Spin Chains and Channels with Memory

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In most studies of quantum channels, it is assumed that the errors in each use of the channel are independent. However, recent investigations of the effect of memory or correlations in error have led to speculation that nonanalytic behavior may occur in the capacity. Motivated by these observations, we connect the study of channels with correlated error to the study of many-body systems. This enables us to use many-body theory to solve some interesting models of correlated error. These models can display nonanalyticities analogous to quantum phase transitions.

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An important problem in quantum information is the determination of the channel capacity of noisy quantum channels [1]. In a typical scenario, we wish to send information over many uses of a noisy quantum channel. At a cost of lowering the information content per particle, it can be shown that quantum error correction can essentially eliminate all errors. This leads to quantum analogues of the channel capacity—the optimal rate at which information may be transferred with vanishing error in the limit of many uses. In this work, we will be concerned with the capacity for transmission of quantum information [2], which we denote $Q(\mathcal{E})$ for a channel $\mathcal{E}$.

Most works on quantum channels assume that the noise is independent in successive transmissions. However, this is never exactly true, and in many realistic systems there can be correlations in the noise. Such channels are termed "memory channels," and their capacity may be significantly affected by memory effects (see [3] for a recent experiment). Consequently, quantum memory channels have received considerable attention recently (see, e.g., [4–7] and references therein). In most models that have been considered explicitly (e.g., [5]), the correlations in the noise are modeled by a small number of memory parameters. Initial investigations of prototype models [5,6] have suggested that the capacity may undergo sharp, nonanalytic changes at certain values of these parameters. These investigations have not been conclusive, however, as with current techniques these models cannot be analyzed in the relevant setting of a very large (in fact, infinite) number of channel uses.

The main aim of this work is to show that, for a variety of interesting models, this obstacle can be overcome by relating the study of memory channels to the study of many-body physics. An advantage of the framework that we introduce is that the criticality of the underlying many-body systems can manifest itself as nonanalytic behavior in the channel capacity of the corresponding memory channels, thus proving the conjectured existence of such effects in memory channels. This suggests, as demonstrated here, that the methods of many-body theory may be used in the study of memory channels to obtain new results. In the first part of the Letter, we discuss a general framework for the channels that we consider, discussing how a capacity formula may be derived. We then provide an explicit bound applicable to random unitary [8] channels in terms of a thermodynamic quantity. In the case of dephasing channels that are random applications of orthogonal dephasing unitaries, this is exact.

Mapping many-body systems to memory channels.—A standard way of describing noise is to assume that each transmitted "system" particle interacts via a unitary U with its own environment. In order to introduce memory effects, we will modify this approach by asserting that the environment particles are initially prepared in the thermal or ground state of an interacting many-body Hamiltonian. The spatial correlations in the environment then lead to correlations in the noise. We will assume that, once the environment has been defined by the parent Hamiltonian, no further dynamics occur other than the system-environment interaction. It is important to note that many of the noise models considered in the literature [4–7] can be reexpressed in precisely this way.

For the case of a memoryless quantum channel, it has been shown [2] that the quantum capacity given by

$$Q(\mathcal{E}) = \lim_{n\to\infty} \frac{I(\mathcal{E}^{\otimes n})}{n}, \tag{1}$$

where $\mathcal{E}$ is the channel acting on a single transmission, and $I(\mathcal{E}) := \sup_\rho\{S[\mathcal{E}(\rho)] - S[I \otimes \mathcal{E}(|\psi\rangle\langle\psi|)]\}$ is the coherent information of the quantum channel $\mathcal{E}$, where S denotes the von Neumann entropy, ρ is a state, and $|\psi\rangle\langle\psi|$ is a purification of ρ. Finally, $\mathcal{E}^{\otimes n}$ represents the uncorrelated channel that acts on n inputs. For channels with correlations, the channel on n inputs $\mathcal{E}_n$ differs from $\mathcal{E}_1^{\otimes n}$, and one has to describe the memory channel by a sequence of channels $\{\mathcal{E}_n\}$, describing the action of the channel for each number of inputs n. Given that Eq. (1) is the memoryless quantum capacity, one may anticipate that

$$Q(\{\mathcal{E}_n\}) := \lim_{n\to\infty} \frac{I(\mathcal{E}_n)}{n} \tag{2}$$

is the quantum capacity of a memory channel. While this will certainly not be true in general, Eq. (2) is always an upper bound on the channel capacity, and one can derive conditions for equality which are often satisfied.

