# Dynamics of Excitations in a One-Dimensional Bose Liquid

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We show that the dynamic structure factor of a one-dimensional Bose liquid has a power-law singularity defining the main mode of collective excitations. Using the Lieb-Liniger model, we evaluate the corresponding exponent as a function of the wave vector and the interaction strength.

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Progress in the ability to manipulate ultracold atomic gases stimulates the interest in fundamental properties of one-dimensional (1D) Bose liquids [1]. The quantity characterizing the collective excitations in these systems, the dynamic structure factor (DSF), is now directly accessible experimentally using the Bragg spectroscopy technique [2]. The very first such measurements [3] clearly showed that the resonance in DSF is wider in 1D than it is in higher dimensions. The goal of this Letter is to elucidate the nature of the resonance in a 1D system of interacting bosons.

In the absence of interactions, bosons occupy the lowest-energy single-particle state at zero temperature. An external field that couples to the particle density would excite bosons from the ground state. The corresponding absorption spectrum reflects the free boson's dispersion relation $\epsilon(q)$. Accordingly, DSF at zero temperature is given by $S(q, \omega) \propto \delta(\omega - \epsilon(q))$.

In dimensions higher than one, this behavior remains largely intact even in the presence of interactions. Bosons still form a condensate, and excitations of the system are very well described in terms of Bogoliubov quasiparticles [4]. Interactions merely affect their spectrum: $\epsilon(q) \propto q$ at small $q$. The quasiparticle decay rate scales with $q$ as $1/\tau_q \propto q^5$ [4]; hence, the quasiparticle peak in $S(q, \omega)$ at $\omega = \epsilon(q)$ is well defined, $1/\tau_q \ll \epsilon(q)$.

In 1D, the effect of interactions is dramatic: quantum fluctuations destroy the condensate. Long-wavelength ($q \to 0$) excitations of a 1D Bose liquid are often described in hydrodynamic approximation [5] (see [6] for a recent review). However, the shape of the peak in DSF cannot be addressed using this approach: in hydrodynamics the peak has zero width.

In this Letter we study DSF of a 1D Bose liquid beyond the hydrodynamic approximation. We consider the Lieb-Liniger (LL) model [7]: $N$ identical spinless bosons with contact repulsive interaction placed on a ring with circumference $L$,

$$H = -\frac{1}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + c\sum_{i<j}\delta(x_i - x_j). \tag{1}$$

The model is integrable [7,8]. The integrability allows one to relate the parameters of the hydrodynamic description [5,6], the sound velocity $v$ and the parameter $K$, to the concentration $n = N/L$ and the dimensionless interaction strength $\gamma = mc/n$ [9]. Finding dynamic correlation functions, such as DSF, in a closed form remains a challenge [8]. The most impressive progress so far was achieved by combining a finite-$N$ numerics with the algebraic Bethe ansatz [10]. Here we study the singular behavior of DSF analytically.

DSF is defined by

$$S(q, \omega) = \int dxdt e^{i(\omega t - qx)}\langle \rho(x, t)\rho(0, 0)\rangle, \tag{2}$$

where $\rho(x) = \sum_i \delta(x - x_i)$ is the density operator. We show that DSF exhibits power-law singularities at the Lieb modes $\epsilon_{1,2}(q)$ [7,11]; see Fig. 1(a). In particular, DSF *diverges* at $\omega \to \epsilon_1(q)$ as

$$S(q, \omega) \sim \frac{m}{q}\left|\frac{\delta\epsilon}{\omega - \epsilon_1}\right|^{\mu_1}[\theta(\epsilon_1 - \omega) + \nu_1\theta(\omega - \epsilon_1)]; \tag{3}$$

see Fig. 1(b). Here $\delta\epsilon(q) = \min\{\epsilon_1 - \epsilon_2, vq\}$. Note that the divergence occurs within the continuum.

