Quantum Gravity Boundary Terms from the Spectral Action of Noncommutative Space

Ali H. Chamseddine[1,3] and Alain Connes[2,3,4]

[1]Physics Department, American University of Beirut, Lebanon
[2]College de France, 3 rue Ulm, F75005, Paris, France
[3]Institut des Hautes Etudes Scientifique F-91440 Bures-sur-Yvette, France
[4]Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240, USA

(Received 12 May 2007; published 17 August 2007)

We study the boundary terms of the spectral action of the noncommutative space, defined by the spectral triple dictated by the physical spectrum of the standard model, unifying gravity with all other fundamental interactions. We prove that the spectral action predicts uniquely the gravitational boundary term required for consistency of quantum gravity with the correct sign and coefficient. This is a remarkable result given the lack of freedom in the spectral action to tune this term.

DOI: 10.1103/PhysRevLett.99.071302 PACS numbers: 04.62.+v, 02.40.Gh, 11.10.Nx, 11.30.Ly

It has been known since the 1960s [1] that in the Hamiltonian quantization of gravity it is essential to include boundary terms in the action, as this allows us to define consistently the momentum conjugate to the metric. This makes it necessary to modify the Einstein-Hilbert action by adding to it a surface integral term so that the variation of the action becomes well defined and yields the Einstein field equations. The reason for this manipulation is that the curvature scalar R contains second derivatives of the metric, which are removed after integrating by parts to obtain an action which is quadratic in first derivatives of the metric. These surface terms are canceled by modifying the Euclidean action to [2], [3]

$$I = -\frac{1}{16\pi}\int_M d^4x\sqrt{g}R - \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}K,$$

where ∂M is the boundary of M, h_{ab} is the induced metric on ∂M, and K is the trace of the second fundamental form on ∂M. We use the sign convention according to which R is positive for the sphere and K is positive for the ball. Notice that there is a relative factor of 2 and a fixed sign between the two terms, and that the boundary term has to be completely fixed. This is a delicate fine-tuning and is not determined by any symmetry, but only by the consistency requirement. There is no known symmetry that predicts this combination and it is always added by hand.

In the noncommutative geometric approach to the formulation of a unified theory of all fundamental interactions including gravity, the starting point is the replacement of the Riemannian geometry of space-time with noncommutative geometry. The basic data of noncommutative geometry consist of an involutive algebra $\mathcal{A}$ of operators in Hilbert space $\mathcal{H}$, which plays the role of the algebra of coordinates, and a self-adjoint operator D in $\mathcal{H}$ [4] which plays the role of the inverse of the line element. The spectrum of the standard model indicates that the algebra is to be taken as $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$ where the algebra $\mathcal{A}_F$ is finite dimensional, $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, and $\mathbb{H} \subset M_2(\mathbb{C})$ is the algebra of quaternions. The algebra $\mathcal{A}$ is a tensor product which geometrically corresponds to a product space. The spectral geometry of $\mathcal{A}$ is given by the product rule

$$\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F, \qquad D = D_M \otimes 1 + \gamma_5 \otimes D_F,$$

where $L^2(M, S)$ is the Hilbert space of L^2 spinors, and D_M is the Dirac operator of the Levi-Civita spin connection on M. The Hilbert space of quarks and leptons fixes the choice of the Dirac operator D_F and the action of $\mathcal{A}_F$ in $\mathcal{H}_F$. The operator D_F anticommutes with the chirality operator γ_F on $\mathcal{H}_F$. The spectral geometry does not change if one replaces D by the equivalent operator

$$D = D_M \otimes \gamma_F + 1 \otimes D_F, \tag{1}$$

but this equivalence fails when M has a boundary and it is only the latter choice which has conceptual meaning since γ_5 no longer anticommutes with D_M when $\partial M \neq \varnothing$. The noncommutative space defined by a spectral triple has to satisfy the basic axioms of noncommutative geometry. This approach shares a common feature with Euclidean quantum gravity in that the Riemannian manifold is taken to be Euclidean in order for the line element, which is the inverse of the Dirac operator, to be compact. It is then assumed that one obtains the Lorentzian results by analytically continuing the expressions obtained by performing the path integral to Minkowski space. A fundamental principle in the noncommutative approach is that the usual emphasis on the points $x \in M$ of a geometric space is now replaced by the spectrum of the operator D. The spectral action principle states that the physical action depends only the spectrum of the Dirac operator, which is geometrical. Indeed, it was shown that all the fundamental interactions including gravity are unified in the spectral action [5]

