{\kappa}(t) = \int_0^\infty \frac{2J(\omega)\cos((\omega-\nu)t)}{(\omega-\nu)\pi m_{\rm eff}}d\omega,$$
$$\xi(t) = \sum_i \lambda_i\left[\left(x_i(0) - \frac{\lambda_i x(0)}{\tilde{m}_i\tilde{\omega}_i^2}\right)\cos(\tilde{\omega}_i t) + \frac{p_i(0)}{\tilde{m}_i\tilde{\omega}_i}\sin(\tilde{\omega}_i t)\right].$$

$\tilde{\kappa}(t)$ is peaked on a short time scale ω_c^{-1}. Its magnitude is characterized through the effective friction constant

$$\kappa_{\rm eff} = \int_0^\infty \tilde{\kappa}(t)dt = 2\kappa\left(\delta - \frac{3\gamma x_m^2}{2m\Omega^2}\right)(1 + \mathcal{O}(\nu/\omega_c)).$$

The factor of 2 difference between $\kappa_{\rm eff}$ and the damping constant of the undriven harmonic system accounts for the fact that in the rotating frame there are bath modes above and below $\omega = 0$ [see Eq. (2)] whereas for the undriven case the frequencies are strictly positive. Thus oscillators with frequency ω have the spectral density $J(\omega + \nu)$ and modes with negative frequencies have significant contribution to noise even at low temperatures. We use a detailed balance condition to determine the effective temperature of the bath as seen by a detector in the rotating frame, e.g., a two level system with level separation $\hbar\Omega_m$

$$P(\Omega_m, T)/P(-\Omega_m, T) = \exp(\hbar\Omega_m\beta_{\rm eff}). \quad (9)$$

Here $P(\omega, T) = J(\omega + \nu)(1 + n(\omega + \nu, T))$ is the probability for a quantum $\hbar\omega$ to be emitted to the bath in the rotating frame. The effective temperature is enhanced at low T and finite even at $t = 0$. This accounts for the fact that what a detector in the rotating frame regards as (quasienergy) absorption can actually be (energy) emission in the lab frame. In the case of constant acceleration in relativistic context this behavior is known as the Unruh effect [17].

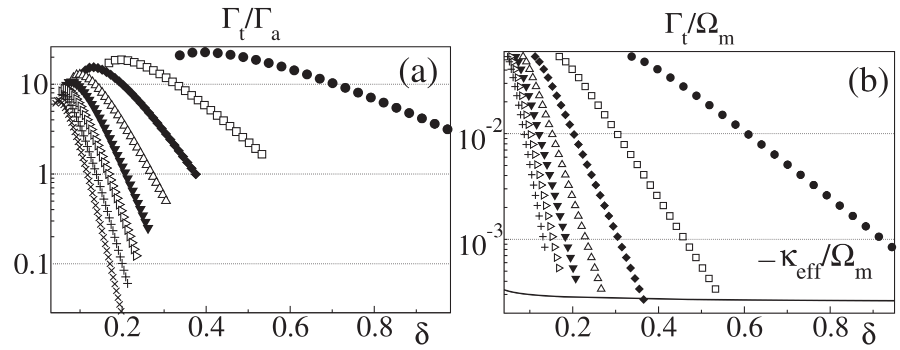

FIG. 3. (a) The ratio of tunneling and activation rates from the small well at the avoided level crossings. (b) Corresponding tunneling rates compared to $\kappa_{\rm eff}$ (where $\Gamma_t/\Gamma_a > 1$). Driving frequency sweep at $F_0 = 0.7F_c(\nu)$; values of Ω (in GHz): 1(●), 2(□), 3(◆), 4(△), 5(▼), 6(▷), 7(+), 8(×), at parameters specified in text.

The barrier crossing problem for systems described by a quantum Langevin equation is well studied in the context of chemical reactions. For low damping, $\kappa_{\rm eff} \ll \Omega_{\rm m}$ mean-first-passage time theory predicts the activation rate

$$\Gamma_{\rm a}^{-1} = \frac{\beta_{\rm eff}}{\kappa_{\rm eff}} \int_0^{S(E_{\rm s})} dS e^{-\beta_{\rm eff}E(S)} \int_{E(S)}^{E_{\rm s}} dE' \frac{e^{-\beta_{\rm eff}E'}}{S(E')}, \qquad (10)$$

where $S(E) = \oint p(x, E)dx$. In the traditional low temperature limit $\kappa_{\rm eff}S(E_{\rm s}) \ll k_B T_{\rm eff} \ll E_{\rm s} - E_{\rm m}$ the activation rate becomes

$$\Gamma_{\rm a} = \kappa_{\rm eff}\beta_{\rm eff}\frac{\Omega_{\rm m}}{2\pi}\exp(-(E_{\rm s} - E_{\rm m})\beta_{\rm eff})S(E_{\rm s}). \qquad (11)$$

In our case, the noise temperature $k_B T_{\rm eff}$ can be larger than the barrier height $E_{\rm s} - E_{\rm m}$. In this limit we obtain from Eq. (10)

$$\Gamma_{\rm a} = \kappa_{\rm eff}[F(\beta_{\rm eff}(E_{\rm s} - E_{\rm m}))]^{-1} \qquad (12)$$

where $F(x) = \int dx(\exp(x) - 1)/x \equiv {\rm Ei}(x) - \log(x)$.

Summarizing, in the rotating frame, as a consequence of driving, the bath appears with a quality factor $\Omega_{\rm m}/\kappa_{\rm eff}$ reduced by approximatively a factor of 2 and an enhanced effective temperature $T_{\rm eff}$. Moreover, the bath shifts the detuning δ. We show that experimental observation of MDQT could still be possible. At the level anticrossings we calculate the WKB tunneling rate from the ground state and the activation rate from Eq. (12), see Fig. 3(a) where we have considered a Josephson junction with $\kappa = 10^{-4}\Omega$, the temperature $T = 10$ mK, shunt capacitance $C = 2 \times 10^{-12}$ F, and $\gamma = m\Omega^2/24$. The values of δ where these anticrossings occur are found by minimizing $|\cot(S_{4'3'}(E_{\rm m}))|$ and are in agreement with the weak driving result [12], $\delta = 3\gamma n/(2m^2\Omega^3)$, $n \in \mathbb{N}$. We observe that the quantum tunneling rate can be one order of magnitude larger than the activation rate in the limit of relatively small detuning δ and low damping. By increasing the value of $\alpha = m\Omega/\hbar$, we observe a reduction of the ratio Γ_t/Γ_a as expected, since α measures the number of quantized levels in the system and thus the "classicality" of its behavior. In Fig. 3 we have $\alpha \in (2, 20)$, while in the experiment of Ref. [13] α was larger than 100, at higher temperature and smaller quality factor, such that MDQT was probably masked by thermal activation. We expect that at the values of Fig. 3 the experiment we propose should produce direct evidence for MDQT.

In conclusion we have investigated macroscopic dynamical tunneling by mapping it onto tunneling between two potential surfaces. We compared this process with the activation over the barrier using the mean-first-passage time approach. The values obtained suggest that dynamical tunneling can be singled out from the background of activation processes. We have proposed an experiment realizable within existing technology to demonstrate dynamical tunneling by monitoring the switching rate between the two dynamical states while tuning a parameter of the external driving.

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