The exponent $\mu_1$ and the coefficient $\nu_1$ in Eq. (3) depend on the dimensionless momentum $Q = q/mc$ and the interaction strength $\gamma$. We were able to compute $\mu_1$ and $\nu_1$ in two limiting cases:

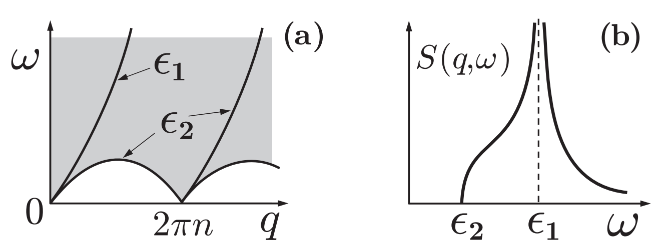


FIG. 1. (a) Shaded area indicates the region in $(\omega, q)$ plane where $S(q, \omega) \neq 0$ at zero temperature. DSF exhibits power-law singularities along the solid lines. (b) Sketch of the dependence of the structure factor on $\omega$ at a fixed $q < 2\pi n$.

 

$$\mu_1 = 1 - (2K)^{-1}, \qquad \nu_1 = 1 \tag{4}$$

for $Q \gg (\gamma K)^{-1} \sim \max\{1, \gamma^{-1/2}\}$ and arbitrary $\gamma$, and

$$\mu_1 = (\delta/\pi)(1 - \delta/2\pi), \qquad \nu_1 = \frac{\sin(\delta^2/4\pi)}{\sin(\delta - \delta^2/4\pi)} \tag{5}$$

for $\gamma \gg 1$ and arbitrary $Q$ (here $\delta = 2\arctan Q$). According to Eq. (5), $\mu_1 \approx 2Q/\pi$ at $Q \to 0$; we expect that $\mu_1 \propto Q$ at small $Q$ for any $\gamma$.

The line $\omega = \epsilon_1(q)$ has a "replica," $\omega = \epsilon_2(q)$, at $q > 2\pi n$. Here DSF does not diverge, but still has a power-law nonanalyticity. The singular part of DSF has the form $\delta S(q, \omega) \propto |\omega - \epsilon_2|^{\mu_2}$ with the exponent

$$\mu_2 = 2K + (2K)^{-1} - 1 \tag{6}$$

at arbitrary $\gamma$ and $Q \gg \max\{1, \gamma^{-1}\}$, and

$$\mu_2 = (\delta/\pi)(1 + \delta/2\pi) \tag{7}$$

at $\gamma \gg 1$ and arbitrary $Q$.

Lieb's holelike (according to Bethe-ansatz classification) mode $\epsilon_2(q)$ serves as the lower boundary of the support [11] of $S(q, \omega)$ at $q < 2\pi n$; see Fig. 1. Here DSF is given by

$$S(q, \omega) \sim \frac{m}{q}\left[\frac{\omega - \epsilon_2}{\delta\epsilon}\right]^{\mu_2}\theta(\omega - \epsilon_2). \tag{8}$$

For $\gamma \gg 1$ the exponent $\mu_2$ here is given by Eq. (7).

Equations (3)–(8) represent the main result of this Letter. The shape of $S(q, \omega)$ near $\omega = \epsilon_1(q)$, see Eq. (3), differs qualitatively from the Lorentzian quasiparticle peak in higher dimensions. In 1D, the collective mode is characterized by a power-law divergence of $S(q, \omega)$. This divergence is protected by the integrability and associated with it absence of three-particle collisions [8]. It is smeared only at a finite temperature $T$,

$$\max\{S(q, \omega)\}_{\text{fixed } q} \propto T^{-\mu_1(q)}, \qquad T \ll \delta\epsilon. \tag{9}$$

An apparent saturation of the height of the peak with the decrease of $T$ would provide a direct measure of three-particle scattering (absent in LL model) or recombination [12] rates.

In the remainder of the Letter we outline the derivation of the above results. We start with the limit of large $q$. Consider the state $\rho_q^\dagger|0\rangle$, where $\rho_q^\dagger = \sum_k \psi_{k+q}^\dagger \psi_k$ is the Fourier component of the density operator ($\psi_p^\dagger$ creates a boson with momentum $p$), and $|0\rangle$ is the ground state of the Bose liquid. Without interactions, all bosons in $|0\rangle$ occupy the single-particle state with $k = 0$. The operator $\rho_q^\dagger$ annihilates one such boson while creating another in the empty state with momentum $q$. With interactions present, the occupation number falls off [13] rapidly with $k$ at $k \gtrsim mv$. Therefore, for $q \gg mv$ the state $\rho_q^\dagger|0\rangle$ still contains a single particle at momentum close to $q$, as well as a "hole" in the quasicondensate with much smaller momentum. This observation suggests to approximate