$$I = \mathrm{Tr}\, f\left(\frac{D}{\Lambda}\right) + \langle \Psi, D\Psi \rangle,$$

where Tr is the usual trace of operators in the Hilbert space $\mathcal{H}$, Λ is a cutoff scale, and f is a positive function. The action is then uniquely defined and the only arbitrariness one encounters is in the first few coefficients in the spectral expansion since higher coefficients are suppressed by the high-energy scale. This remarkable action includes the gravitational Einstein-Hilbert term with the square of the Weyl tensor, the $SU(3)_c \times SU(2)_w \times U(1)_Y$ gauge interactions, the Higgs couplings including the spontaneous symmetry breaking, all coming with the correct signs as well as a relation between the gauge couplings and Higgs couplings. The geometrical model is valid at the unification scale and relates the gauge coupling constants to each other and to the Higgs coupling. When these relations are taken as boundary conditions valid at the unification scale in the renormalization group (RG) equations, one gets a prediction of the Higgs boson mass to be around 170 ± 10 GeV, the error being due to our ignorance of the physics at unification scale. In addition, there is one relation between the sum of the square of fermion masses and the W particle mass square which enables us to predict the top quark mass compatible with the measured experimental value. It also accommodates small neutrino masses through the seesaw mechanism, thanks to a more subtle choice ([6]) of the chirality operator γ_F which gives to the geometry F a KO dimension which is congruent to 6 modulo 8. The charge conjugation operator J for the product geometry (1) is then given by $J = J_M \gamma_5 \otimes J_F$ which commutes with the operator D given by (1) since in even dimension J_M commutes with D_M while in dimension 6 modulo 8, J_F anticommutes with γ_F.

The results were derived for manifolds without boundary. We stress that definition of the noncommutative space corresponding to the physical space-time must satisfy the restrictive axioms of noncommutative geometry. Once this is done, there is essentially no freedom left in determining the spectral action, except for the three coefficients of the Mellin transform of the function f. These correspond to the cosmological constant, the Newton constant, and the gauge couplings and where the dependence on the energy scale is governed by the renormalization group equations. Because of these constraints, it is essential to find out whether the boundary terms of the spectral action agree with the modifications dictated by the consistency of quantum gravity. This is a severe test of the spectral action principle as there is no freedom present in tuning the surface terms to reproduce the desired results with correct signs and numerical values. It is the purpose of this work to show that the spectral action does pass all tests predicting the correct modification of the boundary terms. We can go further and make the mass scale Λ appearing in the Dirac operator dynamical by replacing it with a dilaton field. We have recently shown that in this case the spectral action becomes almost scale invariant and gives the same low-energy limit as the Randall-Sundrum model as well as providing a model for extended inflation [7]. In other words, the simple form of the spectral action is capable of producing all the desirable features of unified theories including gravity with the correct physical predictions.

The Dirac operator in the spectral action must satisfy the Hermiticity condition

$$\langle \Psi, D\Psi \rangle = \langle D\Psi, \Psi \rangle.$$

These are satisfied provided the following "natural" boundary condition is imposed [8–10]

$$\Pi_- \Psi|_{\partial M} = 0,$$

where the projection operator Π_- is given by $\Pi_- = \frac{1}{2} \times (1 - \chi)$, where $\chi = \gamma_n \gamma_5$ satisfies $\chi^2 = 1$. The Clifford algebra is defined by $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$ and we denote by n the unit *inward* normal and γ_n the corresponding Clifford multiplication. Although one can keep the discussion general, it will be more transparent to specialize to the case where the dimensions of the continuous part of the noncommutative space is taken to be four. A local system of coordinates on M will be denoted by x^μ, $\mu = 1, \ldots, 4$, and on ∂M will be denoted by y^a, $a = 1, 2, 3$. Let the functions $x^\mu(y^a)$ be given by the embedding of the hypersurface in M and let $e^\mu_a = \frac{\partial x^\mu}{\partial y^a}$, then the metric $g_{\mu\nu}$ on M induces a metric h_{ab} on the hypersurface such that $h_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b$ and where n^μ is orthogonal to e^μ_a so that $g_{\mu\nu} n^\mu e^\nu_a = 0$. It is convenient to define $n_\mu = g_{\mu\nu} n^\nu$ so that $n_\mu e^\mu_a = 0$. We now define the inverse functions e^a_μ by $e^\mu_a e^b_\mu = \delta^b_a$ which satisfies the condition $e^\mu_a e^a_\nu = \delta^\mu_\nu - n^\mu n_\nu$ to be consistent with $n_\mu e^\mu_a = 0$. We therefore can write [11]

