## Full Control by Locally Induced Relaxation

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We demonstrate a scheme for controlling a large quantum system by acting on a small subsystem only. The local control is mediated to the larger system by some fixed coupling Hamiltonian. The scheme allows us to transfer arbitrary and unknown quantum states from a memory to the large system ("upload access") as well as the inverse ("download access"). We study the sufficient conditions of the coupling Hamiltonian and give lower bounds on the fidelities for downloading and uploading.

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The unitarity of quantum mechanics implies that the information lost by an open system during its dynamical evolution must be contained in the environment and, possibly, in the correlations between the system and the environment [1]. In the context of repetitive applications of the same quantum transformation this fact has been exploited to achieve noise protection [2], cooling, state preparation [3,4], and quantum state transfer [5]. The mathematical aspects related with the convergence of the associated trajectories are well studied (see Ref. [6], and references therein). It has, however, been largely overlooked that the relaxation effects associated with these approaches can also imply full control by acting on a local subsystem only. This is related with the "asymptotic completeness" property introduced by Kümmerer and Maassen [4,7]. It is also analogous to what happens in the case of the universal quantum interface of Ref. [8] where the local control is mediated to the whole system by some fixed coupling Hamiltonian. Once this is achieved, apart from cooling and state preparation, it is also possible to perform arbitrary quantum data processing (e.g., measurements, unitary rotations).

In this Letter we discuss an explicit protocol for universal control of a composite system by operating on it with a simple repetitive local quantum transformation and we provide lower bounds for fidelities obtainable after finitely many steps. Furthermore, turning the problem into a graph theoretic one we provide an easy-to-check sufficient criterion to verify if a given global network Hamiltonian is capable of mediating control.

The results presented here pave the way to new applications of quantum control and quantum computation. Arguably our downloading and uploading protocols (see below) may be useful for the control of quantum hard drives and quantum RAM [9], or CCD-like application for the external control of permanently coupled arrays of sensors (see caption of Fig. 1). A large scale experimental realization of the scheme discussed in this Letter is not realistic at the moment since it requires the ability of performing full quantum computation on a large memory system. However, proof-of-principle tests could be probably realized in hybrid quantum networks by exploiting the methods proposed in Ref. [10] (e.g., controlling a permanently coupled optical lattice—where local control is generally difficult—by coupling it to a fully controllable array of trapped ions).

*Protocol.*—We consider a tripartite finite dimensional Hilbert space  $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_{\bar{C}} \otimes \mathcal{H}_M$ . Full control (i.e., the ability to prepare states and apply unitary transformations) is assumed on systems *C* and *M*, but no (direct) control is given on system  $\bar{C}$ . Systems *C* and  $\bar{C}$  are coupled by some time-independent Hamiltonian *H*. We will show that under certain assumptions, if the system  $C\bar{C}$  is initialized in an arbitrary state we can transfer ("download") this state into the system *M* by applying some simple operations which act locally on *M* and *C*. Likewise, by initializing the system *M* in the correct state, we can "upload"



FIG. 1. The systems *C* and  $\overline{C}$  are coupled through the timeindependent Hamiltonian *H*. The system *C* is controlled by performing regular swap operations  $S_{\ell}$  between it and a quantum memory  $M_{\ell}$ . In CCD-like application  $\overline{C}$  is an array of permanently coupled sensors that are used to probe external signals (schematically sketched by the wiggling curve): information from the sensor is extracted through *C*. arbitrary states on the system  $C\bar{C}$ . These two schemes ensure full controllability of  $C\bar{C}$ : for instance, one can perform an arbitrary quantum operation on such a system by transferring its state into M, applying the equivalent operation there, and transferring the resulting state back to  $C\bar{C}$ . In this context M functions as a quantum memory and must be at least as large as the system  $C\bar{C}$ . As sketched in Fig. 1 we can imagine it to be split into sectors  $M_{\ell}$ ,  $\mathcal{H}_M =$  $\bigotimes_{\ell=1}^{L} \mathcal{H}_{M_{\ell}}$  with the  $\mathcal{H}_{M_{\ell}}$  being isomorphic to  $\mathcal{H}_C$  (i.e., dim  $\mathcal{H}_{M_{\ell}} = \dim \mathcal{H}_C$ ).

