Electroweak Radiative Corrections to Muon Capture

Andrzej Czarnecki
Department of Physics, University of Alberta Edmonton, AB T6G 2J1, Canada

William J. Marciano
Brookhaven National Laboratory, Upton, New York 11973, USA

Alberto Sirlin
Department of Physics, New York University, 4 Washington Place, New York, New York 10003, USA
(Received 26 April 2007; published 16 July 2007)

Electroweak radiative corrections to muon capture on nuclei are computed and found to be sizable. They enhance the capture rates for hydrogen and helium by 2.8% and 3.0%, respectively. As a result, the value of the induced pseudoscalar coupling, $g_P^{\rm exp}$, extracted from a recent hydrogen $1S$ singlet capture experiment is increased by about 21% to $g_P^{\rm exp} = 7.3 \pm 1.2$ and brought into good agreement with the prediction of chiral perturbation theory, $g_P^{\rm theory} = 8.2 \pm 0.2$. Implications for helium capture rate predictions are also discussed.

DOI: 10.1103/PhysRevLett.99.032003 PACS numbers: 12.15.Lk, 23.40.−s, 36.10.Dr

The study of muon capture by nuclei, $\mu^- N \rightarrow \nu_\mu N'$, has played an important role in the development of weak interaction physics [1,2]. Used primarily in the past to explore nuclear structure and its effects on weak interactions, muon capture can now be employed to test quantum chromodynamics (QCD) and its basic chiral symmetries [3]. In addition, it can provide a possible window or constraint on new high mass scale physics [4], beyond standard model expectations, such as additional gauge bosons, charged Higgs scalars, leptoquarks, etc. Of course, to be competitive with other precision low energy experimental tests of the standard model, both theory and experiment for muon capture must be known to a fraction of a percent.

Here, we would like to advance the theory of muon capture to that high level of precision by including standard model electroweak radiative corrections and estimating their degree of reliability. From our previous work [5–9] on neutron (and nuclear) β-decay, one can anticipate that such quantum loop effects are relatively large, ~2%–3%, and therefore important for any precision confrontation between muon capture theory and experiment. As we shall show, that indeed is the case.

We begin by recalling the basics of muon capture. Negative muons, μ^-, are stopped in matter. They bind electromagnetically with nuclei and quickly cascade down to the lowest energy atomic orbitals. There, primarily from $1S$ states, the muon's final fate is to undergo either ordinary muon decay, $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$, or weak capture $\mu^- N \rightarrow \nu_\mu N'$ on the nucleus.

Ordinary decay in orbit occurs essentially at the same rate as in vacuum (modulo bound state time dilation and other small effects [10,11]). The already well-known "free" muon lifetime has been recently remeasured [12], thereby leading to the improved world average

$$\tau_\mu = 2.197\,019(21) \times 10^{-6} \ \text{sec}. \tag{1}$$

Further improvement by a factor of 10 is expected.

The competing weak capture reaction, $\mu^- N \rightarrow \nu_\mu N'$, proceeds via W boson exchange with the nucleus. Because of an overlap flux factor from its atomic wave function at the origin and a factor of Z (nuclear charge) corresponding to the number of protons that can induce capture, the overall capture rate scales very roughly as Z^4. In hydrogen ($Z = 1$), the capture rate is predicted to be very small. From the $1S$ singlet (spin 0) $\mu^- p$ state, it is only about 0.16% of the ordinary decay rate and for the triplet (spin 1) bound state configuration, it is a tiny 0.0025%. Those small rates make experimental hydrogen capture studies difficult, which is unfortunate, since hydrogen theory is very clean. Decay and capture rates become comparable for $Z \simeq 10$, while at much higher Z, capture dominates.

An interesting technique used to obtain muon capture rates involves comparing free and bound μ^- lifetimes,

$$\Gamma(\mu^- N \rightarrow \nu_\mu N') = \frac{1}{\tau_\mu^{\rm bound}} - \frac{1}{\tau_\mu^{\rm free}}, \tag{2}$$

(after making small bound state lifetime corrections). Using an ingenious application of that lifetime technique, the MuCap collaboration [13] at PSI recently reported a precise measurement of the $1S$ singlet capture rate in hydrogen,

$$\Gamma(\mu^- p \rightarrow \nu_\mu n)_{1S}^{\rm singlet} = 725.0 \pm 13.7({\rm stat}) \pm 10.7({\rm syst})/{\rm sec}. \tag{3}$$

That already impressive $\pm 2.4\%$ level of accuracy is expected to further improve to better than $\pm 1\%$ as additional data are analyzed.

