Resonant Excitation and Nonlinear Evolution of Waves in the Equatorial Waveguide in the Presence of the Mean Current

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We study interactions of planetary waves propagating across the equator with trapped Rossby or Yanai modes, and the mean flow. The equatorial waveguide with a mean current acts as a resonator and responds to planetary waves with certain wave numbers by making the trapped modes grow. Thus excited waves reach amplitudes greatly exceeding the amplitude of the incoming wave. Nonlinear saturation of the excited waves is described by an amplitude equation with one or two attracting equilibrium solutions. In the latter case spatial modulation leads to formation of characteristic defects in the wave field. The evolution of the envelopes of long trapped Rossby waves is governed by the driven complex Ginzburg-Landau equation, and by the damped-driven nonlinear Schrödinger equation for short waves. The envelopes of the Yanai waves obey a simple wave equation with cubic nonlinearity.

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In [1,2] we studied interactions of a pair of waves trapped in the equatorial waveguide and a planetary wave freely propagating across the equator in the linear approximation, and showed that the planetary wave resonantly excites the waveguide modes with significant amplitudes. The nonlinear evolution of the excited modes was described by the parametrically driven Ginzburg-Landau (GL) equation and resulted in nontrivial spatiotemporal organization. Below, by a similar technique, we attack the problem of resonant interaction of planetary waves with the trapped ones via the mean zonal current. Beyond the practical importance of this case (there are persistent zonal currents in the equatorial ocean and atmosphere), the dynamics turns out to be different here. First, the resonance is possible only for a discrete spectrum of planetary waves. Second, the resonant growth of the trapped waves is linear (nonmodal) and not exponential, which leads to the damped-driven GL and nonlinear Schrödinger (NLS) evolution equations instead of parametrically driven ones. Third, the nonlinear evolution of short and long excited waves is qualitatively different, and the evolution of nonlinear Yanai and Rossby waves is also different due to different dispersion properties. It should be also noted that both free wave—trapped wave—mean flow (this Letter), and free wave—trapped wave—trapped wave [1] mechanisms of resonant excitation are relevant for other species of trapped waves, e.g., the coastal edge waves, [3], or topographic Rossby or double Kelvin waves.

As in [1,2], we use the nondimensional 2-layer rotating shallow water model on the equatorial tangent plane, written in terms of barotropic ψ and baroclinic h, $\mathbf{u} = (u, v)$ fields

$$\nabla^2\psi_t + \psi_x = \epsilon[-J(\psi, \nabla^2\psi) - \tfrac{1}{4}(\partial_{xx} - \partial_{yy})(uv) + \tfrac{1}{4}\partial_{xy}(u^2 - v^2)], \qquad (1)$$

$$\mathbf{u}_t + \nabla h + y\hat{\mathbf{z}} \times \mathbf{u} = \epsilon\Big[-J(\psi, \mathbf{u}) + \mathbf{u}\cdot\nabla(\hat{\mathbf{z}} \times \nabla\psi) + \frac{\epsilon}{4}(2h\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\mathbf{u}\cdot\nabla h)\Big], \qquad (2)$$

$$h_t + \nabla\cdot\mathbf{u} = \epsilon\Big[-J(\psi, h) + \frac{\epsilon}{4}\nabla\cdot(h^2\mathbf{u})\Big], \qquad (3)$$

where the subscripts denote partial derivatives, J is the Jacobian, and $\epsilon \ll 1$. It is assumed in (1)–(3) that both layers have the same depth. This assumption is not crucial, but it simplifies calculations; cf. (8) and (9) in [2].

The linear spectrum of the model consists of planetary (= barotropic Rossby) waves which propagate at any angle with respect to the equator

$$\tilde{\psi}_0 = A_\psi e^{i(\theta + ly)} + \text{c.c.}; \qquad \theta = kx - \sigma t, \qquad (4)$$

and have the dispersion relation

$$\sigma = -k/(k^2 + l^2), \qquad (5)$$

and of trapped baroclinic waves

$$(\tilde{u}, \tilde{v}, \tilde{h}) = (iU_m, \phi_m, iH_m)Ae^{i\theta_m} + \text{c.c.}; \quad \theta_m = kx - \sigma_m t \qquad (6)$$

with the dispersion relation

$$\sigma_m^3 - (k^2 + 2m + 1)\sigma_m - k = 0; \qquad m = 0, 1, 2, \ldots, \qquad (7)$$