For downloading we assume that the memory M is initialized in  $|e\rangle_M \equiv \bigotimes_{\ell} |e\rangle_{M_{\ell}}$  with the vectors  $|e\rangle$  to be defined in the following. To download an arbitrary initial state  $|\psi\rangle_{C\bar{C}}$  of  $C\bar{C}$  into the memory M we perform a sequence of unitary gates between M and C, intermitted by the time evolution  $U = \exp[-iHt]$  on  $C\bar{C}$  for some fixed time interval t. More specifically, at step  $\ell$  of the protocol we perform a unitary swap  $S_{\ell}$  between system Cand system  $M_{\ell}$  [11]. The protocol stops after the *L*th swap operation. The resulting global transformation is thus represented by the unitary operator

$$W \equiv S_L U S_{L-1} \cdots U S_{\ell} \cdots U S_1 U. \tag{1}$$

As we will see in the next section, the reduced evolution of the system  $\bar{C}$  under the protocol can be expressed in terms of the completely positive trace preserving (CPT) map  $\tau$ defined in Eq. (4). Our main assumption is that the system  $\bar{C}$  is relaxing [12] under repetitive application of  $\tau$ , i.e.,  $\lim_{n\to\infty} \tau^n(\rho) = \rho_*$  for all initial states  $\rho$ . This behavior is also called mixing [13,14] or absorbing [6]. In what follows we will focus on the case in which  $\rho_*$  is a pure state  $|E\rangle_{\bar{C}}\langle E|$ . When this happens it is possible to show that, for sufficiently large *L*, the transfer of  $|\psi\rangle$  from  $C\bar{C}$  into *M* can be done with arbitrarily high fidelity [5] and the transformation which allows one to recover  $|\psi\rangle$  from *M* can be explicitly constructed.

For uploading an arbitrary input state  $|\psi\rangle$  from *M* to  $C\bar{C}$  one is tempted to revert the downloading protocol. Roughly speaking, the idea is to initialize the memory in the state that it would have ended up in after applying *W* if system  $C\bar{C}$  had started in the state we want to initialize. Then we apply the inverse of *W* given by

$$W^{\dagger} = U^{\dagger} S_1 U^{\dagger} \cdots S_{\ell} U^{\dagger} \cdots S_{L-1} U^{\dagger} S_L.$$
(2)

We will see that indeed this induces a unitary coding on M such that arbitrary and unknown states can be initialized on  $C\bar{C}$ . The reader has probably noticed, however, that the transformation (2) requires backward time evolution of  $C\bar{C}$ ; i.e., one has to wait negative time steps between the swaps. For this reason, even though the transformation originated from  $W^{\dagger}$  is coherently defined at a mathematical level, it cannot be considered as a proper uploading algorithm for transformation associated to Eq. (2) the reverse-downloading protocol. A proper uploading algorithm will be defined in the final part of the paper by

imposing an extra hypothesis on the  $C\bar{C}$  couplings and by adopting a simple change of perspective. For the moment we neglect this point and simply focus on the convergence properties of the downloading and the reversedownloading algorithms associated with Eqs. (1) and (2).

*Cooling.*—We start by showing that the action of W on  $C\bar{C}$  is effectively equivalent to a cooling process which transfers any initial state into  $|e\rangle_C |E\rangle_{\bar{C}}$ . Let  $|\psi\rangle_{C\bar{C}} \in \mathcal{H}_{C\bar{C}}$  be an arbitrary state. We notice that the C component of  $W|\psi\rangle_{C\bar{C}}|e\rangle_M$  is always  $|e\rangle_C$ . Therefore we can write

