Ideal Cylindrical Cloak: Perfect but Sensitive to Tiny Perturbations

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A cylindrical wave expansion method is developed to obtain the scattering field for an ideal two-dimensional cylindrical invisibility cloak. A near-ideal model of the invisibility cloak is set up to solve the boundary problem at the inner boundary of the cloak shell. We confirm that a cloak with the ideal material parameters is a perfect invisibility cloak by systematically studying the change of the scattering coefficients from the near-ideal case to the ideal one. However, because of the slow convergence of the zeroth-order scattering coefficients, a tiny perturbation on the cloak would induce a noticeable field scattering and penetration.

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The exciting issue of exotic materials invisible to electromagnetic (EM) waves was discussed in recent works [1–11]. Based on a coordinate transformation of Maxwell's equations, Pendry *et al.* first proposed an invisibility cloak, which can protect objects inside the cloak from detection [1]: When EM waves pass through the invisibility cloak, the cloak will deflect the waves, guide them around the object, and return them to the original propagation direction without perturbing the exterior field. Numerical methods have been applied to solve the EM problem involving invisibility cloaks [6,9], and an experimental result of the invisibility cloak using metamaterial with simplified material parameters has also recently been reported [7]. Yet, the ideal invisibility cloak has not been confirmed as a perfect cloak, due to the extreme material parameters required (zero or infinity) in the ideal cloak when approaching the inner boundary. Also, numerical methods usually describe the material parameters discretely, which can be computationally intensive in extreme cases. Thus it is preferable to use an analytical or semianalytical method whenever possible.

In this Letter, we will study the scattering for an ideal invisibility cloak. We focus our analysis on the 2D cylindrical cloak, because the wave equation can be simplified in comparison with the 3D case, and a 2D invisibility cloak is more feasible to fabricate [7]. Here we take advantage of the cylindrical geometry of the structure and use the cylindrical wave expansion method to study the device semianalytically. To avoid extreme values (zeros or infinity) of material parameters at the cloak's inner surface, we introduce a small perturbation into the ideal cloak, and allow the perturbation to approach zero to study the scattering problem for the ideal cloak. Such an asymptotic analysis not only can confirm whether the ideal cloak would be perfectly invisible or not, it also provides hints on how sensitive such a device is to finite perturbations. A sensitivity analysis of the invisibility cloak directly determines the possibility of its application. Our studies show that the cylindrical invisibility cloak is very sensitive to tiny perturbations of the material parameters.

First, let us look at the wave equation inside a cylindrical cloak. According to Ref. [1], a simple transformation

$$r' = \frac{b-a}{b} r + a, \qquad \theta' = \theta, \qquad z' = z \tag{1}$$

can compress space from the cylindrical region $0 < r < b$ into the annular region $a < r' < b$, where a is the inner radius of the cloak, b is the outer radius of the cloak, and r, θ, and z (r', θ', and z') are the radial, angular, and vertical coordinates in the original (transformed) system, respectively. Following the approach in Ref. [1], the permittivity and permeability tensor components for the cloak shell can be given as

$$\begin{gathered} \varepsilon_r = \mu_r = \frac{r-a}{r}, \qquad \varepsilon_\theta = \mu_\theta = \frac{r}{r-a}, \\ \varepsilon_z = \mu_z = \left(\frac{b}{b-a}\right)^2 \frac{r-a}{r}, \end{gathered} \tag{2}$$

and air is assumed for the ambient environment and the interior regions. In the following, the transverse-electric (TE) polarized electromagnetic field is considered (i.e., the electrical field only exists in the z direction); however, the transverse-magnetic derivation follows in a similar manner. Throughout the Letter, a $\exp(-i\omega t)$ time dependence is assumed. For the TE-polarized wave, only ε_z, μ_r, and μ_θ are relevant to the following general wave equation governing the E_z field in the cloak's cylindrical coordinate

$$\frac{1}{\varepsilon_z r}\frac{\partial}{\partial r}\left(\frac{r}{\mu_\theta}\frac{\partial E_z}{\partial r}\right) + \frac{1}{\varepsilon_z r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\mu_r}\frac{\partial E_z}{\partial \theta}\right) + k_0^2 E_z = 0, \tag{3}$$

where k_0 is the wave vector of light in vacuum. If we substitute Eq. (2) for ε_z, μ_r, and μ_θ, we find

$$r^2\frac{\partial^2 E_z}{\partial r^2} + r\mu_\theta\frac{\partial E_z}{\partial r} + \varepsilon_z\mu_\theta r^2 k_0^2 E_z + \frac{\mu_\theta}{\mu_r}\frac{\partial^2 E_z}{\partial \theta^2} = 0. \tag{4}$$