where m is the meridional wave number. We are interested in the low-frequency Yanai wave with $m = 0$, $\sigma \le 1$, and in Rossby waves with $m \ge 1$, $\sigma < 1$. The functions U_m, ϕ_m, H_m decay rapidly off the equator ($y = 0$)

$$\phi_m(y) = \frac{\mathcal{H}_m(y)e^{-(y^2/2)}}{\sqrt{2^m m!\sqrt{\pi}}}, \qquad U_m(y) = \frac{\sigma_m y\phi_m - k\phi'_m}{\sigma_m^2 - k^2},$$
$$H_m(y) = \frac{ky\phi_m - \sigma_m\phi'_m}{\sigma_m^2 - k^2}, \tag{8}$$

where $\mathcal{H}_m(y)$ are the Hermite polynomials and prime means y differentiation.

We consider weakly nonlinear interactions of a barotropic wave (4) and a trapped baroclinic wave (6) with the baroclinic equatorial zonal flow which is an exact solution to (1)–(3):

$$u = \epsilon^\alpha(y), \quad h = \epsilon^\alpha(y), \quad v = 0,$$
$$y\bar{u} + \bar{h}_y = 0, \quad \bar{\psi} = 0. \tag{9}$$

The parameter $-1 < \alpha \le 0$ measures the strength of the zonal flow. Note that such flow is still insufficiently strong to change the barotropic and the baroclinic waves at the lowest order. To check that such interactions produce resonant growth of the baroclinic waves with subsequent nonlinear saturation, the first step is to verify the synchronism conditions. As seen from (2) and (3), if the zonal wave number k and the frequency σ of the barotropic wave coincide with those of a trapped baroclinic mode, then the latter is resonantly excited by the barotropic wave—mean flow interaction. By virtue of (5) and (7) this is possible if

$$l^2 = 2m + 1 - \sigma_m^2, \qquad m = 0, 1, \ldots \tag{10}$$

and, hence, for $\sigma_m < 1$ the corresponding barotropic mode always exists. Thus, only a discrete spectrum of barotropic Rossby waves resonates with waveguide modes in the presence of the equatorial current.

The next step is finding time dependence of the amplitude of resonantly excited waves by removal of resonances in (1)–(3); cf. [2]. We first apply it to the direct multi-timescale expansion in ϵ of the form

$$\psi = \psi^{(0)}(x, y, t, T, \ldots) + \psi^{(1)}(x, y, t, T, \ldots), \tag{11}$$

$$(u, v, h) = \epsilon^\alpha(\bar{u}^{(0)}, \bar{v}^{(0)}, \bar{h}^{(0)})(y, t, T, \ldots) + (\tilde{u}^{(0)}, \tilde{v}^{(0)}, \tilde{h}^{(0)})(x, y, t, T, \ldots) + (u^{(1)}, v^{(1)}, h^{(1)}) + \ldots. \tag{12}$$

Here $\psi^{(0)}$ and $(\tilde{u}^{(0)}, \tilde{v}^{(0)}, \tilde{h}^{(0)})$ are the barotropic wave (4) and the trapped mode (6) satisfying the synchronism conditions, the zonal flow (9) is $\epsilon^\alpha(\bar{u}^{(0)}, \bar{h}^{(0)})$. The corrections $(\ldots)^{(1)}$ are assumed to be smaller than the lowest-order fields. The slow time is $T = \epsilon^\gamma t$, $\gamma > 0$. By removing the resonances at the lowest order we get that both the barotropic wave and the zonal flow remain unchanged, and the amplitude of the baroclinic wave grows linearly in time

$$a_0 A_T = -kL_\psi A_\psi, \qquad T = \epsilon^{\alpha+1}t, \tag{13}$$

where

$$a_0 = \int_{-\infty}^{\infty} dy(U_m^2 + \phi_m^2 + H_m^2), \tag{14}$$

$$L_\psi = \int_{-\infty}^{\infty} dye^{ily}[\bar{h}_y^{(0)}H_m - (k\phi_m + 2ilU_m + U'_m)\bar{u}^{(0)}]. \tag{15}$$

The energy of the primary barotropic wave is infinite compared to that of the trapped wave, which explains that the amplitude of the former remains constant. Similarly to [1,2] the growth of the baroclinic mode is energetically compensated by interaction between the primary barotropic mode $\psi^{(0)}$ and the growing barotropic correction $\psi^{(1)}$, which is engendered by the growing baroclinic mode and is determined from

$$\nabla^2\psi_t^{(1)} + \psi_x^{(1)} = \frac{\epsilon}{4}\{-(\partial_{xx} - \partial_{yy})(u^{(0)}v^{(0)}) + \partial_{xy}[(u^{(0)})^2 - (v^{(0)})^2]\}. \tag{16}$$