$$W|\psi\rangle_{C\bar{C}}|e\rangle_{M} = |e\rangle_{C}[\sqrt{\eta}|E\rangle_{\bar{C}}|\phi\rangle_{M} + \sqrt{1-\eta}|\Delta\rangle_{\bar{C}M}], \quad (3)$$

with  $|\Delta\rangle_{\bar{C}M}$  being a normalized vector of  $\bar{C}M$  which satisfies the identity  $_{\bar{C}}\langle E|\Delta\rangle_{\bar{C}M} = 0$ . It is worth stressing that the decomposition (3) is unique and that  $\eta$ ,  $|\phi\rangle_M$ , and  $|\Delta\rangle_{\bar{C}M}$  are typically complicated functions of the input state  $|\psi\rangle_{C\bar{C}}$ . The quantity  $\eta$  plays an important role: it gives us the fidelity between the initial state of  $C\bar{C}$  and the target state  $|e\rangle_C|E\rangle_{\bar{C}}$  of the cooling process. An expression for  $\eta$  can be obtained by focusing on the reduced density matrix of the subsystem  $\bar{C}$ . From our definitions it follows that after the first step of the protocol this is

$$\tau(\rho_{\bar{C}}) \equiv \operatorname{Tr}_{CM}[US_1(|\psi\rangle_{\bar{C}C}\langle\psi|\otimes|e\rangle_M\langle e|)S_1U^{\dagger}]$$
  
=  $\operatorname{Tr}_C[U(\rho_{\bar{C}}\otimes|e\rangle_C\langle e|)U^{\dagger}],$  (4)

with  $\rho_{\bar{C}} \equiv \text{Tr}_{C}[U|\psi\rangle_{\bar{C}C}\langle\psi|U^{\dagger}]$  and  $|\psi\rangle_{\bar{C}C}$  the initial state. Reiterating this expression we notice that the state of  $\bar{C}$  after *L* steps can be obtained by successive application of the map (4). Consequently, Eq. (3) gives  $\eta = {}_{\bar{C}}\langle E|\tau^{L-1}(\rho_{\bar{C}})|E\rangle_{\bar{C}}$ , which, according to the mixing properties of  $\tau$  given at the beginning of the section, shows that  $\eta \to 1$  for  $L \to \infty$ . Specifically, we can use [12] to claim that for all input states  $|\psi\rangle$  the following inequality holds:

$$|\eta - 1| \le \|\tau^{L-1}(\rho_{\bar{C}}) - |E\rangle_{\bar{C}} \langle E|\|_1 \le K \kappa^{L-1} (L-1)^{d_{\bar{C}}},$$
(5)

where *K* is a constant which depends upon  $d_{\bar{c}} \equiv \dim \mathcal{H}_{\bar{c}}$ and  $\kappa \in ]0, 1[$  is the second largest of the moduli of eigenvalues of the map  $\tau$ .

Coding transformation.—Let us now derive the decoding or encoding transformation that relates states on the memory *M* to the states of  $C\bar{C}$ . Such a transformation was already derived in a general mathematical context for infinite memory [6]. For the finite case, the idea is to apply the decomposition (3) to each element of a given orthonormal basis { $|\psi_k\rangle_{C\bar{C}}$ } of  $\mathcal{H}_{C\bar{C}}$ , and to define the linear operator *D* on  $\mathcal{H}_M$  which, for all *k*, performs the transformation

$$D|\psi_k\rangle_M = |\phi_k\rangle_M. \tag{6}$$

In this expression  $|\psi_k\rangle_M$  are orthonormal vectors of M used to represent the states  $|\psi_k\rangle_{C\bar{C}}$  of  $\mathcal{H}_{C\bar{C}}$  on M (formally they are obtained by a partial isometry from  $\bar{C}C$  to M). The vectors  $|\phi_k\rangle_M$  instead are connected to the  $|\psi_k\rangle_{C\bar{C}}$  through Eq. (3). Typically, for finite values of *L*, the  $|\phi_k\rangle_M$  will not be orthogonal. However, it is possible to show that they become asymptotically orthogonal in the limit of  $L \gg 1$ . To see this we use the unitarity of the transformation *W* and the orthogonality of  $|\psi_k\rangle_{C\bar{C}}$ . Indeed from Eq. (3) one can easily verify the following identity:

$$\delta_{kk'} = \sqrt{\eta_k \eta_{k'M}} \langle \phi_k | \phi_{k'} \rangle_M + \sqrt{(1 - \eta_k)(1 - \eta_{k'})}_{\bar{C}M} \langle \Delta_k | \Delta_{k'} \rangle_{\bar{C}M}.$$
(7)