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Equation (4) can be solved by a separation of variables $E_z = \Psi(r)\Theta(\theta)$ and the introduction of a constant l:

$$(r-a)^2\frac{\partial^2\Psi}{\partial r^2} + (r-a)\frac{\partial\Psi}{\partial r} + \left[\left(\frac{b}{b-a}\right)^2(r-a)^2k_0^2 - l^2\right]\Psi = 0, \quad (5)$$

$$\frac{\partial^2\Theta}{\partial\theta^2} + l^2\Theta = 0. \quad (6)$$

Equation (5) is the l-order Bessel differential equation, and the general solution of Eq. (6) is $\exp(il\theta)$. Therefore, there exists a simple set of solutions to E_z in the cloak shell of the form

$$F_l(k_1(r-a))\exp(il\theta) \quad (7)$$

where $k_1 = k_0 b/(b-a)$, F_l is the l-order Bessel function, and l is an integer number as required by the rotational boundary condition.

Let us consider the scattering problem in which an arbitrary wave is incident on the cloak. According to the rigorous scattering theory [12], the incident field in the 2D case can be expanded in the cloak's coordinates with the following expression:

$$E_z^{\rm in} = \sum_l \alpha_l^{\rm in} J_l(k_0 r)\exp(il\theta), \quad (8)$$

where J_l is the lth-order Bessel function of the first kind. The scattering field can also be expanded as

$$E_z^{\rm sc} = \sum_l \alpha_l^{\rm sc} H_l(k_0 r)\exp(il\theta), \quad (9)$$

where H_l is the l-order Hankel function of the first kind.

We note that the scattering coefficients cannot be directly obtained for the ideal cloak since $\varepsilon_z \to 0$, $\mu_r \to 0$, and $\mu_\theta \to \infty$ when $r \to a$, and the Bessel function of the second kind in Eq. (7) has a singularity at $r = a$. In order to circumvent this, we introduce a small perturbation to the ideal cloak which we refer to as the near-ideal cloak; see Fig. 1. We expand the inner boundary of the cloak shell slightly, so that it is located at $r = a + \delta$, where δ is a very small positive number. However, the material parameters are still calculated according to Eq. (2) as if the inner boundary is unchanged. The outer boundary remains fixed at $r = b$. When $\delta \to 0$, our model will be equivalent to the ideal cloak. Now the electric field in each region can be given by

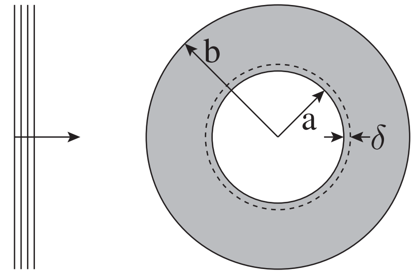

FIG. 1. The schematic of a near-ideal invisibility cloak: The distribution of the material parameter is the same as the ideal one shown in Eq. (2), and the outer boundary is still fixed at $r = b$. However, the actual inner boundary is at $r = a + \delta$, where δ is a very small positive number.

$$\begin{aligned}(b<r)E_z &= \sum_l \alpha_l^{\rm in} J_l(k_0 r)\exp(il\theta) + \alpha_l^{\rm sc} H_l(k_0 r)\exp(il\theta)\\ (a+\delta<r<b)E_z &= \sum_l \alpha_l^1 J_l(k_1(r-a))\exp(il\theta) + \alpha_l^2 H_l(k_1(r-a))\exp(il\theta)\\ (r<a+\delta)E_z &= \sum_l \alpha_l^3 J_l(k_0 r)\exp(il\theta)\end{aligned} \quad (10)$$

where $\alpha_l^i (i = 1, 2, 3)$ are the expansion coefficients for the resulting field inside the cloak.

