Second-Order QCD Corrections to the Thrust Distribution in Electron-Positron Annihilation

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We compute the next-to-next-to-leading-order (NNLO) QCD corrections to the thrust distribution in electron-positron annihilation. The corrections turn out to be sizable, enhancing the previously known next-to-leading-order prediction by about 15%. Inclusion of the NNLO corrections significantly reduces the theoretical renormalization scale uncertainty on the prediction of the thrust distribution.

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Three-jet production cross sections and related event shape distributions in e^+e^- annihilation processes are classical hadronic observables which can be measured very accurately and provide an ideal proving ground for testing our understanding of strong interactions. The deviation from simple two-jet configurations is proportional to the strong coupling constant, so that by comparing the measured three-jet rate and related event shapes with the theoretical predictions, one can determine the strong coupling constant α_s.

The theoretical prediction is made within perturbative QCD, expanded to a finite order in the coupling constant. This truncation of the perturbative series induces a theoretical uncertainty from omitting higher-order terms. It can be quantified by the renormalization scale dependence of the prediction, which is vanishing for an all-order prediction. The residual dependence on variations of the renormalization scale is therefore an estimate of the theoretical error.

So far the three-jet rate and related event shapes have been calculated [1,2] up to the next-to-leading order (NLO), improved by a resummation of leading and subleading infrared logarithms [3,4] and by the inclusion of power corrections [5].

QCD studies of event shape observables at LEP [6] are based around the use of NLO parton-level event generator programs [7]. It turns out that the current error on α_s from these observables [8] is dominated by the theoretical uncertainty. Clearly, to improve the determination of α_s, the calculation of the next-to-next-to-leading-order (NNLO) corrections to these observables becomes mandatory. We present here the first NNLO calculation of the thrust distribution, which is an event shape related to three-jet production.

The thrust variable for a hadronic final state in e^+e^- annihilation is defined as [9]

$$T = \max_{\vec{n}}\left(\frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}\right),$$

where p_i denotes the three-momentum of particle i, with the sum running over all particles. The unit vector $\vec{n}$ is varied to find the thrust direction $\vec{n}_T$ which maximizes the expression in parentheses on the right-hand side.

It can be seen that a two-particle final state has fixed $T = 1$; consequently, the thrust distribution receives its first nontrivial contribution from three-particle final states, which, at order α_s, correspond to three-parton final states. Therefore, both theoretically and experimentally, the thrust distribution is closely related to three-jet production.

Three-jet production at tree-level is induced by the decay of a virtual photon (or other neutral gauge boson) into a quark-antiquark-gluon final state. At higher orders, this process receives corrections from extra real or virtual particles. The individual partonic channels that contribute through to NNLO are shown in Table I. All of the tree-level and loop amplitudes associated with these channels are known in the literature [10–13].

For a given partonic final state, thrust is computed according to the same definition as in the experiment, which is applied to partons instead of hadrons. At leading order, all three final state partons must be well separated from each other, to allow T to differ from the trivial two-parton limit. At NLO, up to four partons can be present in the final state, two of which can be clustered together, whereas at NNLO, the final state can consist of up to five partons, such that as many as three partons can be clustered

TABLE I. Partonic contributions to the thrust distribution in perturbative QCD.

LO	$\gamma^* \to q\bar{q}g$	tree-level
NLO	$\gamma^* \to q\bar{q}g$	one-loop
	$\gamma^* \to q\bar{q}gg$	tree-level
	$\gamma^* \to q\bar{q}q\bar{q}$	tree-level
NNLO	$\gamma^* \to q\bar{q}g$	two-loop
	$\gamma^* \to q\bar{q}gg$	one-loop
	$\gamma^* \to q\bar{q}q\bar{q}$	one-loop
	$\gamma^* \to q\bar{q}q\bar{q}g$	tree-level
	$\gamma^* \to q\bar{q}ggg$	tree-level

together. The more partons in the final state, the better one expects the matching between theory and experiment to be.

