

Rheology and Contact Lifetimes in Dense Granular Flows

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We study the rheology and distribution of interparticle contact lifetimes for gravity-driven, dense granular flows of noncohesive particles down an inclined plane using large-scale, three dimensional, granular dynamics simulations. Rather than observing a large number of long-lived contacts as might be expected for dense flows, brief binary collisions predominate. In the hard-particle limit, the rheology conforms to Bagnold scaling, where the shear stress is quadratic in the strain rate. As the particles are made softer, however, we find significant deviations from Bagnold rheology; the material flows more like a viscous fluid. We attribute this change in the collective rheology of the material to subtle changes in the contact lifetime distribution involving the increasing lifetime and number of the long-lived contacts in the softer particle systems.

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The rheology of granular materials is relevant to many areas of nature and industry, from mountain avalanches and mud slides, to grain transport and storage [1,2]. We study a particularly simple type of flow reminiscent of avalanches: gravity-driven, dense granular flow down a rough, inclined plane. This geometry is the archetypal granular flow with which one can study the relation between the stress state and the dynamics and structure, i.e., the constitutive relation of the granular material. Indeed, a number of recent, well-controlled experiments [3–6], and large-scale simulations [7–10] have motivated numerous theories that capture some of the features of inclined plane flows [11–19]. However, it remains a challenge to reconcile the various observations and results on the nature of particle collisions and contacts, and the mode of stress transmission in dense flows in general [20–24].

To date, the most generally accepted treatment of granular rheology is still the physical picture put forward by Bagnold over 50 years ago [25,26]. Bagnold described a mechanism of momentum transfer between particles in adjacent layers that assumes instantaneous binary collisions between the particles during the flow. Under this assumption, the inverse strain rate is the only relevant time scale in the problem leading to a constitutive relation between the shear stress σ and strain rate $\dot{\gamma}$ of the form

$$\sigma = \kappa \dot{\gamma}^2, \quad (1)$$

where κ is independent of $\dot{\gamma}$.

Despite recent concerns regarding the validity of Bagnold's original experiments [27], and the applicability of the theory [28], Eq. (1) has proved rather successful in predicting the steady-state rheology of dense granular flow down an inclined plane [3,8]. This is somewhat at odds with the original understanding of the Bagnold theory as it is based on the physics of rapidly flowing hard spheres where binary collisions predominate. Dense flows [29], on

the other hand, are thought to be controlled by enduring, multiple interparticle contacts forming extended stress-bearing structures [11] and/or large length-scale cooperative dynamics [16,30,31]. In fact, the stresses arising from such contact forces in dense flows are practically an order of magnitude larger than the kinetic stresses associated with velocity fluctuations [8].

In this Letter, we analyze both the constitutive relation and the interparticle contact dynamics of dense flows of noncohesive granular particles away from the fluid-solid transition. To discuss interparticle dynamics we introduce two time scales. The first is the lifetime of interparticle contacts and the second is the inverse strain rate in the material. The inverse strain rate or shear time sets the fundamental time scale over which particle rearrangement events occur during the flow.

We find that the typical contact lifetimes in the system are of the same order of magnitude as the binary collision time and insensitive to the location of the pair of contacting particles within the flowing pile. In contrast, the shear time is strongly dependent on the height and varies considerably over the parameter space of interparticle interactions studied here. The scenario that emerges is somewhat counterintuitive: Even in dense flows the dynamics at the microstructural scale remain predominately that of short-time, binary collisions. When the interparticle contact lifetime is significantly shorter than the shear time, the system is effectively in the hard-particle limit, and the Bagnold constitutive relation, Eq. (1), holds. Remarkably, the Bagnold relation is extremely sensitive to the development of a small population of long-lived contacts on the time scale of the shear time. Changes in the interparticle interaction that lead to the enhancement of these longer-lived contacts also lead to the breakdown of the Bagnold relation. In such cases it must be supplemented by an additional term linear in the strain rate. Thus the functional form of the constit-

utive law is sensitively dependent on changes in the long-time tail of the interparticle contact-time distribution.

Our simulations are based on the model developed by Cundall and Strack [32], and Walton [33], and has been shown to quantitatively match experimental data [3,8,9]. We study $N = 35,900$ monodisperse spheres of diameter d and mass m , flowing on a rough base of length $20d$, width $20d$, and tilted an angle θ with respect to gravity. We use periodic boundary conditions in the flow and vorticity directions. The height h of the flowing pile is between $90d < h < 100d$ depending on the angle of inclination. For most of the results presented here, the inelastic collisions are modeled as a Hookean spring and dashpot for forces both normal (n) and tangential (t) to the interparticle contact plane, but similar results were found for Hertzian contacts [34]. At existing contacts we include a friction model that obeys the Coulomb yield criterion with a coefficient of friction $\mu = 0.5$.