The tangential fields E_z and H_θ (which can be obtained from E_z), should be continuous across the interfaces at $r = a + \delta$ and $r = b$, and the orthogonality of $\exp(il\theta)$ allows waves in each Bessel order to decouple. Thus, we can have the following four equations:

$$\alpha_l^{in} J_l(k_0 b) + \alpha_l^{sc} H_l(k_0 b) = \alpha_l^1 J_l(k_1(b-a)) + \alpha_l^2 H_l(k_1(b-a)) \quad (11a)$$

$$\alpha_l^1 J_l(k_1\delta) + \alpha_l^2 H_l(k_1\delta) = \alpha_l^3 J_l(k_0(a+\delta)) \quad (11b)$$

$$k_0\alpha_l^{\rm in} J_l'(k_0 b) + k_0\alpha_l^{\rm sc} H_l'(k_0 b) = \frac{k_1}{\mu_\theta(b)}\alpha_l^1 J_l'(k_1(b-a)) + \frac{k_1}{\mu_\theta(b)}\alpha_l^2 H_l'(k_1(b-a)) \quad (11c)$$

$$\frac{k_1}{\mu_\theta(a+\delta)}\alpha_l^1 J_l'(k_1\delta) + \frac{k_1}{\mu_\theta(a+\delta)}\alpha_l^2 H_l'(k_1\delta) = k_0\alpha_l^3 J_l'(k_0(a+\delta)) \quad (11d)$$

which is a set of linear equations. Thus each order expansion coefficient in each material region can be exactly solved. In turn we can obtain the fields in each region.

As a direct result of this set linear equations, we can prove that when $\delta \to 0$, $\alpha_l^{\rm sc} = \alpha_l^2 \to 0$, $\alpha_l^1 = \alpha_l^{\rm in}$, and $\alpha_l^3 \to 0$ for any $\alpha_l^{\rm in}$; i.e., the ideal cloak is a perfect invisibility cloak. First, it can be assumed that $|\alpha_l^i (i = {\rm sc}, 1, 2, 3)|$ must be finite. Otherwise, the scattering field would be infinite if the incident field has the lth-order component. Second, due to $k_1(b-a) = k_0 b$ and $k1 = k_0\mu_\theta(b)$, when $\delta \to 0$, Eqs. (11a) and (11c) become $(\alpha_l^{\rm in} - \alpha_l^1)J_l(k_0 b) + (\alpha_l^{\rm sc} - \alpha_l^2)H_l(k_0 b) = 0$ and $(\alpha_l^{\rm in} - \alpha_l^1)J_l'(k_0 b) + (\alpha_l^{\rm sc} - \alpha_l^2)H_l'(k_0 b) = 0$, respectively. Since b can be arbitrary and the Bessel functions are not always zeros, $\alpha_l^{\rm in} = \alpha_l^1$ and $\alpha_l^{\rm sc} = \alpha_l^2$ must be satisfied. Third, from Eq. (11b), we can obtain the following inequality:

$$|\alpha_l^2 H_l(k_1\delta)| \leq |\alpha_l^3 J_l(k_0(a+\delta))| + |\alpha_l^1 J_l(k_1\delta)|. \quad (12)$$

When $\delta \to 0$, the right side of the above inequality approaches a finite value but $|H_l(k_1\delta)|$ approaches infinity on the left side. Thus, $|\alpha_l^2|$ must approach zero. Finally, from Eq. (11b), we can also obtain that $|\alpha_l^2 H_l'(k_1\delta)| \leq |\alpha_l^3 \frac{k_0}{k_1} J_l'(k_0(a+\delta))| + |\alpha_l^1 J_l'(k_1\delta)|$. While from Eq. (11d), we have

$$|k_0\alpha_l^3 J_l'(k_0(a+\delta))| \leq \left|\frac{k_1}{\mu_\theta(a+\delta)}\alpha_l^1 J_l'(k_1\delta)\right| + \left|\frac{k_1}{\mu_\theta(a+\delta)}\alpha_l^2 H_l'(k_1\delta)\right|. \quad (13)$$

Since $\mu_\theta(a+\delta) \to \infty$ and the right side of the above inequality approaches zero when $\delta \to 0$, we obtain that $|\alpha_l^3| \to 0$. Consequently, this argument proves that the scattering field and the field in the interior region of the cloak are zero when $\delta = 0$; i.e., the ideal cloak is a perfect invisibility cloak.

Although we have just confirmed that the ideal cloak can provide perfect invisibility, further study of the near-ideal cloak by the above analytical method illuminates how sensitive the parameter δ is to the performance of the cloak. As an example, we use the same material parameters in Ref. [6] where the inner radius of the cloak is $a = 0.1$ m, the outer radius of the cloak is $b = 0.2$ m, and the frequency of the incident plane wave is 2 GHz. Similarly, we also consider a plane wave incident on the cloak, where the expansion coefficients in Eq. (8) are

$$\alpha_l^{\rm in} = i^l A \exp(-ik_0 r_1 \cos(\varphi + \theta_1) - il\varphi), \quad (14)$$