The two-loop $\gamma^* \to q\bar{q}g$ matrix elements were derived in [10] by reducing all relevant Feynman integrals to a small set of master integrals using integration by parts [14] and Lorentz invariance [15] identities, solved with the Laporta algorithm [16]. The master integrals [17] were computed from their differential equations [15] and expressed analytically in terms of one- and two-dimensional harmonic polylogarithms [18].

The one-loop four-parton matrix elements relevant here [12] were originally derived in the context of NLO corrections to four-jet production and related event shapes [19,20]. One of these four-jet parton-level event generator programs [20] is the starting point for our calculation, since it already contains all relevant four-parton and five-parton matrix elements.

The four-parton and five-parton contributions to three-jet-like final states at NNLO contain infrared real radiation singularities, which have to be extracted and combined with the infrared singularities [21] present in the virtual three-parton and four-parton contributions to yield a finite result. In our case, this is accomplished by introducing subtraction functions, which account for the infrared real radiation singularities, and are sufficiently simple to be integrated analytically. Schematically, this subtraction reads

$$d\sigma_{\rm NNLO} = \int_{d\Phi_5} (d\sigma^R_{\rm NNLO} - d\sigma^S_{\rm NNLO}) + \int_{d\Phi_4} (d\sigma^{V,1}_{\rm NNLO} - d\sigma^{VS,1}_{\rm NNLO}) + \int_{d\Phi_5} d\sigma^S_{\rm NNLO} + \int_{d\Phi_4} d\sigma^{VS,1}_{\rm NNLO} + \int_{d\Phi_3} d\sigma^{V,2}_{\rm NNLO},$$

where $d\sigma^S_{\rm NNLO}$ denotes the real radiation subtraction term coinciding with the five-parton tree-level cross section $d\sigma^R_{\rm NNLO}$ in all singular limits [22]. Likewise, $d\sigma^{VS,1}_{\rm NNLO}$ is the one-loop virtual subtraction term coinciding with the one-loop four-parton cross section $d\sigma^{V,1}_{\rm NNLO}$ in all singular limits [23]. Finally, the two-loop correction to the three-parton cross section is denoted by $d\sigma^{V,2}_{\rm NNLO}$. With these, each line in the above equation is individually infrared finite, and can be integrated numerically.

Systematic methods to derive and integrate subtraction terms were available in the literature only to NLO [24,25], with extension to NNLO in special cases [26]. In the context of this project, we fully developed an NNLO subtraction formalism [27], based on the antenna subtraction method originally proposed at NLO [20,25]. The basic idea of the antenna subtraction approach is to construct the subtraction terms from antenna functions. Each antenna function encapsulates all singular limits due to the emission of one or two unresolved partons between two color-connected hard partons. This construction exploits the universal factorization of phase space and squared matrix elements in all unresolved limits. The individual antenna functions are obtained by normalizing three-parton and four-parton tree-level matrix elements and three-parton one-loop matrix elements to the corresponding two-parton tree-level matrix elements. Three different types of antenna functions are required, corresponding to the different pairs of hard partons forming the antenna: quark-antiquark, quark-gluon, and gluon-gluon antenna functions. All these can be derived systematically from matrix elements [28] for physical processes.

The factorization of the final state phase space into antenna phase space and hard phase space requires a mapping of the antenna momenta onto reduced hard momenta. We use the mapping derived in [29] for the three-parton and four-parton antenna functions. To extract the infrared poles of the subtraction terms, the antenna functions must be integrated analytically over the appropriate antenna phase spaces, which is done by reduction [30] to known phase space master integrals [31].

We tested the proper implementation of the subtraction by generating trajectories of phase space points approaching a given single or double unresolved limit. Along these trajectories, we observe that the antenna subtraction terms converge towards the physical matrix elements, and that the cancellations among individual contributions to the subtraction terms take place as expected. Moreover, we checked the correctness of the subtraction by introducing a lower cut (slicing parameter) on the phase space variables, and observing that our results are independent of this cut (provided it is chosen small enough). This behavior indicates that the subtraction terms ensure that the contribution of potentially singular regions of the final state phase space does not contribute to the numerical integrals, but is accounted for analytically. A detailed description of the calculation will be given elsewhere [32].