The interaction of $\psi^{(1)}$ with the zonal flow and the baroclinic wave in (2) and (3) arrests the growth of A. The level of saturation depends on the strength of the initial zonal flow. The maximum level of saturation $\mathcal{O}(\epsilon^{-(1/2)})$ is reached for the zonal flow $\mathcal{O}(\epsilon^{-(1/2)})$. In this case, the equation describing both the initial stage of linear growth and its saturation arises from the rearranged asymptotic expansions, [2,3]:

$$\psi = \psi_0(x, y, t, T_1, T_2, \ldots) + \epsilon^{1/2}\psi_1(x, y, t, T_1, T_2, \ldots) + \ldots,$$
$$(u, v, h) = \epsilon^{-(1/2)}(u_0, v_0, h_0)(x, y, t, T_1, T_2, \ldots) + (u_1, v_1, h_1)(x, y, t, T_1, T_2, \ldots) + \ldots, \tag{17}$$

where $T_n = \epsilon^{n/2}t$, ψ_0 contains both primary $\psi^{(0)}$ and secondary $\psi^{(1)}$ barotropic waves, and the lowest-order terms in (u, v, h) are the same as in (12) with $\alpha = -\frac{1}{2}$, $\gamma = \frac{1}{2}$. Eliminating resonances while finding the baroclinic correction (u_1, v_1, h_1) gives

$$A_{T_2} + pA + q|A|^2A = -c_0A_\psi, \qquad c_0 = \frac{kL_\psi}{a_0}. \tag{18}$$

The amplitude of the primary barotropic wave and the zonal flow remain unchanged at the leading order. The real parts of p, q are

$$\mathrm{Re}\, p = \frac{1}{8|l|\sigma a_0}\left|\int_{-\infty}^{+\infty} dyF_1(y)e^{ily}\right|^2, \tag{19}$$

$$\mathrm{Re} q = \frac{1}{16|\bar{l}|\sigma a_0}\left|\int_{-\infty}^{+\infty} dyF_2(y)e^{i\bar{l}y}\right|^2, \ \text{if } \bar{l}^2 = l^2 - 3k^2 > 0,$$
$$\mathrm{Re} q = 0, \ \text{if } \bar{l}^2 = l^2 - 3k^2 < 0, \tag{20}$$

where

$$F_1 = (\phi_m \bar{u}_0)'' - 2k(U_m \bar{u}_0)' + k^2 \phi_m \bar{u}_0,$$
$$F_2 = (\phi_m U_m)'' - 2k(U_m^2 + \phi_m^2)' + 4k^2 \phi_m U_m. \qquad (21)$$

The expressions for Imp, Imq are similar, but rather cumbersome.

Hence, Re$p > 0$, Re$q \geq 0$ and saturation always takes place. The "linear" saturation produced by RepA is due to the interaction of the secondary barotropic mode ψ_1 with the zonal flow, while the "nonlinear" one produced by Re$q|A|^2A$ is due to the interaction of ψ_1 with the baroclinic wave. The mechanism of saturation is different for short (Re$q = 0$) and long (Re$q \neq 0$) waves.

By renormalizing A and T_2 the number of parameters in (18) may be reduced:

$$A_T + e^{i\xi}A + e^{i\eta}|A|^2A = c|A_\psi|, \qquad \xi = Arg p,$$
$$\eta = Arg q, \qquad \mathrm{Im} c = 0. \qquad (22)$$

For time-independent solutions, a cubic equation for $|A|^2$ follows:

$$|A|^6 + 2\cos\chi|A|^4 + |A|^2 - c^2|A_\psi|^2 = 0,$$
$$\chi = \xi - \eta, \qquad (23)$$

having either three positive roots, or a single positive root. In the case of a single root, it is always stable, and in the case of three roots, the largest and the smallest ones are stable, while the intermediate one is unstable. Stable solutions are attracting and, depending on the coefficients, the origin may lie in the domain of attraction of either the smaller or the larger root; see Fig. 1.