$$g_{\mu\nu} = h_{ab} e^a_\mu e^b_\nu + n_\mu n_\nu.$$

The inverse metric is also defined by $h^{ab} = g^{\mu\nu} e^a_\mu e^b_\nu$ and the inverse relation is

$$g^{\mu\nu} = h^{ab} e^\mu_a e^\nu_b + n^\mu n^\nu.$$

This shows that any tensor can be projected into the hypersurface using the completeness relations for the basis $\{e^a_\mu, n_\mu\}$. We finally define on ∂M,

$$\chi = -\frac{\sqrt{h}}{3!} \epsilon^{abc} \gamma_a \gamma_b \gamma_c, \qquad \gamma_5 = \chi \gamma_n,$$

which satisfy $\chi^2 = 1$, $\chi\gamma^a = \gamma^a \chi$, $\chi\gamma^n = -\gamma^n \chi$, $\gamma_5^2 = 1$, $\chi\gamma_5 = -\gamma_5 \chi$. The normal vector n^μ satisfies the properties

$$n_{\mu;\nu} = -K_{ab} e^a_\mu e^b_\nu, \qquad e^\mu_{a;\nu} e^\nu_b = \Gamma^c_{ab} e^\mu_c + K_{ab} n^\mu,$$

where the covariant derivative ; ν is the space-time covariant derivative and Γ^c_{ab} is the Christoffel connection of the metric h_{ab}, and K_{ab} is the extrinsic curvature whose symmetry follows from the relation $e^\mu_{a;b} = e^\mu_{b;a}$.

The bosonic part of the spectral action is then obtained by using the identity [5]

$$\mathrm{Tr}\,[f(D^2/m^2)] \simeq \sum_{n\geq 0} f_{4-n} a_n(D^2/m^2),$$

where f_n are related to the Mellin transforms of the function f. The Seeley-deWitt coefficients $a_n(P,\chi)$ are geometrical invariants. These were calculated for Laplacians which are the square of the Dirac operator, for manifolds with boundary. To evaluate these terms, we first write the Laplacian in the form

$$\begin{aligned} P = D^2 &= -(g^{\mu\nu}\partial_\mu\partial_\nu + \mathbb{A}^\mu + \mathbb{B}) \\ &= -(g^{\mu\nu}\nabla'_\mu\nabla'_\nu + E), \end{aligned}$$

where $\nabla'_\mu = \partial_\mu + \omega'_\mu$ and $\omega'_\mu = \frac{1}{2} g_{\mu\nu}(\mathbb{A}^\nu + g^{\rho\sigma}\Gamma^\nu_{\rho\sigma})$. It is convenient to write the Dirac operator in the form

$$D = \gamma^\mu \nabla_\mu - \Phi,$$

where $\nabla_\mu = \partial_\mu + \omega_\mu$ and ω_μ is the torsion free spin connection. The boundary conditions for D^2 are then equivalent to [9,10]

$$\mathcal{B}_\chi \Psi = \Pi_-(\Psi)|_{\partial M} \oplus \Pi_+(\nabla'_n + S)\Pi_+(\Psi)|_{\partial M} = 0,$$

where

$$\begin{aligned} S &= \Pi_+(\gamma_n \Phi - \tfrac{1}{2}\gamma_n \gamma^a \nabla'_a \chi)\Pi_+, \\ \nabla'_a \chi &= \partial_a \chi + [\omega'_a, \chi] = K_{ab}\chi\gamma^n\gamma^b + [\theta_a, \chi], \end{aligned}$$

and where $\theta_a = \omega'_a - \omega_a$. We then have the relations

$$\begin{aligned} E &= \gamma^\mu \nabla_\mu \Phi - \Phi^2 - \tfrac{1}{2}\gamma^{\mu\nu}\Omega_{\mu\nu}, \\ \Omega_{\mu\nu} &= \partial_\mu \omega'_\nu - \partial_\nu \omega'_\mu + \omega'_\mu \omega'_\nu - \omega'_\nu \omega'_\mu. \end{aligned}$$