The resulting numerical programme, EERAD3, yields the full kinematical information on a given multiparton final state. It can thus be used to compute any infrared-safe observable related to three-particle final states at $\mathcal{O}(\alpha_s^3)$. As a first application, we present results for the NNLO corrections to the thrust distribution here. Our results are complete up to the pure-singlet corrections (which come from $\gamma^* \to ggg$ and related final states, and appear first at NNLO). In the context of $\mathcal{O}(\alpha_s^3)$ corrections to four-jet final states [19] and in the two-loop three-parton matrix elements [10], these were shown to yield only a marginal numerical contribution to the full coefficient at this order. They are therefore ignored here, and will be presented elsewhere for completeness. Without them, the theoretical expression for the thrust distribution through to NNLO can be expressed by three dimensionless coefficients (A, B, C), which depend only on T and not on the scale of the process or on coupling constants and quark charges. These are obtained as coefficients of the thrust distribution normalised to the tree-level cross section σ_0 for $e^+e^- \to q\bar{q}$, evaluated for $\alpha_s = \alpha_s(Q)$, where Q is the center-of-mass energy of the process

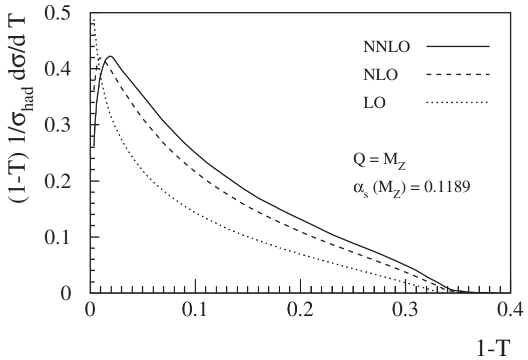

FIG. 1. Thrust distribution at $Q = M_Z$.

$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} = \left(\frac{\alpha_s}{2\pi}\right)\frac{dA}{dT} + \left(\frac{\alpha_s}{2\pi}\right)^2\frac{dB}{dT} + \left(\frac{\alpha_s}{2\pi}\right)^3\frac{dC}{dT}.$$

The experimentally measured thrust distribution

$$\frac{1}{\sigma_{\rm had}}\frac{d\sigma}{dT}$$

can then be expressed in terms of A, B, C by expanding $\sigma_{\rm had}$ around σ_0. If the renormalization scale of α_s is chosen to be $\mu \neq Q$, additional terms proportional to powers of $\ln(\mu^2/Q^2)$ appear, which are again expressed in terms of A, B, C and of the coefficients of the QCD β-function.

In the numerical evaluation, we use $M_Z = 91.1876$ GeV and $\alpha_s(M_Z) = 0.1189$ [8]. Figure 1 displays the perturbative expression for the thrust distribution at LO, NLO, and NNLO, evaluated for $\mu = Q = M_Z$. It can be seen that inclusion of the NNLO corrections enhances the thrust distribution by around (15–20)% over the range $0.03 < (1 - T) < 0.33$. Outside this range, one does not expect the perturbative fixed-order prediction to yield reliable results. For $(1 - T) > 0.33$, the leading-order prediction vanishes due to kinematical constraints from having only three partons in the final state; at NLO, $(1 - T) > 0.42$ is kinematically forbidden, and the spherical limit $T \to 0.5$ is reached only for infinitely many partons in the final state. For $(1 - T) \to 0$, the convergence of the perturbative series is spoiled by powers of logarithms $\ln(1 - T)$ appearing in higher perturbative orders, thus necessitating an all-order resummation of these logarithmic terms [3,4], and a matching of fixed-order and resummed predictions [33].