Condition for Eq. (2).—This section may be omitted by readers not concerned with detailed proofs. We outline the derivation of two conditions on the many-body system which are sufficient to demonstrate that Eq. (2) is the quantum capacity. These conditions are satisfied by a variety of many-body systems, including matrix product states [9] and quasifree bosonic systems [10]. The conditions are derived as follows. We consider a translation invariant chain of length N, split into $v = N/(l+s)$ sections, each consisting of one live block of length l and one spacer block of length $s := \delta l \ll l$. We define two related channels: the live channel and the product channel. The live channel is $\mathcal{E}_{\mathrm{live}} := A \to \mathrm{Tr}_{\mathrm{env}}\{U(\rho_{L_1L_2\ldots L_v} \otimes A)U^\dagger\}$, where A is Alice's input, U is the interaction between chain and input, the $L_1, \ldots, L_v$ label the live blocks, and the environment is traced out. The product channel is defined as $\mathcal{E}_{\mathrm{product}} := A \to \mathrm{Tr}_{\mathrm{env}}\{U((\rho_N^l)^{\otimes v} \otimes A)U^\dagger\}$, where ρ_N^l is the reduced state of an individual live block. The next three steps labeled (A), (B), and (C) construct an analogue of the arguments made in Ref. [7] for forgetful channels. (A) Showing that product classical codes are good classical codes for the live channel. Given an achievable rate R for the product channel, then $\forall\, \epsilon > 0$: $\exists N_\epsilon$ such that for $n > N_\epsilon$ channel uses there is a nl-qubit code $\{\rho_i\}_{1,\ldots,\nu}$ and decoding measurement $\{M_i\}_{1,\ldots,\nu}$ for $\nu = \lfloor 2^{nlR} \rfloor$ such that $\forall\, i$: $\mathrm{Tr}\{\mathcal{E}_{\mathrm{product}}(\rho_i)M_i\} \geq 1 - \epsilon$. The same procedure used for the live channel then yields, via the triangular inequality, $\mathrm{Tr}\{\mathcal{E}_{\mathrm{live}}(\rho_i)M_i\} \geq 1 - \epsilon - \frac{1}{2}||\rho_{L_1L_2\ldots L_v} - (\rho_N^l)^{\otimes v}||_1$. This leads to our first condition on the many-body system: It turns out that one can ensure that this error vanishes, for a choice $v = l^5$ [11], provided that it can be shown that $||\rho_{L_1L_2\ldots L_v} - (\rho_N^l)^{\otimes v}||_1 \leq Cvl^E \exp(-Fs)$ for positive constants C, E, F (see [10]). Hence, if this condition holds, the classical codes for product channels are also good codes for the live correlated channel. (B) Computing achievable classical rates. The product channel with live block length l and a total number of spins $N = v(l+s) = l^6(1+\delta)$ has a Holevo quantity given by $\chi(\mathcal{E}_N^l) = \chi(\mathrm{Tr}_{\mathrm{env}}\{U[(\rho_N^l) \otimes \bullet]U^\dagger\})$, where the $\bullet$ acts as a place holder for the channel input and where $\mathcal{E}_X^j$ denotes the effect of the full channel upon a contiguous subset of $j \leq X$ of the input spins. As in [7], we must now understand when this expression converges to the regularized Holevo bound of the full memory channel as $l \to \infty$. Suppose that we have a spin chain of total length $l + \Delta(l)$, where $\Delta(l) > 0$ is any function such that $\lim_{l\to\infty}\Delta(l)/l = 0$. Using subadditivity and the Araki-Lieb inequality, we find $\chi(\mathcal{E}_{l+\Delta}^l) \geq \chi(\mathcal{E}_{l+\Delta}) - 2\Delta\log(d)$. Now we need to show under which conditions this remains true if the subset of l spins is drawn from a much longer chain of length $N = l^6(1+\delta)$. For any input ω to the live block in question, the output states will differ by at most $||\mathrm{Tr}_{\mathrm{env}}\{U[\omega \otimes (\rho_{l+\Delta}^l - \rho_{l^6(1+\delta)}^l)]U^\dagger\}||_1 \leq ||U[\omega \otimes (\rho_{l+\Delta}^l - \rho_{l^6(1+\delta)}^l)]U^\dagger||_1 \leq P(l, \Delta) := ||\rho_{l+\Delta}^l - \rho_N^l||_1$. Combining this with $\chi(\mathcal{E}_{l+\Delta}^l) \geq \chi(\mathcal{E}_{l+\Delta}) - 2\Delta\log(d)$ and the Fannes inequality yields $\lim_{l\to\infty}\chi(\mathcal{E}_{l^6(1+\delta)}^l)/l \geq \chi_\infty - \lim_{l\to\infty}2P[l, \Delta(l)] \times \log(d)$. This gives the second condition on the many-body system: If we can pick $\Delta(l)$ such that $\lim_{l\to\infty}\Delta(l)/l = 0$ and $\lim_{l\to\infty}P[l, \Delta(l)] = 0$, then the regularized Holevo quantity is the correct classical capacity. (C) Coherentification. The final step is to argue that the above arguments for classical coding can be "coherentified" [2] into a quantum code. This analysis does not give new conditions on the many-body system and can be conducted as in Ref. [7] (see [10] for details).