We list the first relevant Seeley-deWitt coefficients for Laplacians which are square of Dirac operators [12]

$$\begin{aligned} a_0(P,\chi) &= \frac{1}{16\pi^2}\int_M d^4x\sqrt{g}\,\mathrm{Tr}\,(1), \qquad a_1(P,\chi) = 0, \\ a_2(P,\chi) &= \frac{1}{96\pi^2}\bigg[\int_M d^4x\sqrt{g}\,\mathrm{Tr}\,(6E + R) \\ &\quad + \int_{\partial M} d^3x\sqrt{h}\,\mathrm{Tr}\,(2K + 12S)\bigg], \\ a_3(P,\chi) &= \frac{1}{384(4\pi)^{3/2}}\int_{\partial M} d^3x\sqrt{h}\,\mathrm{Tr}\,(96\chi E + 3K^2 \\ &\quad + 6K_{ab}K^{ab} + 96SK + 192S^2 - 12\nabla'_a\chi\nabla'^a\chi). \end{aligned}$$

As a warm-up, these results could be applied to the simple case of an ordinary Dirac operator $D = \gamma^\mu(\partial_\mu + \omega_\mu)$. Therefore, in the above formulas we have

$$\begin{aligned} \omega'_\mu &= \omega_\mu, \qquad E = -\tfrac{1}{4}R, \qquad \Phi = 0, \\ S &= -\tfrac{1}{2}K\Pi_+, \qquad \nabla'_a\chi = K_{ab}\chi\gamma^n\gamma^b. \end{aligned}$$

Substituting $\mathrm{Tr}\,(1) = 4$ and $\mathrm{Tr}\,(S) = -K$ we have for the first few terms

$$\begin{aligned} a_0(P,\chi) &= \frac{1}{4\pi^2}\int_M d^4x\sqrt{g}, \\ a_2(P,\chi) &= -\frac{1}{24\pi^2}\left(\int_M d^4x\frac{1}{2}\sqrt{g}R + \int_{\partial M} d^3x\sqrt{h}K\right), \\ a_3(P,\chi) &= \frac{1}{32(4\pi)^{3/2}}\int_{\partial M} d^3x\sqrt{h}(K^2 - 2K_{ab}K^{ab}). \end{aligned}$$

The important point in the above result is the emergence of the combination [2] $-\int_M d^4x\sqrt{g}R - 2\int_{\partial M} d^3x\sqrt{h}K$ as the lowest term of the gravitational action which is known to be the required correction to the Einstein action involving the surface term so as to make the Hamiltonian formalism consistent. This is remarkable because both the sign and the coefficient are correct. The only assumption made is that normal boundary conditions are taken such that they enforce the Hermiticity of the Dirac operator. This is yet another miracle concerning correct signs obtained in the spectral action of the Dirac operator. We also notice that the relative coefficient between R and K depends, in general, on the nature of the Laplacian. The desired answer is true for the square of the Dirac operator, but *not* for a general Laplacian. We note that other boundary conditions may lead to different results [12].

This is a general result and applies to all noncommutative models based on spaces which are the tensor product of the spectral triple of a Riemannian manifold by that of a discrete space. In particular, the above feature also works for the spectral action of the standard model. Indeed, by applying the above formulas to the Dirac operators in the quarks and leptonic sectors with the corresponding boundary conditions one derives the full spectral action with boundary terms included. We just give the results here; the full details will appear in the expanded version of this

Letter [13]. (Note that in [6] we use the opposite sign convention for the scalar R):

$$\begin{aligned} I = {} & \frac{48\Lambda^4}{\pi^2} f_4 \int_M d^4x\sqrt{g} + \frac{8\Lambda^2}{\pi^2} f_2 \left\{ \int_M d^4x\sqrt{g}\left[-\frac{1}{2}R - \frac{1}{4}\left(a|\varphi|^2 + \frac{1}{2}c\right)\right] - \int_{\partial M} d^3x\sqrt{h}K \right\} \\ & + \frac{2\Lambda}{(4\pi)^{3/2}} f_1 \int_{\partial M} d^3x\sqrt{h}[3(K^2 - 2K_{ab}K^{ab})] \\ & + \frac{f_0}{2\pi^2}\left\{ \int_M d^4x\sqrt{g}\left[-\frac{3}{5}C^2_{\mu\nu\rho\sigma} + \frac{11}{30}R^*R^* + a|D_\mu\varphi|^2 + \frac{1}{6}R\left(a|\varphi|^2 + \frac{1}{2}c\right) + g_3^2(G^i_{\mu\nu})^2 + g_2^2(F^\alpha_{\mu\nu})^2 + \frac{5}{3}g_1^2(B_{\mu\nu})^2 \right.\right. \\ & \left.\left. + b|\varphi|^4 + 2e|\varphi|^2 + \frac{1}{2}d - \frac{1}{3}a(|\varphi|^2)^{;\mu}_{;\mu} - \frac{2}{5}R^{;\mu}_{;\mu} \right] \right. \\ & + \frac{f_0}{2\pi^2}\left\{ \int_{\partial M} d^3x\sqrt{h}\left[\frac{1}{3}K\left(a|\varphi|^2 + \frac{1}{2}c\right) + \frac{2}{15}(5RK + 4KR^a_{nan} + 4K_{ab}R^c_{acb} + 18R_{anbn}K^{ab}) \right.\right. \\ & \left.\left.\left. + \frac{4}{315}(17K^3 + 39KK_{ab}K^{ab} - 116K^b_aK^c_bK^a_c) \right]\right]\right\}, \end{aligned}$$