To estimate the theoretical error inherent to the perturbative prediction, we vary the renormalization scale in the interval $\mu \in [M_Z/2; 2M_Z]$. The relative uncertainty at each order

$$\delta = \frac{\max_\mu[\sigma(\mu)] - \min_\mu[\sigma(\mu)]}{2\sigma(\mu = M_Z)}$$

is displayed in Fig. 2. It can be clearly seen how the inclusion of higher-order corrections stabilizes the prediction, and the uncertainty δ is reduced by about 30% between NLO and NNLO. The increase in uncertainty above $1 - T = 0.33$ is due to the vanishing of the

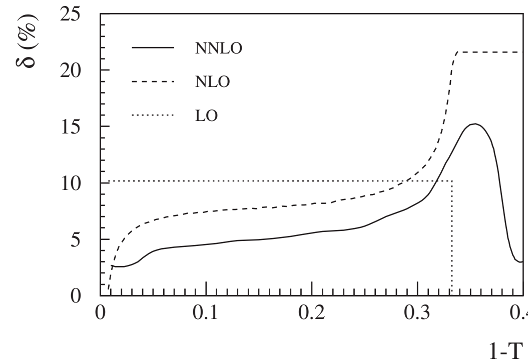

FIG. 2. Relative scale-uncertainty on thrust distribution at different orders in perturbation theory.

leading-order contribution; the perturbative fixed-order description and its theoretical error become unreliable beyond this point.

In Fig. 3, we compare the theoretical NNLO prediction for the thrust distribution to experimental data from the ALEPH experiment [34]. Similar measurements were carried out by all LEP experiments [35]. The NNLO correction is positive and for the same value of α_s, yields a prediction that is larger than at NLO. This indicates the need for an improved determination of α_s from event shape data, which takes the newly computed NNLO corrections into account. Work on this is ongoing.

To obtain a reliable description of the thrust distribution over the full kinematic range, it would be desired to match the NNLO results presented here to the resummed expressions in the two-jet limit, and to include hadronization

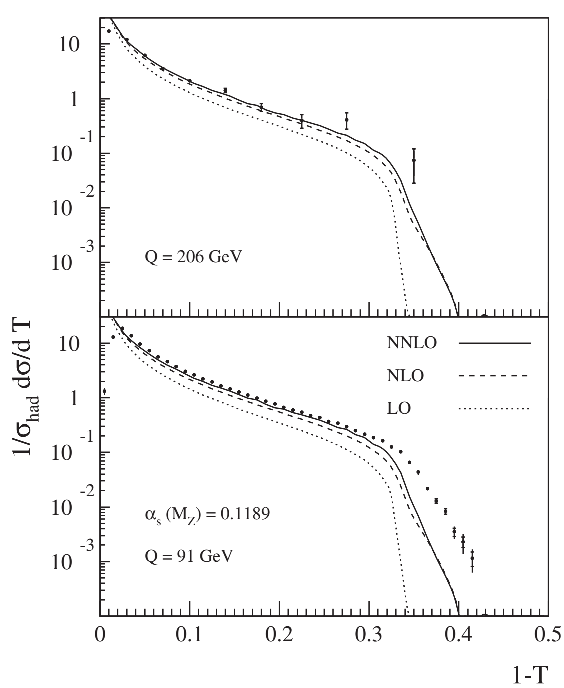

FIG. 3. NNLO thrust distribution compared to experimental data from ALEPH [34].

corrections. Both these improvements are beyond the scope of this letter, and will be addressed in later work.

In this Letter, we presented the first NNLO calculation of event shapes related to three-jet production in e^+e^- annihilation. We developed a numerical program which can compute any infrared-safe observable through to $\mathcal{O}(\alpha_s^3)$, which we applied here to determine the NNLO corrections to the thrust distribution. These corrections are moderate, indicating the convergence of the perturbative expansion. Their inclusion results in a considerable reduction of the theoretical error on the thrust distribution. Our results will allow a significantly improved determination of the strong coupling constant from jet observables.

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