Defining  $\eta_0 \equiv \min_k \eta_k$ , we notice that for sufficiently large *L* this is a strictly positive quantity and converges to 1 see Eq. (5). From the identity (7) it follows then that for  $k \neq k'$  one can write  $|_M \langle \phi_k | \phi_{k'} \rangle_M | \leq (1 - \eta_0) / \eta_0$ . This can now be used to bound the eigenvalues  $\lambda_j$  of the linear operator  $D^{\dagger}D$ . Indeed the Cauchy-Schwartz inequality yields  $|\lambda_j - 1| \leq d_{C\bar{C}}(1 - \eta_0) / \eta_0$ , with  $d_{C\bar{C}} \equiv \dim \mathcal{H}_{C\bar{C}}$ . Take now a polar decomposition D = PV with *P* being positive semidefinite. According to [15] (p. 432), *V* is the best unitary approximation of *D*. In our case it satisfies the relations

$$\|D - V\|_{2}^{2} = \sum_{j} [\sqrt{\lambda_{j}} - 1]^{2} \leq \sum_{k} |\lambda_{j} - 1|$$
$$\leq d_{C\bar{C}} (d_{C\bar{C}} - 1)(1 - \eta_{0}) / \eta_{0}.$$
(8)

This is a key equation: thanks to Eq. (5), it shows that D can be approximated arbitrary well by the unitary operator V for  $L \rightarrow \infty$ .

*Fidelities.*—In what follows we will use  $V^{\dagger}$  and V as our downloading and reverse-downloading transformation, respectively. In particular,  $V^{\dagger}$  will be used to recover the input state  $|\psi\rangle_{C\bar{C}}$  of the chain after we have (partially) transferred it into M through the unitary W (i.e., we first act on  $|\psi\rangle_{C\bar{C}}|e\rangle_M$  with W, and then we apply  $V^{\dagger}$  on M). Vice versa, in order to upload a state  $|\psi\rangle_M$  on  $C\bar{C}$  by using the reverse-downloading protocol we first prepare  $C\bar{C}$  in  $|eE\rangle_{C\bar{C}}$  [16], then we apply to it the unitary transformation V and finally we apply  $W^{\dagger}$ .

With these choices the fidelity for downloading a state  $|\psi\rangle_M$  is simply given by  $F_d(\psi) \equiv {}_M \langle \psi | V^{\dagger} \rho_M V | \psi \rangle_M$  where  $\rho_M$  is the state of the memory after W, i.e.,

$$\rho_{M} \equiv \operatorname{Tr}_{C\bar{C}}[W(|\psi\rangle_{C\bar{C}}\langle\psi|\otimes|e\rangle_{M}\langle e|)W^{\dagger}]$$
$$= \eta|\phi\rangle_{M}\langle\phi| + (1-\eta)\sigma_{M}$$
(9)

(here we used Eq. (3) and introduced the density matrix  $\sigma_M \equiv \text{Tr}_{\bar{C}}[|\Delta\rangle_{\bar{C}M}\langle\Delta|]$ ). Analogously the fidelity for transferring a state  $|\psi\rangle$  from M to  $C\bar{C}$  through the reverse-downloading protocol is given by

$$F_{\rm rd}(\psi) = {}_{C\bar{C}} \langle \psi | {\rm Tr}_{M} [ W^{\dagger} V(|\psi\rangle_{M} \langle \psi | \otimes |eE\rangle_{C\bar{C}} \langle eE|) V^{\dagger} W ] |\psi\rangle_{C\bar{C}}.$$