The final step is to include spatial modulation. We again consider a zonal current $\sim\epsilon^{-(1/2)}$ and introduce in (17) the spatial modulation scales $X_n = \epsilon^{n/2}x$, $n = 1, 2, \ldots$, [2]. The modulation equations combining two leading orders for A and A_ψ are

$$(\partial_{T_1} + c_g^{bt}\partial_{X_1})A_\psi - \epsilon^{1/2}\frac{i}{2}(\sigma^{bt})''\partial^2_{X_1X_1}A_\psi = 0, \qquad (24)$$

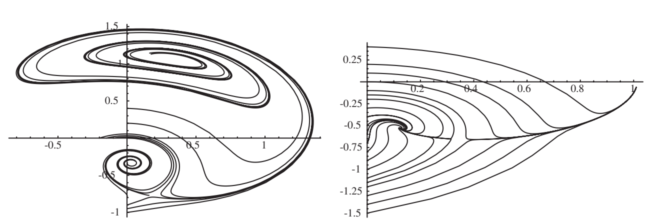

FIG. 1. Phase portraits of the system (22) with $\xi = 19\pi/20$, $c|A_\psi| = 0.3$ in the ReA-ImA plane with and without nonlinear saturation; left panel: $\eta = \pi/2$; Re$q = 0$; right panel: $\eta = 0.4\pi$; Re$q \neq 0$. Stable stationary solutions correspond to the foci, the unstable ones correspond to the saddle points.

$$(\partial_{T_1} + c_g^{bc}\partial_{X_1})A + \epsilon^{1/2}\left[-\frac{i}{2}(\sigma^{bc})''\partial^2_{X_1X_1}A + pA + q|A|^2A\right] = -\epsilon^{1/2}c_0A_\psi. \qquad (25)$$

Here $\sigma^{bt,bc}$ are the frequencies of the barotropic and the baroclinic waves, $c_g^{bt,bc} = (\sigma^{bt,bc})'$ are the corresponding zonal group velocities, and prime denotes differentiation with respect to k. A crucial remark is that the group velocity of the Yanai wave may differ significantly from the group velocity of the barotropic Rossby wave of the same frequency, e.g., $|k| \ll 1$, $c_g^{bc} \approx \frac{1}{2} \ll c_g^{bt} \approx -\frac{1}{|k|}$ for long waves. On the contrary, the group velocities of the baroclinic and the barotropic Rossby waves of the same frequency are practically the same.

In the case of Yanai wave excitation, the only situation where barotropic and baroclinic waves can significantly interact is that of "gentle" spatial modulation when the fields depend on X_2, and not on X_1, and on T_2, and not on T_1. In this case dispersion effects are weak, and we get a nondispersive propagation of modulations of the barotropic wave

$$\partial_{T_2}A_\psi + c_g^{bt}\partial_{X_2}A_\psi = 0, \qquad (26)$$

and the damped-driven simple wave equation with cubic nonlinearity for the baroclinic wave

$$\partial_{T_2}A + c_g^{bc}\partial_{X_2}A + pA + q|A|^2A = -c_0A_\psi. \qquad (27)$$

By choosing the reference frame moving with the group velocity of the barotropic wave c_g^{bt}, this system is reduced to a single damped-driven nonlinear simple wave equation with space-dependent forcing.

In the case of Rossby wave excitation, by choosing the reference frame moving with the common group velocity we get the linear Schrödinger equation for barotropic modulations

$$\partial_{T_2}A_\psi - \frac{i}{2}(\sigma^{bt})''\partial^2_{X_1X_1}A_\psi = 0, \qquad (28)$$

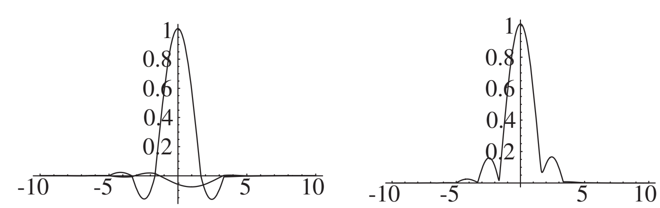

FIG. 2. Excitation of a long Yanai wave, DNS of the normalized envelope equations (26) and (27): left panel: profiles of Re$A(X_2)$ (larger values) and Im$A(X_2)$; right panel: Abs$A(X_2)$. Profiles at $T_2 = 30$ in a refererence frame moving with the barotropic wave; $\eta = -0.4\pi$, $\xi = 19\pi/20$, $c = 1$. Barotropic wave is Gaussian with maximum amplitude 0.4. The forcing sweeps the domains of attraction of both stationary states. Domain-wall defects appear in locations where ReA and ImA intersect simultaneously with the zero level.