$$\rho_q^\dagger \approx \int dx d^\dagger(x)\psi(x), \tag{10}$$

where $d^\dagger(x) = L^{-1/2}\sum_{|k|<k_0} e^{-ikx}\psi_{q+k}^\dagger$ creates a high-momentum particle and $\psi(x)$ creates a long-wavelength hole; here, $k_0 \sim mv$ is the high-momentum cutoff. The $d$ particle is described by the Hamiltonian

$$H_d = \int dx d^\dagger(x)[\epsilon_1(q) - iv_d\partial_x]d(x), \qquad v_d = q/m. \tag{11}$$

Here we took into account that $\epsilon_1(p) \approx p^2/2m$ at large $p$ [11] and linearized the dispersion relation around $p = q$. We treat the long-wavelength bosons in the conventional hydrodynamic approximation [5,6],

$$\psi(x) = [n + \pi^{-1}\partial_x\varphi]^{1/2}e^{i\vartheta(x)}. \tag{12}$$

The fields $\varphi$, $\vartheta$ obey $[\varphi(x), \vartheta(y)] = i(\pi/2)\,\mathrm{sgn}(x - y)$ and their dynamics is governed by the Hamiltonian

$$H_0 = \frac{v_0}{2\pi}\int dx\left[\frac{(\partial_x\varphi)^2}{K^2} + (\partial_x\vartheta)^2\right], \qquad v_0 = \frac{\pi n}{m}. \tag{13}$$

Equations (2), (10), and (12) yield DSF in the form

$$S(q, \omega) = \int dx dt e^{i\omega t}\langle B(x, t)B^\dagger(0, 0)\rangle \tag{14}$$

with $B^\dagger(x) \propto d^\dagger(x)e^{i\vartheta(x)}$. Evaluation of Eq. (14) with the quadratic Hamiltonian $H_d + H_0$ is straightforward and yields Eq. (3) with $\mu_1$ and $\nu_1$ given by Eq. (4). The decomposition Eq. (10) is applicable for $q \gg k_0 \sim mv$, hence the restriction on $Q$ in Eq. (4). On the other hand, the constraint $|k| < k_0$ on the momentum of $d$ particle limits the applicability of Eq. (3) to $|\omega - \epsilon_1(q)| \lesssim qv$.

We now extend the above derivation to the vicinity of the mode $\epsilon_2(q)$ at $q \gg \max\{2\pi n, n\gamma\}$. At these momenta, $\epsilon_2(q) = \epsilon_1(q - 2\pi n)$ is a replica of mode $\epsilon_1(q)$. At a given energy $\omega \approx \epsilon_2(q)$ the relevant excitation includes, in addition to $d$ particle, the $2\pi n$-momentum excitation of the quasicondensate Eq. (13). In hydrodynamics [5,6], such excitation corresponds to $\psi(x) \propto e^{i\vartheta(x) - 2i[\pi nx + \varphi(x)]}$ instead of Eq. (12). DSF is still given by Eq. (14) with $B^\dagger(x) \propto d^\dagger(x)e^{i\vartheta(x) - 2i[\pi nx + \varphi(x)]}$ and with the replacement $\epsilon_1 \to \epsilon_2$ in Eq. (11). Evaluation of Eq. (14) then yields a power law for the singular part of $S(q, \omega)$ with the exponent $\mu_2$ given by Eq. (6).