In the case of helium, the capture rate for $\mu^- {}^3{\rm He} \rightarrow \nu_\mu {}^3{\rm H}$ has been even better measured by directly detecting the charged final state ^{3}H. For the statistical combination of singlet (spin 0) and triplet (spin 1) $1S$ $\mu^-{}^3$He bound states,

$$\Gamma(\mu^-{}^3\text{He} \to \nu_\mu{}^3\text{H})_{\text{stat}} = \tfrac{1}{4}\Gamma(\mu^-{}^3\text{He} \to \nu_\mu{}^3\text{H})_{\text{singlet}} + \tfrac{3}{4}\Gamma(\mu^-{}^3\text{He} \to \nu_\mu{}^3\text{H})_{\text{triplet}}, \quad (4)$$

a long standing result [14]

$$\Gamma(\mu^-{}^3\text{He} \to \nu_\mu{}^3\text{H})^{\text{exp}}_{\text{stat}} = 1496(4)/\,\text{sec}, \quad (5)$$

represents a remarkable $\pm 0.3\%$ determination.

The standard model theoretical prediction for the basic $\mu^- p \to \nu_\mu n$ capture rate depends on four relativistic form factors that result from nucleon matrix elements of the V-A weak quark charged current,

$$\langle n|\bar{d}\gamma_\alpha(1-\gamma_5)u|p\rangle = \bar{u}_n(p_2)\bigg[F_1(q^2)\gamma_\alpha + \frac{i}{2m_N}F_M(q^2)\sigma_{\alpha\beta}q^\beta - g_A(q^2)\gamma_\alpha\gamma_5 - \frac{1}{m_\mu}g_P(q^2)q_\alpha\gamma_5\bigg]u_p(p_1), \quad (6)$$

where $q \equiv p_2 - p_1$, $m_N \equiv \frac{m_p+m_n}{2}$. Two other form factors, scalar and pseudotensor, are in general possible, but are negligibly small in the standard model (arising from isospin violation). They should, however, be included in general searches for "new physics" effects [4,15]. In terms of the above form factors, the capture rate is given by (modulo radiative corrections, discussed later)

$$\Gamma(\mu^- p \to \nu_\mu n)|_{\text{singlet}} = |\psi(0)|^2 \frac{G_\mu^2|V_{\text{ud}}|^2}{2\pi}\frac{E_\nu^2}{M^2}(M-m_n)^2\bigg\{\frac{2M-m_n}{M-m_n}F_1 + \frac{2M+m_n}{M-m_n}g_A - \frac{g_P}{2} + (2M+2m_n-3m_\mu)\frac{F_M}{4m_N}\bigg\}^2,$$

$$\Gamma(\mu^- p \to \nu_\mu n)|_{\text{triplet}} = |\psi(0)|^2 \frac{G_\mu^2|V_{\text{ud}}|^2}{24\pi}\frac{E_\nu^2}{M^2}(M-m_n)^2\bigg\{\bigg[g_P - \frac{2m_n}{M-m_n}(F_1-g_A) + (2M+2m_n-m_\mu)\frac{F_M}{2m_N}\bigg]^2 + 2\bigg[g_P + \frac{2M}{M-m_n}(F_1-g_A) - m_\mu\frac{F_M}{2m_N}\bigg]^2\bigg\}. \quad (7)$$

M denotes the mass of the $\mu^- p$ atom. We neglect the binding effect and use $M \equiv m_p + m_\mu$. The $\mu^- p$ hydrogenic wave function at the origin is

$$|\psi(0)|^2 = \frac{\mu^3\alpha^3}{\pi}(1 - 4\alpha\mu r_p) \simeq \frac{\mu^3\alpha^3}{\pi}(1 - 0.005), \quad (8)$$

where $\mu \equiv \frac{m_p m_\mu}{m_p+m_\mu}$ is the reduced mass and we have accounted for the proton charge distribution with the radius $r_p = \frac{0.862}{\sqrt{6}}$ fm (see [16,17] for a more detailed discussion).