Explicit computation of capacities.—Even if the many-body system can be well understood, the explicit computation of the capacity may still be difficult, as it depends upon the interaction of each system with its environment U. Judicious choice of U will allow us to obtain analytically solvable models. To this end, we will choose U to be of the form of a controlled-PHASE gate, denoted by U_z, where the environmental particles act as controls. In this case, it becomes possible to write down explicit formulas for Eq. (2) in terms of properties of the many-body environment that share a close relationship with thermodynamical quantities. For all other random unitary noise, the approach leads to lower bounds, although they are not always exact—see the concluding section for discussion. For d-dimensional systems, the controlled-PHASE gate is defined as $U_z = \sum_{k=1}^d |k\rangle\langle k| \otimes Z(k)$, where $Z(k) := \sum_r \exp(2\pi i kr/d)|r\rangle\langle r|$, and the first tensor factor acts on the environment. This interaction leads to channels that are probabilistic applications of $Z(k)$ unitaries on the system particles, with the (correlated) probabilities determined by the diagonal elements of the environment state. Now Eq. (2) can be written as

$$Q(\{\mathcal{E}_n\}) = \log d - \lim_{n\to\infty} \frac{S[\mathrm{Diag}(\rho_{\mathrm{env}})]}{n}, \tag{3}$$

where $\mathrm{Diag}(\rho_{\mathrm{env}})$ is the state obtained by eliminating all off-diagonal elements of the state of the environment in the computational basis. Hence, computing the capacity of our channel $\{\mathcal{E}_n\}$ reduces to computing the regularized diagonal entropy of the environment. Although this is unlikely to be generally computable, it is amenable to a great deal of analysis using many-body theory.

Proof sketch of Eq. (3).—The proof utilizes the Choi-Jamiolkowski representation of the quantum channel. Given any quantum operation $\mathcal{E}$ acting upon a d-level quantum system, one may form the corresponding Choi-Jamiolkowski (CJ) state $J(\mathcal{E}) = I \otimes \mathcal{E}(|+\rangle\langle+|)$, where $|+\rangle = (1/\sqrt{d})\sum_{i=1\ldots d}|ii\rangle$. The proof of Eq. (3) follows from three steps: (i) We argue that, for the kinds of channels we consider, a copy of $J(\mathcal{E})$ allows one to physically implement the channel exactly; (ii) for any channel that can be implemented using $J(\mathcal{E})$, we argue that the quantum channel capacity $Q(\mathcal{E})$ of the channel equals $D_1[J(\mathcal{E})]$, the one-way distillable entanglement of the state $J(\mathcal{E})$; (iii) we then use known results on D_1.

Step (1).—For simplicity, we describe the argument for channels $\mathcal{E}$ that are random applications of Pauli rotations on a single qubit. The argument generalizes easily to Pauli channels on many qubits. Suppose that you have $J(\mathcal{E})$ and you want to implement one action of $\mathcal{E}$ upon an input state ρ. This can be achieved by teleporting ρ through $J(\mathcal{E})$. This will leave you with a state $\mathcal{E}(\sigma_i \rho \sigma_i^\dagger)$, with the Pauli error σ_i depending upon the outcome of the teleportation measurement. As $\mathcal{E}$ is a random Pauli channel, we can now "undo" the error by applying the inverse of σ_i. Hence, we have $\sigma_i \mathcal{E}(\sigma_i \rho \sigma_i^\dagger)\sigma_i = \mathcal{E}(\rho)$, and we have implemented one action of $\mathcal{E}$.