At $\gamma \gg 1$, one can take an advantage of the exact mapping [14] of the LL model onto fermions with $\delta''(x)$ interaction. The mapping generalizes the famous duality [15] between impenetrable bosons and free fermions and is based on the elementary identity

$$2\arctan(p/mc) = \pi - 2\arctan(mc/p). \tag{15}$$

The second term in the right-hand side here is the scattering phase shift $\theta_s(p)$ of the symmetric wave function [16] of two particles with relative momentum $p$ interacting via $V_B = c\delta(x)$ potential. Adding $\pi$ to $\theta_s$ converts the symmetric wave function into the antisymmetric one. On the other hand, the left-hand side of (15) is the phase shift $\theta_a(p)$ of the antisymmetric wave function of two particles interacting via potential $V_F = -2/(m^2c)\delta''(x)$. In view of the integrability of the LL model, the two-particle phase shifts contain a complete information about the Bethe-ansatz wave function of the many-body problem. Thus, for any *bosonic* eigenstate of the LL model (1) there is a dual *fermionic* eigenstate of the Hamiltonian

$$H_F = -\frac{1}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} - \frac{2}{m^2c}\sum_{i>j}\delta''(x_i - x_j) \qquad (16)$$

that has the same energy. The two wave functions coincide in one of the sectors, say $x_1 < x_2 \ldots < x_N$, but differ by their symmetry with respect to the permutation of particles' coordinates. Since the density operator does not permute particles, its matrix elements between any two many-body eigenstates of Eq. (1) are identical to those evaluated with the corresponding dual eigenstates of $H_F$. In particular, the DSF for the LL model coincides with that for the fermionic model Eq. (16).

It is convenient to rewrite Eq. (16) in the second-quantized representation,

$$H_F = \sum_p \xi_p \psi_p^\dagger \psi_p + \sum_k \frac{V_k}{2L}\rho_k\rho_{-k}, \qquad V_k = \frac{2k^2}{m^2c}. \quad (17)$$

Here the operator $\psi_p^\dagger$ creates a fermion with momentum $p$ and energy $\xi_p = p^2/2m$ and $\rho_k = \sum_p \psi_{p-k}^\dagger \psi_p$.

Strong repulsion between the original bosons corresponds to a weak interaction in the dual fermionic model Eqs. (16) and (17). In the limit $c \to \infty$ Eqs. (16) and (17) describe free fermions. In this limit the structure factor differs from zero only in a finite interval, $\epsilon_2 < \omega < \epsilon_1$ with $\epsilon_{1,2}(q) = v_0 q \pm q^2/2m$. A weak ($\propto 1/\gamma$) residual interaction between fermions leads to corrections to $\epsilon_{1,2}$; for example, the Fermi velocity $v_0$ is replaced by the sound velocity $v = v_0/K \approx v_0(1 - 4/\gamma)$ [9,11]. Rather than discussing these modifications, we concentrate here on the singularities in $S(q, \omega)$.

DSF is proportional to the dissipative response to a field that couples to density. In the fermionic representation, the absorption of a quantum with energy $\omega$ and momentum $q$ is due to excitation of particle-hole pairs; there is just one such pair in the limit $c \to \infty$. At $\omega \to \epsilon_1$, the hole is created just below the Fermi level while the particle has momentum close to $k_F + q$; here $k_F = \pi n/m$ is the Fermi momentum. In the presence of interactions, such process is accompanied by a creation of multiple low-energy particle-hole pairs near the two Fermi points $p = \pm k_F$. Similar to the well-known phenomenon of the Fermi-edge singularity in the x-ray absorption spectra of metals [17], the proliferation of low-energy pairs leads to power-law singularities in the response function at the edges of the spectral support.

The Fermi-edge singularity relies crucially on the sharpness of the distribution function which is smeared at a finite temperature, hence Eq. (9). Note that in an integrable model Eq. (16) there is no relaxation of excited fermions [18] as three-particle collisions are absent. Therefore, at $T = 0$ there is no smearing of the power-law singularities in $S(q, \omega)$ even at finite $\omega$ [18,19].

The derivation of Eqs. (3), (5), (7), and (8) follows the method of Ref. [20]. Consider first the limit $\omega \to \epsilon_1$. We truncate the continuum of single-particle states to three narrow subbands [20] of the width $k_0 \ll q$: $d$ subband around $p = q$ that hosts a single particle in the final state of the transition, and two subbands, $\alpha = \pm$, around the right (left) Fermi points $p = \pm k_F$ that accommodate low-energy particle-hole pairs. After linearization of the spectrum within each subband, the resulting effective Hamiltonian takes the form

$$H = H_d + H_0 + H_{\rm int} \qquad (18)$$

with $H_d$ given by Eq. (11) and

$$H_0 = \int dx \sum_\alpha \psi_\alpha^\dagger(x)[-i\alpha v \partial_x]\psi_\alpha(x) \qquad (19)$$