For Hookean contacts the coefficient of restitution parameterizes the dissipative nature of the interparticle collisions and is given by $e_n = \exp(-\gamma_n t_{\text{col}}/2)$. Here t_{col} is the binary collision time,

$$t_{\text{col}} = \pi[2k_n/m - \gamma_n^2/4]^{-1/2}, \quad (2)$$

where k_n and γ_n parameterize the elastic and dissipative normal interparticle forces, respectively. By simultaneously adjusting these parameters, we fix $e_n = 0.88$ for the results presented here. Varying e_n has little effect on the lifetime distributions and the conclusions of this work. For the tangential interactions we set $k_t = 2k_n/7$ and $\gamma_t = 0$.

We explore a range of spring constants k_n , from 10^3 – 10^9 mg/d . For $k_n \leq 2 \times 10^5 \text{ mg/d}$, the time step for numerical integrations $\delta t \sim 10^{-4} \tau$, where time is in units of $\tau = \sqrt{d/g}$. δt is adjusted accordingly for larger k_n , to ensure accurate dynamics. After reaching a dynamical steady state, characterized by time-independent total energy, where the mean flow velocity ranged from 10 – $100 \sqrt{d/g}$ depending on the system parameters, we collected data over a period of $\sim 10^6 \delta t$.

In Figure 1 we show the distributions, $P(\tau_c^* \equiv \tau_c/t_{\text{col}})$, of two-particle contact lifetimes, τ_c , normalized by the binary collision time t_{col} , for different k_n . Under this time rescaling all distributions exhibit a prominent short-time peak near the binary collision time $\tau_c^* = 1$, and an approximately exponential decay towards longer contact lifetimes. As k_n is reduced this exponential tail becomes broader indicating an increasing density of enduring contacts. The data in Fig. 1 are depth averaged over contacts in the flowing pile away from the bottom boundary and the top saltating layer, at an inclination angle, $\theta = 23^\circ$. The distributions are qualitatively unchanged over the range of accessible angles. We return to the more subtle dependence of τ_c on θ in our discussion of Fig. 4(b).

The normalized mean contact lifetime exhibits essentially no depth dependence as shown in Fig. 2(a). Given the

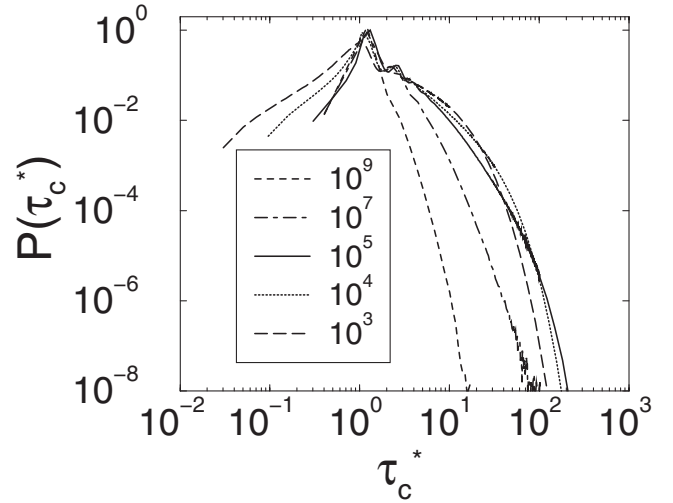


FIG. 1. Dependence of the distributions $P(\tau_c^*)$ of the depth-averaged normalized contact lifetimes, $\tau_c^* \equiv \tau_c/t_{\text{col}}$, on particle stiffness $k_n = 2 \times [10^9, 10^7, 10^5, 10^4, 10^3] \text{ mg/d}$. Data for $\theta = 23^\circ$, but the distributions are only weakly sensitive to θ .

stress-free boundary condition at the free surface of the pile, $\dot{\gamma} \rightarrow 0$ there so the normalized inverse strain rate, $\dot{\gamma}^{*-1} \equiv \dot{\gamma}^{-1}/t_{\text{col}}$, shown in Fig. 2(b), must be height dependent. It also depends strongly on k_n , varying by several orders of magnitude in our data set.