Three of the four form factors in Eq. (6) are very well determined at $q^2 = 0$ from conserved vector-current constraint (CVC) and neutron β decay [8,18], $F_1(0) = 1$, $F_M(0) = 3.706$, $g_A(0) = 1.2695(29)$. Extrapolating to $q_0^2 = -0.88m_\mu^2$, as appropriate for μ^- capture on hydrogen, one finds

$$F_1(q_0^2) = 0.976(1), \quad F_M(q_0^2) = 3.583(3), \quad g_A(q_0^2) = 1.247(4), \quad (9)$$

where the errors include estimated q^2 evolution uncertainties.

In the case of the induced pseudoscalar coupling, $g_P(q_0^2)$, partially conserved axial-vector current (PCAC) and chiral perturbation theory predict [19–22]

$$g_P(q_0^2) = \frac{2m_\mu g_{\pi pn}(q_0^2)F_\pi}{m_\pi^2 - q_0^2} - \frac{1}{3}g_A(0)m_\mu m_N r_A^2, \quad (10)$$

which for $g_{\pi pn} = 13.05(20)$, $F_\pi = 92.4(4)$ MeV, and $r_A^2 = 0.43(3)$ fm^2 implies

$$g_P(q_0^2) = 8.2 \pm 0.2. \quad (11)$$

That prediction is expected to be very reliable, depending only on the chiral properties of QCD and principles of PCAC. Nevertheless, it would be very useful to have a first-principles lattice QCD calculation of $g_P(q_0^2)$ [as well as $g_A(q_0^2)$]. Of course, it is also very important to verify the prediction in Eq. (11) experimentally.

Employing the above form factors at q_0^2 and allowing for the variation $g_P(q_0^2) = 8.2 + \delta g_P$, one obtains from Eq. (7) the singlet $1S$ capture rate on hydrogen,

$$\Gamma(\mu^- p \to \nu_\mu n)^{\text{singlet}}_{1S} = 692.3(3.4)(1 + \text{RC(H)})(1 - 0.0108\delta g_P)^2/\,\text{sec}. \quad (12)$$

$G_\mu = 1.166\,371(6) \times 10^{-5}$ GeV^{-2} (the Fermi constant obtained from the free muon lifetime [12]), $V_{\text{ud}} = 0.9738$ and a 0.5% reduction from the finite proton size have been incorporated into Eq. (12). The $1 + \text{RC(H)}$ factor represents the effect of electroweak radiative corrections, which up until this work have not been seriously considered in discussions of muon capture [23,24]. If we set $\text{RC(H)} = 0$ and compare Eq. (12) with Eq. (3), we find $g_P(q_0^2) = 6.0 \pm 1.2$ which is about 2σ below the prediction in Eq. (11); however, that result is not very meaningful since we expect the radiative corrections to be sizable.

In the case of helium, the tree level theoretical prediction for muon capture is not as pristine. When compared with the same input parameters, two distinct approaches give somewhat different results. The first is based on an elementary particle prescription which treats ^{3}He and ^{3}H as initial and final particle states [3,25]. It then employs form factors analogous to those in Eq. (6) (but defined with an additional minus sign for all but F_1) at $q^2 = -0.954m_\mu^2$ appropriate for μ^- capture on $^3\text{He} \to {}^3\text{H}$. Using CVC for the vector form factors and PCAC to relate axial-vector and pseudoscalar form factors, the analysis leads to what has been viewed as a rather reliable ^{3}He capture rate prediction. It depends primarily on the input

$$g_A(q^2 = -0.954m_\mu^2)_{^3\mathrm{He}\to{}^3\mathrm{H}} = 1.052 \pm 0.005, \tag{13}$$

obtained by evolving $g_A(0)_{^3\mathrm{He}\to{}^3\mathrm{H}} = 1.212$, obtained from tritium β decay [26,27], to $q^2 = -0.954m_\mu^2$.

The second method for calculating the capture rate for $\mu^- {}^3\mathrm{He} \to \nu_\mu {}^3\mathrm{H}$ uses an impulse approximation to combine the basic $\mu^- p \to \nu_\mu n$ captures within ^{3}He [3,28]. It has been argued that when supplemented by meson exchange current corrections [29], this method agrees with the above (elementary particle) approach. However, a close scrutiny of the most detailed impulse approximation study [30] reveals some difference in their predictions.