where $(r_1, \theta_1, 0)$ is the coordinate of the phase reference point, A is the amplitude of the plane wave, and φ is the incident angle [13]. Here the phase reference point is set at $r_1 = 4a$ and $\theta_1 = \pi$, the amplitude is $A = 1$, and the incident angle is $\varphi = 0$ (i.e., the plane wave propagates from left to right). We use 31 Bessel terms $(-15 \leq l \leq 15)$ to calculate the scattering field for the near-ideal cloak with $\delta = 10^{-5}a$. The number of expansion terms is sufficient for convergence of the calculated fields. Figure 2 shows the snapshot of the resulting electric-field distribution (i.e., the real part of the electric-field phasor), and the corresponding norm in the vicinity of the cloaked object. The electric-field distribution clearly demonstrates the cloaking effect of the near-ideal cloak to the incident plane wave. However, the norm of the electric-field [Fig. 2(b)] reveals that there is still a little bit of the field in the cloak interior and an obvious scattering ripple around the cloak. The amplitude of the resulting electric-field at the center is 0.197. The snapshot of the scattering field [Fig. 2(c)] shows that it propagates almost isotropically in all angles. Even though the amplitude of the scattering field is much smaller than that of the incident plane wave, the interference of the incident plane wave and the scattering field creates the ripples in the norm [Fig. 2(b)].

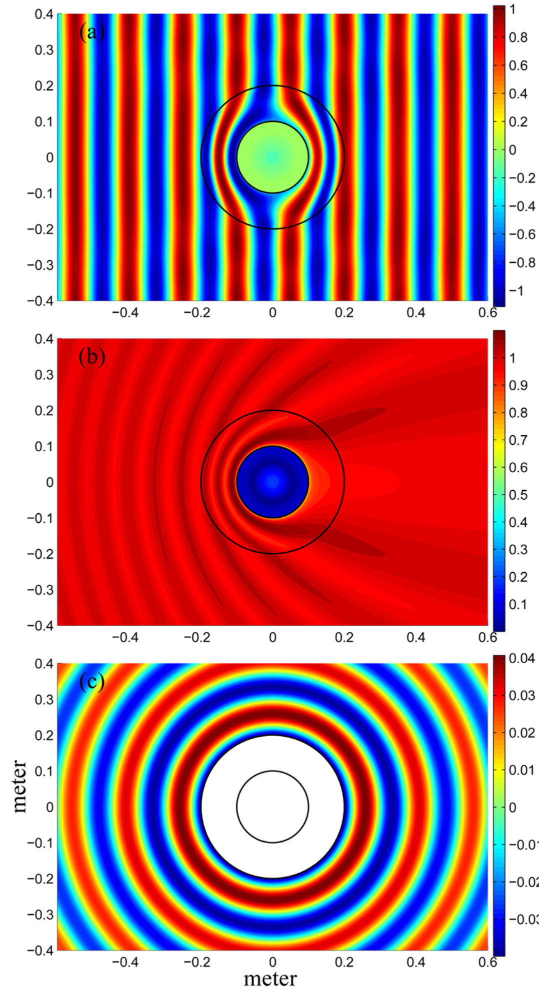

FIG. 2 (color online). (a) Snapshot of the resulting electric-field distribution, (b) the corresponding norm in the vicinity of the cloaked object, and (c) the snapshot of the corresponding scattering field outside the cloak for the near-ideal cloak with $\delta = 10^{-5}a$ and when a plane wave is perpendicularly incident on the cloak. The black lines outline the cloak shell. Axis unit: meter.

Since each order expansion coefficient of the scattering field is only relevant to each order expansion coefficient of the incident field [cf. Eq. (11)], we can define the scattering coefficient for each order as

$$c_l^{\rm sc} = \frac{\alpha_l^{\rm sc}}{\alpha_l^{\rm in}}. \quad (15)$$

These coefficients for the field inside the cloak $c_l^{(i)} = \alpha_l^i/\alpha_l^{\rm in}$, $i = 1, 2, 3$ can also be defined in the same way. To study the ideal cloak, we move δ closer to 0. The amplitude and the phase of these coefficients for $10^{-8}a < \delta < 10^{-2}a$ are shown in Fig. 3, where (a),(b) and (c),(d) correspond to the cases of $l = 0$ and $l = 1$, respectively.