with $\psi_\alpha(x) = L^{-1/2}\sum_k e^{i(k-\alpha k_F)x}\psi_k$. The last term in the right-hand side of Eq. (18) describes interaction,

$$H_{\rm int} = \sum_\alpha U_\alpha \int dx \rho_\alpha(x)\rho_d(x), \qquad (20)$$

where $\rho_\alpha = \psi_\alpha^\dagger \psi_\alpha$ and the coupling constants $U_\alpha$ are related to the parameters of the initial Hamiltonian (17); see Eq. (21) below.

Note that $H_{\rm int}$ does not include the direct interaction between the right and left movers. Indeed, the corresponding coupling constant $V_{2k_F} = 8\pi v_0/\gamma$ is small in the limit $\gamma \gg 1$. In the absence of such interaction, the remaining coupling constants $U_\alpha$ are set by the requirement [19] that the two-particle scattering phase shifts for the effective Hamiltonian (18)–(20) with linearized spectrum reproduce those for the original model (16) and (17). In the latter case, the phase shifts $\delta_\pm \equiv \theta_a(q + k_F \mp k_F)$ (see [16]) are given by

$$\delta_\pm = 2\arctan\frac{V_{q+k_F\mp k_F}}{2(v_d \mp v)}.$$

In the limit $\gamma \gg 1$ taken at a constant $Q = q/mc$, one finds $\delta_\pm = \delta$. In order to reproduce these phase shifts, the coupling constants in Eq. (20) must be equal to

$$U_\pm = -(v_d \mp v)\delta, \qquad \delta = 2\arctan Q. \qquad (21)$$

In terms of the effective Hamiltonian (18)–(21), the structure factor is given by Eq. (14) with $B^\dagger(x) = d^\dagger(x)\psi_+(x)$.

Following the steps familiar from the theory of the Fermi-edge singularity [21], we arrive at Eq. (3) with the exponent given by Eq. (5).

The power law Eq. (8) with the exponent given by Eq. (7) is obtained in a similar fashion. The only difference is that at $q < 2k_F$ the $d$ subband is centered at momentum $p = k_F - q$; i.e., it is below the Fermi level and hosts a single *hole* with velocity $v_d = v - q/m$. The hole is relatively slow, $|v_d| < v$, which leads [20] to $S(q, \omega) = 0$ at $\omega < \epsilon_2(q)$. When $q \to 2k_F - 0$, the center of $d$ subband is approaching $-k_F$. At larger $q$, one returns to the particle-like $d$ subband, but the density operator $B^\dagger$ in Eq. (14) is now given by $B^\dagger(x) = d^\dagger(x)\psi_-(x)$.

The "bosonic" route of evaluation of DSF described first, and the "fermionic" one described second, have a common region of applicability corresponding to both $Q$ and $\gamma$ being large. In this limit the fermionic calculation yields $\mu_1 \to 1/2$ and $\mu_2 \to 3/2$; see Eqs. (5) and (7). This is in agreement with the strong-repulsion limit ($K \to 1$) of the result of the bosonic calculation; see Eqs. (4) and (6). Note that at the special point $q = 2\pi n$ the exponent $\mu_2$ can be found using the hydrodynamic approximation (indeed, $\epsilon_2 \to 0$ at $q \to 2\pi n$, hence $\omega \to \epsilon_2$ limit is accessible within the effective low-energy description). The hydrodynamics yields $\mu_2 = K - 1 \approx 4/\gamma$ [6,22], in agreement with the corresponding limit of Eq. (7).

The peculiarity of 1D Bose liquid is that an arbitrarily weak repulsion between particles destroys condensation. This renders the perturbation theory developed for higher dimensions [4] inapplicable. The well-known alternative method based on the hydrodynamic description of low-energy excitations [5] also has its limitations, yielding infinitely narrow resonance in DSF at small $q$ [6]. In this Letter we demonstrated the existence of power-law singularities in DSF. The two complementary methods of analytic evaluation of DSF developed here allowed us to find the corresponding exponents in several regimes. Evaluation of the exponents in the entire range of parameters remains a challenging problem.

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