Normalizing to $V_{\mathrm{ud}} = 0.9738$, the elementary particle model (EPM) approach predicts [28]

$$\Gamma(\mu^- {}^3\mathrm{He} \to \nu_\mu {}^3\mathrm{H})^{\mathrm{EPM}}_{\mathrm{stat}} = 1492(21)(1 + \mathrm{RC(He)})(1 - 0.013\delta g_P)/\sec, \tag{14}$$

while the impulse approximation study by Marcucci *et al.* [30] updated to a central value of $g_A = 1.2695(29)$ and $g_P = 8.2 + \delta g_P$ gives

$$\Gamma(\mu^- {}^3\mathrm{He} \to \nu_\mu {}^3\mathrm{H})^{\mathrm{IA}}_{\mathrm{stat}} = 1462(8)(7)_{g_A}(1 + \mathrm{RC(He)})(1 - 0.013\delta g_P)/\sec. \tag{15}$$

Again, we have allowed for inclusion of electroweak radiative corrections, RC(He), appropriate for capture. In both Eqs. (14) and (15) we assume the same dependence on δg_P as given in Refs. [4,28]. For clarity, we note that the values reported by Marcucci *et al.* [30] are larger than in Eq. (15) because these authors identified G_V^2 with a parameter $G_V'^2 \equiv 1.024|V_{\mathrm{ud}}|^2 G_\mu^2$, extracted from superallowed beta decays, in which inner radiative corrections of 2.4% were already included. In Eq. (15), we have factored out this 2.4% effect and included it in the overall RC(He) to be discussed below.

The prediction in Eq. (14) is in very good agreement with Eq. (5), $\Gamma(\mu^- {}^3\mathrm{He} \to \nu_\mu {}^3\mathrm{H})^{\mathrm{exp}}_{\mathrm{stat}} = 1496(4)/\sec$, if we naively set RC(He) = 0. That agreement has been viewed as a success of theory and used to constrain [4] new physics appendages to the standard model. On the other hand, Eq. (15) only agrees with experiment if one includes the +2.4% radiative correction contained in their G_V^2 value.

Now, we consider the electroweak radiative corrections (RC). They naturally divide into two contributions. The first set is essentially common to all semileptonic weak charged current amplitudes normalized in terms of G_μ, the Fermi constant obtained from the free muon lifetime. The second type are QED corrections to the muonic atom wave function. As pointed out by Goldman [31], those latter effects are dominated by vacuum polarization corrections to the Coulombic bound state interaction.

Making the above division, $\mathrm{RC}(N) = \mathrm{RC}(N)_1 + \mathrm{RC}(N)_2$, we find from the detailed studies of neutron decay [8,9] (neglecting terms of relative order $\alpha m_\mu/m_N$) that the $\mathcal{O}(\alpha)$ electroweak radiative corrections to the muon capture rate on hydrogen are given by

$$\mathrm{RC\,(H)}_1 = \frac{\alpha}{2\pi}\left[4\ln\frac{m_Z}{m_p} - 0.595 + 2C + g(m_\mu, \beta_\mu = 0)\right], \tag{16}$$

where $m_Z = 91.1875$ GeV, $m_p = 0.938$ GeV, $C = 0.829$, and the quantity $g(m_\mu, \beta_\mu = 0)$ can be obtained from Eq. (20b) in Ref. [32] by replacing $m_e \to m_\mu$, ignoring bremsstrahlung and taking the nonrelativistic (zero muon velocity) $\beta_\mu = 0$ limit. In that way one finds $g(m_\mu, \beta_\mu = 0) = 3\ln\frac{m_p}{m_\mu} - \frac{27}{4} = -0.199$. In total, Eq. (16) gives 0.0223. Summing up higher order leading logs along the lines of Refs. [7,8] enhances that correction somewhat to $\mathrm{RC(H)}_1 = 0.024(4)$, where we have included a fairly generous estimate of the uncertainty. It corresponds to roughly a $\pm100\%$ variation in C and conservatively allows for $\mathcal{O}(\alpha m_\mu/m_p)$ corrections that we have not computed.