Step (2).—Our aim is to show that, for channels that may be physically implemented using $J(\mathcal{E})$, the one-way distillable entanglement of $J(\mathcal{E})$, $D_1[J(\mathcal{E})]$, is equivalent to $Q(\mathcal{E})$. In Ref. [12], it was essentially shown that $Q(\mathcal{E}) \geq D_1[J(\mathcal{E})]$. For the converse inequality, consider the specific protocol: (a) Alice prepares many perfect EPR pairs and encodes one half according to the code that achieves the quantum capacity $Q(\mathcal{E})$. (b) She teleports the encoded qubits through the copies of $J(\mathcal{E})$, informing Bob of the outcome so that he can undo the effect of the Paulis. (c) This effectively implements the channel $\mathcal{E}$ between Alice and Bob. (d) Bob decodes the optimal code, thereby sharing perfect EPR pairs with Alice, at the rate determined by $Q(\mathcal{E})$. As this is a specific one-way distillation protocol, this means that $Q(\mathcal{E}) \leq D_1[J(\mathcal{E})]$. These arguments extend straightforwardly to any channel that is a mixture of Paulis on many qubits.

Step (3).—The CJ states of our channel are the so-called maximally correlated state, for which the distillable entanglement is known and is given by the Hashing bound [13] $D_1[J(\mathcal{E})] = S[\mathrm{Tr}_B\{J(\mathcal{E})\}] - S[J(\mathcal{E})]$, where S is the von Neumann entropy. Hence, for such channels $\mathcal{E}$, this expression is also the single copy coherent information. In our cases, we are interested in the regularized value of this quantity, which is given by Eq. (3). This expression for the coherent information has an interesting interpretation for the dephasing interactions that we consider—it represents the classical information lost to the environment that is needed to correct the errors [14].

The simplicity of Eq. (3) enables one to immediately write down many models for which Eq. (2) can both be calculated and also represents the quantum capacity.

Classical environments.—We discuss briefly two cases. If the environment consists of classical systems described by a classical Markov chain, then in a large number of cases Eq. (3) can be written explicitly with a simple expression that represents the entropy rate of the Markov chain [15]. Related results on Markov chain models have been obtained using different methods in Ref. [16]. In general, a classical environment is represented by a diagonal state, and the second term of Eq. (3) is precisely the entropy. Hence, in this case the capacity becomes

$$Q(\{\mathcal{E}_n\}) = 1 - \log_2(e)\left(1 - \beta\frac{\partial}{\partial\beta}\right)\lim_{n\to\infty}\frac{1}{n}\ln Z_n, \qquad (4)$$

where Z_n is the partition function for n environment spins, $\beta = 1/k_B T$, and the $\log_2(e)$ converts from nats to bits. Equation (4) shows that we can now use results from classical statistical physics to compute the capacity—any classical spin-chain models with sufficiently decaying correlations that can be solved exactly will lead to memory channels that can be "solved exactly."

Quantum environments.—For quantum environments, Eq. (3) represents the entropy that results when every environment qubit is completely dephased. Although this quantity is not standard in statistical physics, we expect that it may be amenable to the techniques of many-body theory. Here we provide support for this claim by solving analytically a class of quantum environments inspired by recent work on matrix product states (MPSs) [17]. For MPSs, the two conditions required to prove Eq. (2) are satisfied except at transition points [9].

In Ref. [18], it was shown that there are Hamiltonians that exhibit quantum phase transitions and have ground states that are MPSs involving only rank-1 matrices. We will now show that for MPSs involving rank-1 matrices a full analytical treatment of the quantum channel capacity of the associated memory channel becomes possible. To this end, we demonstrate that the diagonal elements of such rank-1 MPSs are given by the probabilities of microstates in related classical Ising chains. For simplicity, we will focus on a translationally invariant MPS for a 1D system of 2-level particles with periodic boundary conditions. Generalization to other rank-1 MPSs is straightforward. Such an environment state is characterized by two matrices Q_0 and Q_1 and is given by $|\psi\rangle = \sum_{i_1,\ldots,i_N}\mathrm{Tr}\{Q_{i_1}\ldots Q_{i_N}\}\times |i_1\ldots i_N\rangle$. The unnormalized state resulting from dephasing each qubit is

$$\rho = \sum_{i_1,\ldots,i_N}\mathrm{Tr}\left\{\prod_{k=1}^{N}(Q_{i_k}\otimes Q_{i_k}^*)\right\}|i_1\ldots i_N\rangle\langle i_1\ldots i_N|. \qquad (5)$$