As shown in Fig. 3, over several orders of magnitude in k_n , the mean normalized contact time remains nearly constant while the maximum contact time $\tau_{c,\text{max}}$, extracted from the distributions in Fig. 1, decreases with increasing stiffness, reflecting the narrowing of the contact-time distributions as the particles become harder. Additionally, the average number of contacting neighbors per particle n_c , shown in Fig. 3(b), tends to the binary collision limit ($= 0$) as the particles become harder. Thus, from a coordination point of view the flows are not densely packed. Taken in combination, the data presented in Figs. 1–3 show that increasing the stiffness of the interparticle interaction reduces the weight of the exponentially small long-time tail of the contact lifetime distribution. This narrowing of the

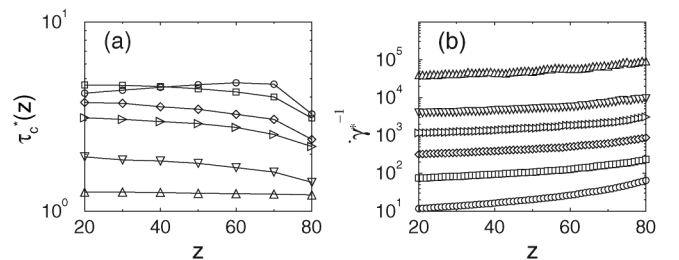


FIG. 2. Depth profiles of, (a) the normalized contact time $\tau_c^*(z)$ (line is a guide to the eye) and, (b) the normalized shear time $\dot{\gamma}^{*-1} \equiv \dot{\gamma}^{-1}/t_{\text{col}}$, for different particle stiffness, $k_n = 2 \times [10^9(\Delta), 10^7(\nabla), 10^6(\triangleright), 10^5(\diamond), 10^4(\square), 10^3(\circ)] \text{ mg/d}$. $z = 0$ defines the bottom of the pile. Flow at $\theta = 23^\circ$.

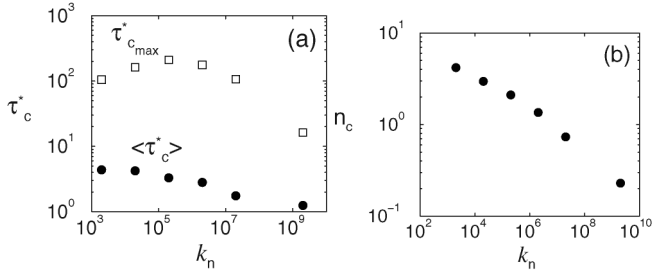


FIG. 3. Dependence on the particle stiffness k_n , of, (a) the depth-averaged contact lifetime $\langle \tau_c^* \rangle$ (filled circles) and maximum contact time $\tau_{c\max}^*$ (open squares), both normalized by t_{col} , and (b) the depth-averaged number of contact neighbors per particle, n_c . Flow at $\theta = 23^\circ$.

contact lifetime distribution is not observable from the more coarse measure of the mean contact lifetime distribution, which remains close to the binary collision time t_{col} . Thus the particle dynamics are *always* dominated by binary collisions, but changing the effective particle stiffness causes subtle changes in the number of rare, enduring contacts.

Since the interparticle contact lifetime is generically small compared to the shear time, one might imagine that momentum transport is dominated by effectively instantaneous binary collisions, and might expect that the Bagnold constitutive law holds. To test this we now turn to a characterization of the granular rheology by fitting the velocity profile of the flowing material to the prediction of a modified Bagnold relation [35] of the form

$$\sigma = \kappa \dot{\gamma}^2 + \beta \dot{\gamma}, \quad (3)$$

where the coefficients κ and β are determined by a least-squares fit to the velocity data. To characterize the devia-

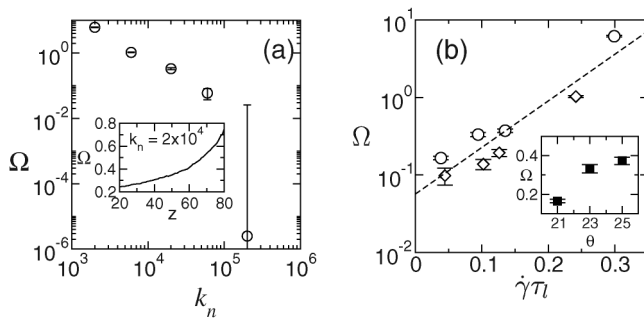


FIG. 4. (a) Dependence of the parameter Ω on the stiffness k_n for $\theta = 23^\circ$. For higher k_n the best fit is consistent with $\Omega \rightarrow 0$ (data not shown). Taken from the middle of the pile at $h/2$. Inset: Height dependence of Ω , for $k_n = 2 \times 10^4 \text{ mg/d}$. (b) Ω vs $\dot{\gamma} \tau_l$ as defined in the text for Hookean (\circ : $k_n = 2 \times \{10^3, 10^4\} \text{ mg/d}$) and Hertzian (\diamond : $k_n = 2 \times \{10^4, 10^5\} \text{ mg/d}$) systems at inclination angles shown in the inset, which shows the dependence of Ω on θ on a linear scale. The dashed line in main panel of (b) is a best-fit exponential. The inclination angle is the most sensitive control of the shear rate $\dot{\gamma}$.

tions from standard Bagnold rheology, we define the ratio of the shear stress in the Bagnold and linear forms to be $\Omega \equiv \beta / \kappa \dot{\gamma}$.