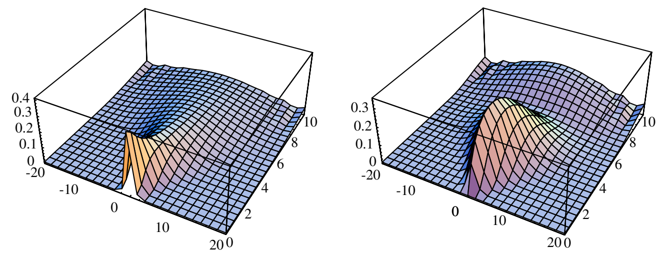

FIG. 3 (color online). Excitation of a long Rossby wave, DNS of the normalized envelope Eqs. (28) and (29): spatiotemporal evolution of A_ψ (*left panel*), and A (*right panel*) for $-20 \leq X_1 \leq 20$, $0 \leq T_2 \leq 10$; $\eta = -0.4\pi$, $\xi = 19\pi/20$. Note the time-lag between A_ψ and A.

and the equation for baroclinic modulations

$$\partial_{T_2} A - \frac{i}{2}(\sigma^{bc})'' \partial^2_{X_1 X_1} A + pA + q|A|^2 A = -c_0 A_\psi. \quad (29)$$

Both for Rossby and Yanai wave excitations the modulation equations are different for waves with small and large zonal wavelength. In the case of short enough Yanai and Rossby waves, respectively; cf. (10) and (20)

$$|k| > \frac{1}{2\sqrt{3}}, \qquad m = 0,$$
$$|k| \geq \sqrt{\frac{2m+1}{3}}, \qquad m = 1, 2, \ldots \quad (30)$$

there is no nonlinear saturation (Re$q = 0$), while it does work for long enough waves. Thus, for short Rossby waves by rescaling A in (29) with the time-depending phase we get the well-known damped—driven NLS equation. This equation, in the case of spatially nonmodulated driver corresponding in our context to a plane incoming barotropic Rossby wave, was a subject of numerous studies following the pioneering work [5]. It is known that depending on the values of damping and forcing it may exhibit chaotic (with different types of chaos, e.g., [6]) or regular behavior [7], and possesses stationary localized soliton solutions in some windows of parameters [8]. (For Re$q = 0$ (22) is a variant of the equation for the flat-locked states studied in the damped-driven NLS literature). Thus, such dynamical patterns are to be also expected in equatorial dynamics.

A driven complex Ginzburg-Landau (CGL) equation arises for long Rossby waves with Re$q \neq 0$. The phase diagram of the undriven CGL is well established [9]. There are some works on driven 1d CGL in the context of turbulence control [10]. However, we are not aware of a systematic study of the driven 1d CGL. On general grounds, in the case with two different flat-locked stationary solutions we expect appearance of the domain-wall like defects, and hence a possibility of the defect chaos. The coherent structures may be sought by the same method as

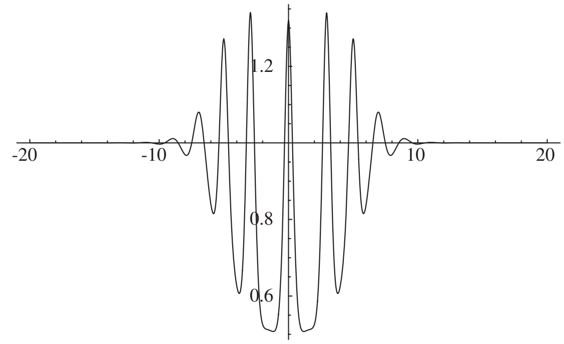

FIG. 4. Excitation of a long Rossby wave by a plane barotropic wave, DNS of the normalized envelope Eq. (29) with $cA_\psi = 0.4$ and two different stationary solutions: Abs$A(X_1)$ at $T_2 = 30$; $\eta = -0.4\pi$, $\xi = 19\pi/20$. Initial $A = ie^{-x^2}$ sweeps the domains of attraction of both stationary states; cf. Figure 1(b); the displayed state is nonstationary; cf. [8] for *stationary* structures of similar form in damped-driven NLS.

in the damped-driven NLS [8]. The appearance of defects is also expected in the Yanai wave case (27).

The detailed analysis of the systems (26)–(29), will be given elsewhere. Figures 2–4 below give some results of direct numerical simulations (DNS) for long waves Re$q \neq 0$ (see [7,8] for the DNS of damped-driven NLS corresponding in our context to the excitation of short Rossby waves). Thus, the equatorial waveguide with a baroclinic zonal current acts as a resonator: it responds to incoming barotropic waves by amplifying the trapped baroclinic Yanai or Rossby waves, which are then saturated at levels largely exceeding the amplitude of the incoming wave. Nontrivial spatiotemporal organization and/or chaotic behavior result for the envelopes of thus excited waves.

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