Relabeling the matrices $A_i = Q_i \otimes Q_i^*$, the diagonal elements in the computational basis are of the form $\mathrm{Tr}\{\prod_k A_k\}$. As the A_i are both rank-1 with unique nonzero eigenvalues a_i, the normalized matrices $\tilde{A}_i = A_i/a_i$ satisfy $\tilde{A}_i^2 = \tilde{A}_i$. Using this idempotency, it is easy to show that, if $|i_1\ldots i_N\rangle$ has l occurrences of 0 and $N-l$ occurrences of 1, and K boundaries between blocks of 0s and blocks of 1s, then the corresponding diagonal element of ρ will be $p(l, n-l, K) = (a^l b^{N-l})\mathrm{Tr}\{(\tilde{A}_0\tilde{A}_1)^K\}/C(N)$, where $C(N)$ is a normalization factor. Noting that $\tilde{A}_0\tilde{A}_1$ is also rank-1, denote its nonzero eigenvalue by c. Hence, the diagonal elements are $p(l, n-l, K) = a^l b^{N-l} c^K/C(N)$. Hence, for channels described by rank-1 MPSs, a, b, and c are the only relevant

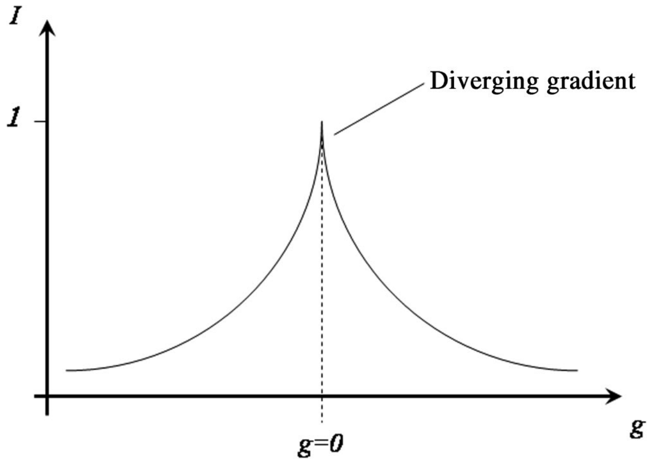

FIG. 1. A sketch of the capacity for an environment that is the ground state of the Hamiltonian $\sum_i 2(g^2-1)\sigma_z^i\sigma_z^{i+1} - (1+g)^2\sigma_x^i + (g-1)^2\sigma_z^i\sigma_x^i\sigma_z^{i+1}$ [18]. The plot's symmetry is expected as the channel is invariant under $g \to -g$. However, near the phase transition $g = 0$, the gradient diverges.

parameters, and we may choose any matrices with those parameters. We choose the matrices of a classical (i.e., diagonal) chain:

$$A_0 = \begin{pmatrix} a & \sqrt{cab} \\ 0 & 0 \end{pmatrix}; \qquad A_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{cab} & b \end{pmatrix}. \tag{6}$$

Here we have assumed that $c > 0$ (which is guaranteed for $N > 5$, as otherwise the state can become nonpositive). It is then easy to check that these matrices have the correct values of a, b, and c, as required. We choose this form because the matrices are essentially the top row and bottom row of a transfer matrix [19] corresponding to a classical Ising chain. Roughly speaking, the parameter c encodes the coupling between adjacent antiparallel spins, and the a and b encode the coupling between adjacent parallel spins.

This connection implies that the limit in Eq. (3) can be computed easily using well-known methods [19]. Figure 1 shows the result for a Hamiltonian presented in Ref. [18] for which the ground state is known to be a rank-1 MPS possessing a nonstandard "phase transition" at $g = 0$ [20], at which some correlation functions are continuous but nondifferentiable, while the ground state energy is actually analytic [18].

Figure 1 shows that this is mirrored in the nonanalyticity of the channel capacity.

Generalizations and future work.—It is important to know whether our approach could prove useful for other interactions. Some generalizations are immediate. For instance, given any channels that are probabilistic applications of unitaries, expression (3) can easily be shown to be an explicit lower bound to the coherent information, and hence, if the environment state has sufficiently decaying correlations, it will also be a lower bound to the channel capacity. It is likely that any channel whose capacity can be bounded by such a simple entropic expression will benefit from similar insights. In the long term, one might speculate that there may be a deeper explanation for these connections—not in terms of entropic expressions appearing in both fields, but in terms of a link between coding and many-body physics.

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