We note that the first two bracketed terms in Eq. (16) (which include QCD perturbative effects) are of short-distance origin and therefore apply to all muon capture rates. Similarly, the g function is essentially unchanged as long as the muon is nonrelativistic and $\mathcal{O}(\alpha m_\mu/m_p)$ contributions are ignored. On the other hand, the quantity C is specific to hydrogen and will be modified by nucleon interactions in multinucleon systems. Rather than try to account for that modification, we assume that our rather conservative error covers those variations and continues to hold, $\mathrm{RC(He)}_1 = 0.024(4)$. If needed, this correction can be used as a good approximation for any muon capture rate. We note that our +2.4% correction happens to coincide numerically with the inner radiative corrections included in the G_V^2 value employed in [30].

At this point we note that the factorization of the radiative corrections comes about because in the formulation of Ref. [8], which we follow; the axial couplings in neutron decay have by definition the same electroweak radiative corrections as the vector ones. Small differences that can result from $q^2 \neq 0$ are included in the theoretical uncertainty or evolution uncertainty of the form factors.

The vacuum polarization correction to the muon bound state wave function [31] must be individually evaluated for different nuclei. A detailed calculation gives $\mathrm{RC(H)}_2 = 1.73\frac{\alpha}{\pi} \simeq 0.004$, which is somewhat smaller than found by Goldman [31]. In the case of helium, we obtain

$RC(He)_2 = 2.92\frac{\alpha}{\pi} \simeq 0.0068$. Overall, we find

$$1 + RC(H) = 1.028(4), \qquad 1 + RC(He) = 1.030(4), \tag{17}$$

which modify the capture rate predictions in Eqs. (12), (14), and (15) to

$$\Gamma(\mu^- p \to \nu_\mu n)_{1S}^{\text{singlet}} = 711.5(3.5)_{g_A}(3)_{\text{RC}}(1 - 0.0108\delta g_P)^2/\text{sec}, \quad \Gamma(\mu^- {}^3\text{He} \to \nu_\mu {}^3\text{H})_{\text{stat}}^{\text{EPM}} = 1537(22)(1 - 0.013\delta g_P)/\text{sec},$$
$$\Gamma(\mu^- {}^3\text{He} \to \nu_\mu {}^3\text{H})_{\text{stat}}^{\text{IA}} = 1506(8)(7)_{g_A}(6)_{\text{RC}}(1 - 0.013\delta g_P)/\text{sec}. \tag{18}$$

For hydrogen, comparison with the experimental results in Eq. (3) leads to $\delta g_p = -0.9 \pm 1.2$,

$$g_P^{\text{exp}} = 7.3 \pm 1.2 \qquad \text{(hydrogen)}. \tag{19}$$

The electroweak radiative corrections have increased the value of g_P^{exp} by about $+21\%$. They bring theory and experiment into agreement. That situation is to be contrasted with the world average $g_P^{\text{exp}} = 10.5 \pm 1.8$ obtained [3] from muon capture on hydrogen before the new MuCap result [13] and our evaluation of the radiative corrections. On its own, that previous world average would have been shifted to $g_P^{\text{exp}} = 11.7 \pm 1.8$ by the radiative corrections, about a 2σ deviation from the chiral perturbation theory prediction. However, including the MuCap [13] result, one finds the new world average from muon capture on hydrogen, $g_P^{\text{exp}} = 8.7 \pm 1.0$, in good agreement with the theoretical prediction of Eq. (11).

For helium, radiative corrections spoil somewhat the good agreement between experiment and the EPM prediction. The new disagreement suggests a smaller value of $g_A(-0.954m_\mu^2)_{\text{He}\to\text{H}}$ is likely or a significantly larger g_P by about 25% in magnitude beyond PCAC predictions (a situation similar to hydrogen if we had used the pre MuCap capture rates). On the other hand the impulse approximation (IA) approach [30] fares much better, leading to $g_P^{\text{exp}} = 8.7 \pm 0.6$ which is also in good agreement with chiral perturbation theory.