where $f_n = \int_0^\infty v^{n-1} f(v)dv$, and

$$a = \mathrm{Tr}\,(3|k^u|^2 + 3|k^d|^2 + |k^e|^2 + |k^\nu|^2),$$
$$b = \mathrm{Tr}\,(3|k^u|^4 + 3|k^d|^4 + |k^e|^4 + |k^\nu|^4),$$
$$c = \mathrm{Tr}\,(|k^{\nu_R}|^2),$$
$$d = \mathrm{Tr}\,(|k^{\nu_R}|^4),$$
$$e = \mathrm{Tr}\,(|k^{\nu_R}|^2|k^\nu|^2).$$

In the above expression, g_1, g_2, and g_3 are the $U(1)$, $SU(2)$, and $SU(3)$ gauge couplings with the corresponding gauge field strengths $B_{\mu\nu}$, $F^\alpha_{\mu\nu}$, and $G^i_{\mu\nu}$, and where the Higgs doublet is φ and the Yukawa fermionic couplings are given by the 3×3 matrices k^u, k^d, k^e, k^ν, and k^{ν_R}. The first few boundary terms depend only on the gravitational fields, while the Higgs field would begin to appear in the a_4 term. Contributions of the vector fields drop out completely if we make the assumption that their normal components vanish on the boundary: $A_n|_{\partial M} = 0$. Remarkably the terms $\frac{1}{6}R(a|\varphi|^2 + \frac{1}{2}c)$ and $\frac{1}{3}K(a|\varphi|^2 + \frac{1}{2}c)$ appear again with the same sign and the same relative factor of 2. This is a proof that the spectral action takes care of its self consistency.

From all these considerations we deduce that the simple requirement of having boundary conditions consistent with the Hermiticity of the Dirac operator is enough to guarantee that the spectral action has all the correct features and expected terms, including correct signs and coefficients.

Finally we note that we can include the effects of introducing a dilaton field to make the mass scale dynamical and obtain an almost scale invariant action. The main results were obtained recently [7] where it was shown that the dilaton interacts only through its kinetic term with a potential generated at the quantum level. The model has the same low-energy sector as the Randall-Sundrum model and the model of extended inflation. In the case of manifolds without boundary, the only modifications needed in the spectral action is the addition of the dilaton terms $\frac{8}{3\pi^2} f_2 \int_M d^4x\sqrt{G}G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. For manifolds with boundary there will be additional terms and these could play some role in cosmological considerations.

The research of A. H. C. is supported in part by the National Science Foundation under Grant No. Phys-0601213 and by the Arab Fund for Economic and Social Development.

[1] For a review, see C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Sect. 21.4–21.8.
[2] See the reviews of S. Hawking, in *Euclidean Quantum Gravity*, edited by G. Gibbons and S. Hawking (World Scientific, Singapore, 1993).
[3] S. Hawking and G. Horowitz, Classical Quantum Gravity **13**, 1487 (1996).
[4] A. Connes, *Noncommutative Geometry* (Academic, New York, 1994).
[5] Ali H. Chamseddine and Alain Connes, Phys. Rev. Lett. **77**, 4868 (1996); Commun. Math. Phys. **186**, 731 (1997).
[6] Ali H. Chamseddine, Alain Connes, and Matilde Marcolli, arXiv:hep-th/0610241.
[7] Ali H. Chamseddine and Alain Connes, J. Math. Phys. (N.Y.) **47**, 063504 (2006).
[8] H. Luckock, J. Math. Phys. (N.Y.) **32**, 1755 (1991).
[9] T. Branson and P. Gilkey, Differential Geometry and Its Applications **2**, 249 (1992).
[10] T. Branson and P. Gilkey, J. Funct. Anal. **108**, 47 (1992).
[11] E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, England, 2004), Chap. 3 and 4.
[12] D. Vassilevich, Phys. Rep. **388**, 279 (2003).
[13] Ali H. Chamseddine and Alain Connes (to be published).