From Fig. 3, it is clear that $c_l^{(1)}$ is always equal to 1 for both cases. That is, the incident field propagates into the

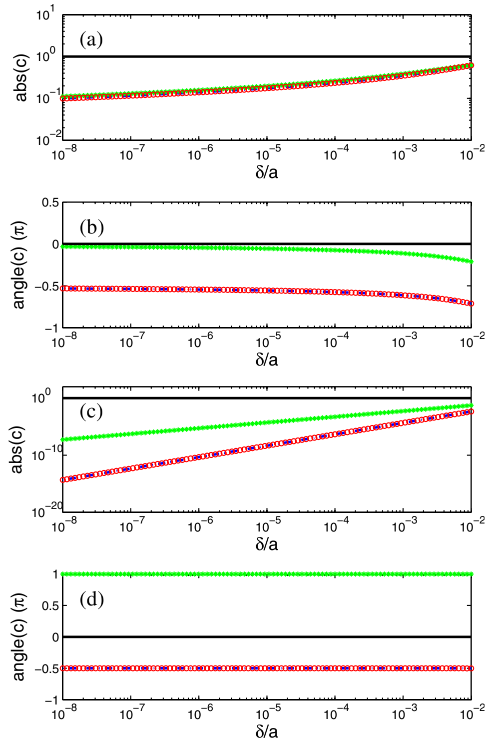

FIG. 3 (color online). The amplitude and phase of the scattering coefficients for the different δ, where (a),(b) and (c),(d) correspond to the cases of $l = 0$ and $l = 1$, respectively. $c_l^{\rm sc}$ is denoted by the blue (or dark gray) point-dashed line, $c_l^{(1)}$ (the solid black line), $c_l^{(2)}$ [the red (or gray) circle–marked], and $c_l^{(3)}$ [the green (or light gray) star–marked].

cloak without any reflection at the outer boundary, which coincides with the explanation of the cloaking effect from the coordination transformation approach [1]. The same behavior for the scattering fields occurs at the outer boundary, where they propagate from inside to outside without any reflection; thus, $c_l^{\rm sc}$ is always equal to $c_l^{(2)}$.

Our computational results also confirm that both $c_l^{\rm sc}$ and $c_l^{(3)}$ approach zero when $\delta \longrightarrow 0$. In particular, compared with the case of $l = 0$, $|c_l^{\rm sc}|$ and $|c_l^{(3)}|$ for $l = 1$ are much smaller, and approach zero more rapidly. This is also observed for the other higher order cases. Thus, in the case of the plane wave incident, where $|\alpha_l^{\rm in}|$ is the same for each order, the dominating term of the scattering field outside of the cloak is of the form of the zeroth-order Hankel function of the first kind. Meanwhile, the resulting field in the interior region has a dominating term of $J_0(k_0 r)$. This explains the near azimuthally invariable distribution of the field in the interior region [see Fig. 2(a)] and the scattering field outside the cloak [see Fig. 2(c)], which has also been mentioned in Ref. [6].

It is worth noting that the zeroth-order scattering coefficients $c_0^{\rm sc}$ and $c_0^{(3)}$ decrease extremely slowly with reduced δ; e.g., when δ is decreased from $10^{-5}a$ to $10^{-8}a$, $|c_0^{\rm sc}|$ decreased only from 0.175 to 0.099. By utilizing the arbitrary calculation precision of the software MATHEMATICA, we found that the convergence of the limit is so slow that even for $\delta = 10^{-99}a$ (i.e., $\varepsilon_z \approx 4 \times 10^{-99}$, $\mu_r \approx 10^{-99}$, and $\mu_\theta \approx 10^{99}$ at the inner boundary in this case), $|c_0^{\rm sc}| = 6.973 \times 10^{-3}$. Therefore, we conclude that even though a cloak with the ideal material parameters in Ref. [1] is a perfect cloak, a nonideal invisibility cloak does not provide a good enough cloaking effect due to the slow convergence of $|c_0^{\rm sc}|$ and $|c_0^{(0)}|$.

In conclusion, we have used the cylindrical wave expansion method to study the electromagnetic scattering properties of a 2D invisibility cloak. A near-ideal model of the invisibility cloak is set up to solve the boundary problem at the inner boundary of the cloak shell. By systematically studying the change of the scattering coefficients from the near-ideal case to the ideal one, we confirm that the cloak with the ideal material parameter is a perfect invisibility cloak. But due to the slow convergence of the scattering coefficients, a tiny perturbation on the cloak would induce a noticeable field scattering and penetration. We also proved that the scattered and penetrated fields are dominated by zeroth-order cylindrical waves. Though our work has focused on the 2D cylindrical cloak, it can be reliably extended to the 3D spherical case. Our method and results are also useful for either designing or detecting this type of the invisibility cloak.

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