In summary, when our calculation of the electroweak radiative corrections to muon capture on hydrogen is combined with a new singlet $\mu^- p$ capture rate measurement, it leads to $g_P^{\text{exp}} = 7.3 \pm 1.2$, which is in very good accord with the prediction of chiral perturbation theory, $g_P^{\text{theory}} = 8.2 \pm 0.2$. That agreement would seem to close a confusing chapter in nuclear physics which has seen decades of disagreement regarding the value of g_P^{exp}. It will be very interesting to watch continuing improvements in the MuCap results.

We thank the MuCap Collaboration for discussions and advising us about their results. A. C. was supported by the Science and Engineering Research Canada. W. J. M. acknowledges support by DOE Grant No. DE-AC02-76CH00016. The work of A. S. was supported in part by the National Science Foundation Grant No. PHY-0245068. A. C. and W. J. M. thank the Institute for Nuclear Theory at the University of Washington for its hospitality and the Department of Energy for partial support.

[1] H. Primakoff, Rev. Mod. Phys. **31**, 802 (1959).
[2] N. C. Mukhopadhyay, Phys. Rep. **30**, 1 (1977).
[3] T. Gorringe and H. W. Fearing, Rev. Mod. Phys. **76**, 31 (2003).
[4] J. Govaerts and J.-L. Lucio-Martinez, Nucl. Phys. A **678**, 110 (2000).
[5] A. Sirlin, Nucl. Phys. **B71**, 29 (1974).
[6] A. Sirlin, Rev. Mod. Phys. **50**, 573 (1978).
[7] W. J. Marciano and A. Sirlin, Phys. Rev. Lett. **56**, 22 (1986).
[8] A. Czarnecki, W. J. Marciano, and A. Sirlin, Phys. Rev. D **70**, 093006 (2004).
[9] W. J. Marciano and A. Sirlin, Phys. Rev. Lett. **96**, 032002 (2006).
[10] H. Überall, Phys. Rev. **119**, 365 (1960).
[11] A. Czarnecki, G. P. Lepage, and W. J. Marciano, Phys. Rev. D **61**, 073001 (2000).
[12] D. B. Chitwood *et al.* (MuLan Collaboration), Phys. Rev. Lett. **99**, 032001 (2007).
[13] V. A. Andreev *et al.* (MuCap Collaboration), Phys. Rev. Lett. **99**, 032002 (2007).
[14] P. Ackerbauer *et al.*, Phys. Lett. B **417**, 224 (1998).
[15] A. Santisteban and R. Pascual, Nucl. Phys. **A260**, 392 (1976).
[16] K. Pachucki, Phys. Rev. A **60**, 3593 (1999).
[17] G. G. Simon, C. Schmitt, F. Borkowski, and V. H. Walther, Nucl. Phys. **A333**, 381 (1980).
[18] W. M. Yao *et al.*, J. Phys. G **33**, 1 (2006).
[19] N. Kaiser, Phys. Rev. C **67**, 027002 (2003).
[20] S. L. Adler and Y. Dothan, Phys. Rev. **151**, 1267 (1966).
[21] L. Wolfenstein, in *High Energy Physics and Nuclear Structure*, edited by S. Devons (Plenum, New York, 1970), p. 661.
[22] V. Bernard, N. Kaiser, and Ulf-G. Meissner, Phys. Rev. D **50**, 6899 (1994).
[23] V. Bernard, T. R. Hemmert, and Ulf-G. Meissner, Nucl. Phys. **A686**, 290 (2001).
[24] S. I. Ando, F. Myhrer, and K. Kubodera, Phys. Rev. C **63**, 015203 (2000).
[25] A. Fujii and Y. Yamaguchi, Prog. Theor. Phys. **31**, 107 (1964).
[26] B. Budick, J. S. Chen, and H. Lin, Phys. Rev. Lett. **67**, 2630 (1991).
[27] J. J. Simpson, Phys. Rev. C **35**, 752 (1987).
[28] J. G. Congleton and H. W. Fearing, Nucl. Phys. **A552**, 534 (1993).
[29] J. G. Congleton and E. Truhlik, Phys. Rev. C **53**, 956 (1996).
[30] L. E. Marcucci, R. Schiavilla, S. Rosati, A. Kievsky, and M. Viviani, Phys. Rev. C **66**, 054003 (2002).
[31] M. R. Goldman, Nucl. Phys. **B49**, 621 (1972).
[32] A. Sirlin, Phys. Rev. **164**, 1767 (1967).