Figure 4(a) shows the dependence of Ω on k_n for $\theta = 23^\circ$. In the hard-particle limit we expect $\Omega \rightarrow 0$. This is practically achieved already for $k_n \geq 2 \times 10^5 \text{ mg/d}$, indicating that this value of k_n is appropriate for modeling hard (glass) particles in a system of this size [8,9]. As the particles are made softer, Ω grows meaning that the constitutive law approaches a linear relation reminiscent of a viscous fluid. Since Ω is inversely dependent on $\dot{\gamma}$, it is no surprise that it grows monotonically as one approaches the free surface as is shown by the inset to Fig. 4(a). We have examined the (weaker) dependence of Ω on all other particle interaction parameters, e.g., friction and inelasticity; these results will be discussed elsewhere [34].

For these dense flows the contact dynamics of *all* systems studied are dominated by binary collisions, that endure for no more than t_{col} , such that $\langle \tau_c \rangle \dot{\gamma} \ll 1$. But their rheology changes dramatically with k_n , going from the quadratic-in-shear-rate Bagnold law for hard particles to a nearly linear or viscous relation for softer particles. The dramatic change in the constitutive relation is controlled by a more subtle feature of the contact-time distribution than simply the mean value. The significant rheological change appears to be due to the growth of the lifetime and number of long-lived contacts in the softer systems. We emphasize that the fraction of such enduring contacts remains small in comparison to the more common short-lived contacts with $\tau_c \sim t_{\text{col}}$.

The growth of a small population of long-lived contacts leads to the formation of transient stress-bearing structures within the flowing material, which can be rheologically significant. These structures span streamlines in the flow and thus elastically transmit stress across their length in proportion to the rate of particle impacts with these structures $\sim \dot{\gamma}$. This picture is also consistent with the rapid-quasistatic transition in shear flows [36]. This reasoning suggests that the shear rate is the appropriate local clock with which to measure contact lifetimes τ_c so that the dimensionless contact lifetime $\tau_c \dot{\gamma}$ is the fundamental quantity controlling deviations from Bagnold rheology in the flowing state. Given similar contact-time distributions, faster flows should then deviate more strongly from the Bagnold prediction than slower flows.

To explore this point, we use the inclination angle of the pile to adjust the overall shear rate and study the resulting change in the observed rheology as parameterized by Ω . In Fig. 4(b) we plot Ω vs $\dot{\gamma} \tau_l$ for soft flows with parameters spanning our range of stable angles and particle stiffnesses. Here, τ_l is the lifetime of long-lived contacts in the system, as defined by the criterion that these contacts are sufficiently far out in the long-time tail of the unnormalized contact-time distribution where $P(\tau_c) = 10^0$. For these systems, where we see pronounced deviations from the

Bagnold constitutive relation, Ω increases with increasing angle as shown by the inset to Fig. 4(b). All variations of Ω can be collapsed onto a single master curve showing that the system's rheology depends (exponentially) on the dimensionless contact lifetime $\tau_l \dot{\gamma}$. The increasing value of these longer-lived contacts measured relative to the fundamental clock-rate $\dot{\gamma}$ drives the system from Bagnold to viscous rheology. Simulations with Ω consistent with 0 or ∞ are omitted from Fig. 4(b), but follow the displayed trend. Attempting to similarly collapse all data sets using the dimensionless mean contact time $\dot{\gamma} \langle \tau_c \rangle$ fails. Only the change in the long-time tail of the contact distribution correlates with the observed changes in the rheology.

In summary, we have provided extensive simulation results on the rheology and interparticle contact statistics of gravity-driven, dense, granular flows of noncohesive grains. We observe a transition from a Bagnold constitutive relation to one reminiscent of a Newtonian fluid as the particles are made softer. Based on our numerical data, and to account for this transition, we suggest a generalized Bagnold relation [35] to quantify the changing rheology. Furthermore, despite the naive guess that the flow in such dense systems should predominantly involve long-lived, stress-bearing structures, it turns out that for both hard and soft particles the microscopic particle dynamics is dominated by frequent, short-time, binary collisions, in accordance with experimental evidence [20]. When examining the entire contact lifetime distribution, however, we note that as the particles are made softer there is a growth in the width of the distribution and that the lifetime of rare, long-lived contacts grows relative to the shear time. Based on the data collapse shown in Fig. 4(b), we propose that the emergence of these atypically long-lived contacts is related to the dramatic change in the granular rheology with particle stiffness and thereby rationalize the surprising result that larger inclination angles, and hence faster flows, are actually less Bagnold in rheology than slower flows at smaller inclination angles. It appears that the constitutive relation of granular media interpolates between these extremes and is controlled by a combination of the particle hardness and flow rate.

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