A bound for  $F_r$  and  $F_{rd}$  follows by noticing that both these quantities satisfy the inequality

$$F(\psi) \ge \eta|_{M} \langle \phi | V | \psi \rangle_{M} |^{2}.$$
(10)

For  $F_d$  this simply comes by expressing it in terms of Eq. (9) and by neglecting a positive contribution proportional to  $1 - \eta$ . For  $F_{rd}$  instead the inequality (10) follows by replacing the trace over M with the expectation value on  $|0\rangle_M$  and by using Eq. (3). One can now estimate the scalar product on the right-hand side of Eq. (10) by observing that  $|_M \langle \phi | V | \psi \rangle_M | \ge |_M \langle \phi | D | \psi \rangle_M | - |_M \langle \phi | D - V | \psi \rangle_M |$ . This can be further bounded by employing the inequality (8) and the fact that  $|_M \langle \phi | D - V | \psi \rangle_M | \le ||D - V||_2$ . If  $|\psi\rangle_M$  is a vector of the basis  $|\psi_k\rangle_M$ , then  $|_M \langle \phi | D | \psi \rangle_M | = 1$  by the definition—see Eq. (6). For generic  $|\psi\rangle_M$  instead some simple algebra yields  $\sqrt{\eta}|_M \langle \phi | D | \psi \rangle_M | \ge \sqrt{\eta_0} - d_{C\bar{C}}(1 - \eta_0)/\eta_0$ . Replacing all this into Eq. (10) we finally get

$$F \ge \eta_0 - 4d_{C\bar{C}}\sqrt{(1-\eta_0)/\eta_0},$$
 (11)

which holds for  $F = F_r$ ,  $F_{rd}$ . This is a lower bound for the fidelity of the downloading and reverse-downloading protocols: it is probably not tight but it is sufficient to show that  $F_r$  and  $F_{rd}$  converge to 1 in the limit of large L [17]. According to Eq. (5) such convergence is exponentially fast in L even though, not surprisingly, the bound deteriorates as the size  $d_{\bar{C}}$  of the controlled system increases.

Uploading protocol.-Let us now come back to the question about the operation  $W^{\dagger}$  requiring a backward time evolution. To define a proper uploading protocol consider a modified scenario in which  $C\bar{C}$  is replaced by a isomorphic system  $C'\bar{C}'$  characterized by the Hamiltonian H' = -H. In this scenario the downloading protocol is described by the operator W' = $S_L U' S_{L-1} \cdots U' S_1 U'$  with  $U' \equiv \exp[-iH't]$  while the corresponding reverse-downloading transformation by the operator  $(W')^{\dagger} = (U')^{\dagger} S_1 (U')^{\dagger} \cdots S_{L-1} (U')^{\dagger} S_L$ . Since  $U' = U^{\dagger}$  the reverse-downloading algorithm of  $C'\bar{C}'$  induces a proper uploading transformation for  $C\bar{C}$ . To ensure that such an algorithm converges it is hence sufficient to study the downloading and reverse-downloading protocols associated with W' and W'<sup> $\dagger$ </sup>. According to our previous results this can be done by focusing on the CPT map  $\tau'(\rho_{\bar{C}}) \equiv \operatorname{Tr}_{C}[U^{\dagger}(\rho_{\bar{C}} \otimes |e\rangle_{C} \langle e|)U]$  which replaces  $\tau$  of Eq. (4). It is then sufficient to assume  $\tau'$  to be ergodic with pure fixed point  $|E\rangle_{\bar{C}}$ . When this happens we can define a transformation D' as in (6) and its unitary part V'. The latter is the coding transformation which will be used for the uploading protocol of  $C\bar{C}$ . Consequently, the fidelity  $F_{\rm up}(\psi)$  associated with such an algorithm is bounded as in Eq. (11) with  $\eta_0$  being lower bounded by Eq. (5) where the parameters K and  $\kappa$  of  $\tau$  have being replaced by the corresponding quantities of  $\tau'$ . It should be noted that the definition of uploading protocol given here is more general than in other schemes relying on timereversal symmetries [4].

A condition for controllability.—The mixing properties of  $\tau$  and  $\tau'$  are typically independent (see, for instance, Ref. [6]): this makes it difficult to give a general condition for the full controllability of  $C\bar{C}$  (i.e., convergence of both the downloading and uploading protocols). Notably, however, a generic statement can be made using a result of Ref. [14]. For the sake of simplicity here we will focus on the case in which  $C\bar{C}$  is a network of coupled spins 1/2 particles. According to [14] we have that (i) if the Hamiltonian H of  $C\bar{C}$  preserves the number of spin excitations and (ii) the vector  $|e\rangle_C |E\rangle_{\bar{C}}$  is the only eigenstate with C in  $|e\rangle_C$  then the map  $\tau$  is mixing with fix point  $|E\rangle_{\bar{C}}$ (here  $|e\rangle_C$  and  $|E\rangle_{\bar{C}}$  represent states with all spins aligned down). However, the Hamiltonian H' = -H associated with  $\tau'$  has the same eigenvectors of H; hence, conditions (i) and (ii) also yield a sufficient criterion for determining that  $\tilde{\tau}$  is mixing with fixed point  $|E\rangle_{\bar{C}}$ .

The analysis further simplifies by focusing on a twosites interaction Hamiltonian. In this case a simple recursive analysis is sufficient to check if H satisfies the condition (ii). For linear chains of spins this was discussed in Ref. [5]: here we generalize this argument to arbitrary topology. To do so, define the graph G with the spins of the network as vertices, and the non-Ising components of H as edges. Introduce also the following color code: a black vertex corresponds to a spin in  $|\downarrow\rangle$  while a white vertex corresponds to a generic spin configuration (i.e., not necessarily  $|\downarrow\rangle$ ). Consider now the case in which the subset C of the graph has all-black vertexes. Our goal is to determine if such a configuration is compatible with being a nontrivial eigenstate of the network Hamiltonian (that is an eigenstate in which not all the vertices are black); if not, then the whole network can be controlled. This can be checked by noticing that the excitations (i.e., the white vertices) tend to propagate along the edges when H is applied to the graph: consequently, only certain distributions of black and white vertices are compatible with the eigenvector structure of H (they must allow certain interference effects that prevents the propagations of the white vertices). To exploit this property we introduce the following cellular graph automata: iff a vertex is black and has exactly one white neighbor, then this neighbor will turn black in the next step. Otherwise, vertices remain unchanged. It follows that the final state associated with a generic initial configuration is the one which is compatible with being an eigenstate of H and has the minimum allowed number of black vertices. Therefore to verify if the whole graph is controllable by operating on C it is sufficient to initialize C in the all-black configuration and  $\overline{C}$  in the all-white configuration and let the graph evolve. If the final result has only black vertices then we conclude that C induces full control on  $C\overline{C}$  (see Fig. 2 for examples of controllable graphs). Note that this method allows us to say something about the structure and the asymptotic dynamics of Hamiltonians that are in general very far from analytic solvability, including disordered systems.

*Conclusion.*—We have shown that for a large class of physically realistic Hamiltonians, a quantum computer can fully control a large system by inducing a relaxation on a



FIG. 2. Examples of graphs coupled by Heisenberg-like interaction that can be controlled by acting on the black qubits only. The lower right graph provides a counterexample.

small subsystem only. Furthermore, we were able to relate this "asymptotic completeness" [4,7] to some easy-tocheck property of the graph representing the coupling Hamiltonian. The fidelity converges exponentially fast with the size of the memory, so the required overhead of resources is low. The results discussed here can be generalized to maps  $\tau$  and  $\tau'$  are mixing with nonpure fixed point  $\rho_*$ . In this case the fidelities of the protocols will be not necessarily optimal. Still one can provide a nontrivial lower bound for these quantities which depends upon the purity of  $\rho_*$ .

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